PACIFIC JOURNAL OF MATHEMATICS
Vol 58, No. 1, 1975

THE SINGLE VALUED EXTENSION PROPERTY ON A BANACH SPACE

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An operator T which maps a Banach space X into itself has the single valued extension property if the only analytic function f which satisfies $(\lambda I - T)f(\lambda) = 0$ is $f = 0$. Clearly the point spectrum of any operator which does not have the single valued extension property must have nonempty interior. The converse does not hold. However, it is shown below that if $\lambda_0 I - T$ is semi-Fredholm and λ_0 is an interior point of the point spectrum of T, then T does not have the single valued extension property.

It will be convenient to use the following definition.

1. DEFINITION. Let $T: X \to X$ be a closed linear operator mapping a Banach space X into itself, and let λ_0 be a complex number. The operator T has the *single valued extension property at* λ_0 if $f = 0$ is the only solution to $(\lambda I - T)f(\lambda) = 0$ that is analytic in a neighborhood of λ_0. Also, T has the *single valued extension property* if it has this property at every point λ_0 in the complex plane.

2. THEOREM. *Let T be a closed linear operator mapping the Banach space X into itself. If T is onto but not one-one, then T does not have the single valued extension property at $\lambda = 0$.*

Proof. First we produce a candidate for f. Choose any x_0 in X with $\|x_0\| = 1$ and $Tx_0 = 0$, which is possible since T is not one-one. Since T is a closed operator and is onto, it is an open mapping. The open mapping theorem implies there exists a $k > 0$ such that for any element $x \in X$ there is a $y \in X$ with $Ty = x$ and $\|y\| \leqq k\|x\|$. Now choose x_n inductively so that $Tx_n = x_{n-1}$ and $\|x_n\| \leqq k\|x_{n-1}\|$. Define $f(\lambda) = \Sigma x_n \lambda^n$. Since $\|x_n\| \leqq k^n$, the sum converges for $|\lambda| < k^{-1}$, and f is analytic in this neighborhood of zero.

Now to show that $(\lambda I - T)f(\lambda) = 0$. Since T is a closed linear operator, so is $\lambda I - T$. Each of the partial sums $\Sigma_{n=0}^{N} x_n \lambda^n$ is in the domain of $\lambda I - T$, since each x_n was chosen from the domain of T. Furthermore

$$(\lambda I - T)\sum_{n=0}^{N} x_n \lambda^n = x_N \lambda^{N+1}$$

and

$$\|x_N \lambda^{N+1}\| \leqq k^N |\lambda|^{N+1}.$$

But as N goes to infinity, $k^N|\lambda|^{N+1}$ converges to zero for $|\lambda| < k^{-1}$. Since $\lambda I - T$ is a closed map, $f(\lambda) = \lim_N \Sigma_{n=0}^{N} x_n\lambda^n$ is in the domain of $\lambda I - T$ and $(\lambda I - T)f(\lambda) = \lim_N x_N\lambda^{N+1} = 0$.

The function $f(\lambda)$ obtained in the proof of Theorem 2 is certainly not unique. However, it is typical of any function g satisfying $(\lambda I - T)g(\lambda) = 0$ in the following sense: Suppose T is any closed operator not having the single valued extension property at $\lambda = 0$, and that g is any analytic function defined near $\lambda = 0$ satisfying $(\lambda I - T)g(\lambda) = 0$. Expand g in a Taylor series around $0: g(\lambda) = \Sigma x_n\lambda^n$. It can be shown that: (i) each x_n is in the domain of T; (ii) $Tx_{n+1} = x_n$ for $n = 0, 1, 2, \cdots$; and (iii) $Tx_0 = 0$.

The above discussion holds also at points $\lambda_0 \neq 0$ if we replace every T by $T - \lambda_0 I$, and every λ by $\lambda - \lambda_0$.

There are more interesting ways to express Theorem 2: If T has the single valued extension property, then T is invertible whenever it is onto. Or again, if T has the single valued extension property, then λ_0 is in the spectrum of T if and only if $\lambda_0 - T$ is not onto. In particular this is true for normal operators, spectral operators, etc.

3. COROLLARY. *Let T be a closed linear operator on a Banach space X and suppose Y is a cb sed invariant subspace. If $TY = Y$ but T is not one-one on Y, then T does not have the single valued extension property at* 0.

Actually, in Corollary 3, Y could be a linear manifold that is not closed, *provided* that it can be given a new norm, larger than the original, for which Y is complete (and hence becomes a Banach space).

4. COROLLARY. *Let Y be the domain of a closed linear operator $S: X \to Z$, where Z is a Banach space. If $TY = Y$ but T is not one-one on Y then T does not have the single valued extension property.*

5. COROLLARY. *If there is a bounded linear operator on X which is onto but not one-one, then the set of bounded operators that do not have the single valued extension property at* 0 *has nonempty interior* (in the norm topology). *And thus the set of operators without the single valued extension property has nonempty interior.*

A special case of the following result appears in Colojoară and Foiaş, Chapter 1.

6. COROLLARY. *Let T be a closed linear operator mapping the Banach space X into itself, and assume that the domain of T is dense in X so that the adjoint T^* exists. If T is bounded below but is not onto, then T^* does not have the single valued extension property. Or alternately, if the range of T is closed and T is one-one but not onto, then T^* does not have the single valued extension property.*

Proof. If T is bounded below, its range is closed, and T is one-one. Thus the range of T^* is the orthogonal set to $\{0\}$, which is all of X^*; that is, T^* is onto. Since T is not onto and the range of T is closed, the null space of T^* is not just $\{0\}$. Thus T^* is onto but not one-one, and so does not have the single valued extension property by Theorem 2.

A point λ is in the limit spectrum of T if and only if there is a sequence x_n with $\|x_n\| = 1$ and $(\lambda I - T)x_n$ converging to 0.

7. COROLLARY. *If the closed linear operator T has the single valued extension property, then the limit spectrum of T^* is the entire spectrum of T^*. Similarly, if T^* has the single valued extension property, then the limit spectrum of T is the entire spectrum of T.*

A closed linear operator is semi-Fredholm if the range is closed, and the dimension $n(T)$ of the null space or the codimension $d(T)$ of the range is finite (or both). First we investigate the case where the null space is finite dimensional, after a preliminary lemma needed in both proofs.

8. LEMMA. *Let T be a closed linear operator with closed range mapping a Banach space X into itself, and let N be its null space. For an arbitrary linear manifold M in X, if $M + N$ is closed, then the image $T(M)$ of M is closed.*

9. THEOREM. *Let T be a semi-Fredholm operator mapping a Banach space X into itself with $n(T)$ finite. If the point spectrum of T contains a neighborhood of zero, then T does not have the single valued extension property at $\lambda = 0$.*

Proof. Let Y be the subset of X given by the intersection of the ranges of T^n for all n, that is,

$$Y = \bigcap_{n=1}^{\infty} T^n X.$$

The proof consists of showing that Y is a closed invariant subspace of X, that T maps Y onto itself, and that T is not one-one on Y. Then, applying Corollary 3, we see that T does not have the single valued extension property.

It is obvious that Y is a linear manifold in X and that it is invariant under T. To show that Y is closed, we need only show that T^nX is closed for all n. Applying Lemma 9 to $M = T^nX$, it follows that $T^{n+1}(X) = T(T^nX)$ is closed if and only if $T^nX + N$ is closed. Now by hypothesis N is a finite dimensional closed subspace of X, and $TX$is closed. Since the sum of a closed subspace with a finite dimensional subspace is always closed, T^2X is closed. By induction T^nX is closed, and hence $Y = \bigcap (T^nX)$ is also closed.

The next step is to show that T maps Y onto itself. Let $R_n = T^nX$. For any y in Y there is an x_n in R_n with $Tx_n = y$. Also, $x_n - x_m$ is in N. Now $R_n \cap N$ is a decreasing sequence of subspaces. Since N is finite dimensional, this sequence is eventually constant. That is, for some $m, R_m \cap N = R_k \cap N$ for all $k \geqq m$. Thus $x_m - x_k$, which is in $R_m \cap N$, is also in R_k. Since x_k was chosen in R_k, it follows that $x_m = x_k + (x_m - x_k)$ is in R_k as well. That is, x_m is in R_k for all $k \geqq m$; therefore x_m is in $Y = \bigcap R_k$. Thus T does map Y onto itself.

It remains to show that the restriction of T to Y is not one-one. It is clear that if $Tx = \lambda x$ for some $\lambda \neq 0$, then x is in Y. Thus any $\lambda \neq 0$ that is in the point spectrum of T is also in the point spectrum of $T \mid Y$. Hence by our hypothesis, for some $r > 0$, every λ with $0 < |\lambda| < r$ is in the spectrum of $T \mid Y$. Since the spectrum of $T \mid Y$ is closed, it also contains 0. But then since $T \mid Y$ is onto, it cannot be one-one.

In summary, the resriction of T to Y maps Y onto itself but is not one-one. Hence T does not have the single valued extension property.

Taylor (1966) has shown the following: Let T be an arbitrary linear operator on a vector space (no topological properties are necessary). If $n(T)$ is finite, then $Y = \bigcap T^nX$ satisfies $TY = Y$. This result can fail if $n(T)$ is infinite.

A slight extension of this theorem is possible. We may replace the assumption that the range of T is closed by: the range of T^k is closed for some k. Since $n(T^k) \leqq kn(T)$ (see Taylor, 1966), we have that $n(T^k)$ is also finite. Then (using the notation of the proof of Theorem 9) $Y = \bigcap T^nX$ is also equal to $\bigcap (T^k)^nX$. The argument of the proof applied to T^k shows that Y is closed and that $T^kY = Y$, but T^k is not one-one on Y. But of course this means that $TY = Y$ and T is not one-one on Y. Hence, by Corollary 3, T does not have the single valued extension property.

10. THEOREM. *Let T be a closed linear operator mapping a Banach space X into itself, and suppose that the codimension of the range $d(T)$ is finite. If the point spectrum of T contains a neighborhood of zero, then T does not have the single valued extension property at $\lambda = 0$.*

Note that if T is a closed operator and the codimension (the dimension of X/R) of the range is finite, then the range is closed and so T is automatically a semi-Fredholm operator. (See Kato, 1966, p. 233, problem 5.7.)

Proof. The idea of this proof is similar to that of Theorem 9. Let $Y = \bigcap T^n X$; it will be shown that Y is a closed subspace invariant under T, and that T maps Y onto itself but is not one-one.

First we show that Y is closed; it is obviously a linear manifold that is invariant under T. Let $R_n = T^n X$. Since $d(T)$ is finite, then $d(T^n) \leqq nd(T)$ is also finite. It follows that $R_n + N = R_n \oplus F_n$ where F_n is a finite dimensional space. We now show that R_n is closed by an induction argument: $R_{n+1} = T(R_n) = T(R_n + N)$. From Lemma 8, R_{n+1} is closed if $R_n + N$ is. But $R_n + N = R_n \oplus F_n$ is closed since R_n is closed and F_n is finite dimensional. Thus $Y = \bigcap R_n$ is closed.

Next it will be shown that T maps Y onto itself. It will be sufficient to show that for some $m, R_k \cap N = R_m \cap N$ for $k \geqq m$; for then we complete the proof as in Theorem 9. For a proof of the contrapositive, suppose that for an infinite number of n there is a z_n in $R_n \cap N$ but not in $R_{n+1} \cap N$. Then $z_n = T^n u_n$, where u_n is not in R_1. But the u_n are linearly independent. For if $\Sigma_{k=1}^{K} a_k u_k = 0$, then taking T^K and recalling that z_k is in N, we get $a_K = 0$. Then recusively we get $a_k = 0$ for $k = K - 1, \cdots, 1$. Thus the u_n form an infinite linearly independent set not in R_1. This contradicts the assumption that the condimension of R_1 is finite. Thus T maps Y onto itself as in the proof of Theorem 9.

Finally, we show that T is not one-one on Y exactly as in the proof of Theorem 9.

In conclusion, Y is a closed subspace of X which is invariant under T, and T maps Y onto itself but is not one-one on Y. Thus T does not have the single valued extension property.

The two theorems above can be summarized to say that: *if T is a semi-Fredholm operator and the point spectrum of T contains a neighborhood of zero, then T does not have the single valued extension property at $\lambda = 0$.*

The requirement that T be semi-Fredholm in the above theorems seems to be crucial. Even though the range is closed and the point spectrum contains a neighborhood of zero, if $n(T) = d(T) = \infty$, T may still have the single valued extension property. (Certain normal operators on a nonseparable Hilbert space will work). An attempt to extend the proofs of Theorems 9 and 10 to operators which are not semi-Fredholm must encounter the following two difficulties: The subspace $Y = \bigcap T^n X$ may fail to be closed, or T may not map Y onto itself.

In Theorem 2 it was shown that an operator T which is onto but not one-one does not have the single valued extension property. Such operators are semi-Fredholm operators with $n(T) \geqq 1$ and $d(T) = 0$. A rather natural extension of this theorem is now possible.

11. COROLLARY. *Let T be a closed linear operator with closed range mapping a Banach space X into itself. If the dimension $n(T)$ of the null space is strictly greater than the codimension $d(T)$ of the range, then T does not have the single valued extension property.*

Proof. From the theory of semi-Fredholm operators, for sufficiently small perturbations $S, n(T+S) - d(T+S) = n(T) - d(T)$ (see Kato, Theorem 5.22). Thus for small λ,

$$\begin{aligned} n(\lambda I - T) &= n(T) - d(T) + d(\lambda I - T) \\ &> d(\lambda I - t) \geqq 0. \end{aligned}$$

That is, $\lambda I - T$ is not one-one for λ sufficiently small. Thus T is a semi-Fredholm operator with point spectrum containing a neighborhood of zero. By Theorem 10, T does not have the single valued extension property.

One would like to show that if $n(T) < d(T)$, then T does have the single valued extension property (at least at 0). Unfortunately, this is not true. For, if S is the right shift on Hilbert space, then $n(S) = 0$ and $d(S) = 1$; if T is any operator, then $n(T \oplus S) = n(T)$ and $d(T \oplus S) = d(T) + 1$. In this way we may extend T to a new operator $T \oplus S \oplus \cdots \oplus S$ with d arbitrarily large and n fixed, without affecting the single valued extendibility (or lack of it).

For a closed linear operator T having a dense domain on a Banach space X there is a unique adjoint operator T^* defined on a total subset of the dual space X^*.

12. COROLLARY. *Let T be a semi-Fredholm operator on X with domain dense in X. If $n(T) < d(T) \leqq \infty$, then T^* does not have the single valued extension property at $\lambda = 0$.*

Proof. For semi-Fredholm operators,

$$n(T^*) = d(T), \quad \text{and} \quad d(T^*) = n(T).$$

Hence $d(T^*) < n(T^*)$, and by Theorem 11, T^* does not have the single valued extension property.

13. COROLLARY. *If T is a closed linear operator on a Banach space with dense domain and closed range, and if both T and T^* have the single valued extension property, then $n(T) = d(T)$.*

14. COROLLARY. *Let T be a closed linear operator on a Banach space with dense domain and with $n(T) = d(T)$ finite. Then T has the single valued extension property near $\lambda = 0$ if and only if T^* does.*

Proof. Since $d(T)$ is finite and T is a closed operator, it follows that the range is closed and hence T is a semi-Fredholm operator. If T does not have the single valued extension property near 0, then $n(\lambda I - T) > 0$ for λ in a neighborhood of zero. Then

$$\begin{aligned} d(\lambda I - T) &= d(T) - n(T) + n(\lambda I - T) \\ &= n(\lambda I - T) > 0. \end{aligned}$$

Thus $n(\lambda I^* - T^*) = d(\lambda I - T)$ is strictly positive in a neighborhood of zero. But then T^* is a Fredholm operator whose point spectrum contains an open set, and so by Theorem 10, T^* does not have the single valued extension property.

Conversely, suppose T does have the single valued extension property. Then from Theorem 10, $n(\lambda I - T) = 0$ in a deleted neighborhood of zero (using the fact that $n(\lambda I - T)$ is constant in a deleted neighborhood of a point where $\lambda I - T$ is semi-Fredholm). Hence $d(\lambda I - T) = 0$ in this neighborhood. This implies that T^* has the single valued extension property near $\lambda = 0$.

Two more concepts are useful at this point. Consider the iterates $T^k, k = 0, 1, 2, \cdots$, of the operator T. The null space $N(T^{k+1})$ always contains $N(T^k)$ and may be strictly larger. But if for some

$p, N(T^{p+1}) = N(T^p)$, then for all $k > p$, $N(T^{k+1}) = N(T^k)$. The smallest p satisfying the above is the *ascent* of T. It may happen that the equation is not satisfied for any p; in this case the ascent is infinite. The *descent* is defined in a similar way with the ranges of T instead of the null spaces. It is the smallest q with $T^{q+1}X = T^qX$, and is infinite if no such q exists.

15. THEOREM. *Let T be a semi-Fredholm operator on X. Then T has the single valued extension property near 0 if and only if the ascent of T is finite.*

Proof. If the ascent is finite, then certainly T has the single valued extension property near 0. For if $(\lambda I - T)f(\lambda) = 0$ and $f(\lambda) = \Sigma x_n \lambda^n$, then x_n is in $N(T^{n+1})$ but not in $N(T^n)$. Hence $N(T^{n+1}) \neq N(T^n)$ for any n.

Suppose that the ascent is infinite. Since T is semi-Fredholm, if the nullity $n(T)$ is infinite, then the deficiency $d(T)$ is finite; and, by Theorem 11, T does not have the single valued extension property. Thus assume that $n(T)$ is finite, and let

$$Y = \bigcap_{n=1}^{\infty} T^n X.$$

As was shown in the proof of Theorem 9, Y is a closed, invariant subspace, and T maps Y onto Y. Since $n(T) < \infty$, the null space N in finite dimensional, and so $(T^kX) \cap N$ is eventually constant. Since the ascent is infinite, $(T^kX) \cap N \neq (0)$, for all k. It then follows that $Y \cap N \neq (0)$; that is, T is not one-one on Y. From Corollary 3, T does not have the single valued extension property.

16. COROLLARY. *If T is a semi-Fredholm operator with domain dense in X, then T^* has the single valued extension property if and only if the descent of T is finite.*

Proof. Since the range of T^k is closed for all k (as was shown in the proofs of Theorems 9 and 10), the null space of T^{*k} is the set of x^* orthogonal to the range of T^k. Hence the ascent of T^* is the descent of T, and the conclusion follows by Theorem 15.

Acknowledgement. This paper is based on the author's Ph.D. dissertation at the University of Illinois.

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Received July 17, 1974.

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