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INFIMUM AND DOMINATION PRINCIPLES IN VECTOR LATTICES

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The main purpose of this paper is to demonstrate (a) that the potential theoretic notions of infimum principle and domination principle are meaningful in a setting of a vector lattice with a monotone map to the dual space, and (b) in this general setting these two principles are equivalent under very weak hypotheses.

THEOREM 1. **If $T: L \to L'$ is a strictly monotone map, L a vector lattice, then the infimum principle implies the domination principle.**

THEOREM 2. **If $T: B \to B'$ is a monotone, coercive hemicontinuous map, B a reflexive Banach space and vector lattice with closed positive cone, then the domination principle implies the infimum principle.**

1. Introduction. The results herein improve theorems proved earlier by the author [7], and are motivated by work of Calvert [4] and Kenmochi and Mizuta [11, 12]. The paper [4] deals with the Sobolev space $W^{1,m}$ and a monotone operator satisfying certain further conditions. The papers [11, 12] deal with functional spaces whose intersection with either $\mathscr{C}$ (continuous compact support functions) or L^2 is a dense subspace and with a monotone operator which is the gradient of a certain convex function.

The present paper shows that the relationship between the infimum and domination principles is independent of the specialized properties of the above mentioned spaces. This relationship depends more on the lattice structure in a linear space in one case and on the lattice structure in a reflexive Banach space in the other case. Results on monotone operators of Browder [2, Theorem 1] and Hartman and Stampacchia [10, Theorem 1.1] are employed in the Banach space case.

2. Difinitions. Let L be a vector lattice with partial order $\ll$, and $L^+ = \{x \in L \mid 0 \ll x\}$. The symbol $x \wedge y$ denotes the infimum of x and y. Let L' be a subspace (possibly improper) of the algebraic dual of L. The usual bi-linear form on $L \times L'$ is denoted $\langle \cdot, \cdot \rangle$, and $L'^+ = \{f \in L' \mid \langle x, f \rangle \geqq 0$ for all $x \in L^+\}$.

DEFINITION 1. A map $T: L \to L'$ is *monotone* if for all $x, y \in L$, $\langle x - y, Tx - Ty \rangle \geqq 0$; T is *strictly monotone* if T is monotone and $\langle x - y, Tx - Ty \rangle = 0$ implies $x = y$.

In many treatments of kernel-free potential theory in Hilbert or Banach functional spaces L, pure potentials are associated with those elements $u \in L$ which satisfy $Tu \in L'^{+}$. See, for example, Beurling and Deny [1], Fowler [8, 9], Kenmochi and Mizuta [11, 12]. We are thus led naturally to the following definition.

DEFINITION 2. The triple (L, T, L') satisfies the *infimum principle* if for each pair $x, y \in L$ such that there exists $z' \in L'$ with $Tx - z' \in L'^{+}$ and $Ty - z' \in L'^{+}$, it follows that $T(x \wedge y) - z' \in L'^{+}$.

Suppose $L = \mathscr{X}(X, \xi)$ is a functional space with $L \cap \mathscr{C}$ dense in L and $\mathscr{C}$, as per Example 1, § 4. If we insist that $z' = T(z)$ with $z \in L$ a potential, then the above infimum principle is that embodied in Theorem 3.2 of Kenmochi and Mizuta [11].

Recall that the cone L'^{+} induces a partial order $\prec$ on L'. By adding certain hypotheses it can be assured that L' is a vector lattice under this order. See, for example, Day [5, pp. 98-99] or Namioka [13, Theorem 6.7, Corollary 7.3, Theorem 8.7]. If L' is a vector lattice, then the above infimum principle is equivalent to the condition:

$$(Tx \wedge Ty) \prec T(x \wedge y) \quad \text{for all } x, y \in L .$$

(The symbol $\wedge$ is used for infimum in both L and L'.) To see this equivalence, merely put $z' = Tx \wedge Ty$. The L^p spaces, $1 \leqq p$ are examples of this phenomenon.

DEFINITION 3. The triple (L, T, L') satisfies the *domination principle* if for each pair $x, y \in L$ such that there exists $z' \in L'$ with $Tx - z' \in L'^{+}$ and $Ty - z' \in L'^{+}$ and $\langle (x - y)^{+}, Tx - z' \rangle = 0$ it follows that $x \ll y$.

This is essentially the domination principle of [11, Theorem 3.3] and [4, p. 480] stated in an arbitrary vector lattice.

In the next two definitions we assume that L is a Banach space with norm $\| \cdot \|$, and L' is the continuous dual.

DEFINITION 4. A map $T: L \to L'$ is *hemi-continuous* if it is continuous from each line segment in L to the weak* topology in L'.

DEFINITION 5. A map $T: L \to L'$ is *coercive* if for all $w \in L$, $\langle v - w, Tv \rangle / \| v \| \to +\infty$ as $\| v \| \to +\infty$.

3. Theorems.

THEOREM 1. *Let L be a vector lattice. If $T: L \to L'$ is strictly* *monotone and the triple* (L, T, L') *satisfies the infimum principle, then it satisfies the domination principle.*

Proof. Let $x, y \in L$, $z' \in L'$ with $Tx - z'$, $Ty - z' \in L'^{+}$ and

$$\langle (x-y)^{+}, Tx - z' \rangle = 0 . \tag{1}$$

By the infimum principle, $T(x \wedge y) - z' \in L'^{+}$ so

$$\langle (x-y)^{+}, z' - T(x \wedge y) \rangle \leqq 0 . \tag{2}$$

Adding (1) and (2) yields

$$\langle (x-y)^{+}, Tx - T(x \wedge y) \rangle \leqq 0 . \tag{3}$$

But $(x-y)^{+} = x - x \wedge y$, so (3) becomes

$$\langle x - x \wedge y, Tx - T(x \wedge y) \rangle \leqq 0 .$$

Since T is strictly monotone, $x = x \wedge y$, i.e., $x \ll y$.

REMARK. The proof of Theorem 1 is essentially that of [4, Proposition 1.4]. However, it is not required that any notion of infimum exist in the dual space.

We now replace L by a reflexive Banach space B and let B' denote the continuous dual. Again, $\ll$ denotes a lattice order on B, and B^{+}, B'^{+} are as above. We do not assume that the norm of B is a lattice norm [13, p. 41] nor a monotone norm [13, p. 18]. However, for our main result we require that the positive cone B^{+} be closed in B. As shown in [13, Cor 4.2], this requirement is equivalent to the condition that $x \in B^{+}$ if and only if $\langle x, f \rangle \geqq 0$ for all $f \in B'^{+}$.

For the purposes of Theorem 2 below, we wish to extend [2, Theorem 1] to a closed convex set E not containing the origin. This will assure the existence of a solution $x_0 \in E$ to the variational inequality

$$\langle x_0 - x, Tx_0 \rangle \leqq 0 \text{ for all } x \in E .$$

The following lemma permits this simple extension. It is essentially the result given for separable Banach spaces in [6, p. 274].

LEMMA. *Let* $w \in B$ *be fixed and* $E = K + w$ *where* K *is a closed convex set in* B *with* $0 \in K$. *Let* $T\colon B \to B'$ *be monotone, hemi-continuous and coercive as defined in* § 2. *Then there exists* $x_0 \in E$ *such that*

$$\langle x_0 - x, Tx_0 \rangle \leqq 0 \textit{ for all } x \in E .$$

Proof. Define $A: B \to B'$ be $A(u) = T(u + w)$. Clearly A is monotone and hemi-continuous. If $\|v\| \geqq \|w\|$ we have for $u = v - w$,

$$\begin{aligned}\frac{\langle v - w, Tv\rangle}{2\|v\|} &\leqq \frac{\langle v - w, Tv\rangle}{\|v\| + \|w\|} \\ &\leqq \frac{\langle v - w, Tv\rangle}{\|v - w\|} = \frac{\langle u, T(u + w)\rangle}{\|u\|} \\ &= \frac{\langle u, Au\rangle}{\|u\|} .\end{aligned}$$

Thus the coercive condition on T entails $\langle u, Au\rangle/\|u\| \to +\infty$ as $\|u\| \to +\infty$. Since $0 \in K$ it follows from [2, Theorem 1] that there exists $u_0 \in K$ such that

$$\langle u_0 - y, Au_0\rangle \leqq 0 \text{ for all } y \in K .$$

Put $x_0 = u_0 + w$, $x = y + w$. Then $x, x_0 \in E$, $Tx_0 = Au_0$ and the result follows.

THEOREM 2. *Let the triple (B, T, B') satisfy the domination principle with B a reflexive Banach space and vector lattice, B^+ closed, and T a monotone, hemi-continuous, coercive map. Then (B, T, B') satisfies the infimum principle.*

Proof. Let $x, y \in B$, $z' \in B'$ with $Tx - z', Ty - z' \in B'^+$. Define

$$E = x \wedge y + B^+ = \{u \in B \mid x \wedge y \ll u\} .$$

It is immediate that E is convex and closed. Define $S: B \to B'$ by $Su = Tu - z'$. Clearly S is monotone, hemi-continuous and coercive since T has these properties. By the lemma applied to S, there exists $x_0 \in E$ such that

$$\langle u - x_0, Sx_0\rangle \geqq 0 \text{ for all } u \in E . \tag{4}$$

But for any $v \in B^+$, $v = (v + x_0) - x_0$ and $v + x_0 \in E$, so (4) entails $Sx_0 \in B'^+$, i.e.,

$$Tx_0 - z' \in B'^+ . \tag{5}$$

It remains to show $x_0 = x \wedge y$. Now $x_0 \wedge x \ll x_0$, so $x_0 - (x_0 \wedge x) \in B^+$, and (5) entails

$$\langle x_0 - (x_0 \wedge x), Tx_0 - z'\rangle \geqq 0 . \tag{6}$$

But $x_0 \in E$, $x \in E$ and the definition of E imply $x \wedge y \ll x_0$ and $x \wedge y \ll x$, so $x \wedge y \ll x_0 \wedge x$, i.e., $x_0 \wedge x \in E$. Thus (4) entails

$$\langle x_0 - x_0 \wedge x, Tx_0 - z'\rangle \leqq 0 . \tag{7}$$

By (6) and (7) we see $\langle x_0 - x_0 \wedge x, Tx_0 - z' \rangle = 0$, i.e., $\langle (x_0 - x)^+, Tx_0 - z' \rangle = 0$. Thus, (5) and the assumption that $Tx - z' \in B'^+$ and the domination principle yield $x_0 \ll x$. Analogously, $x_0 \ll y$, so $x_0 \ll x \wedge y$. By definition of $x_0 \in E$ we have $x \wedge y \ll x_0$. Since a lattice order is necessarily anti-symmetric, it follows that $x_0 = x \wedge y$ and thus $T(x \wedge y) - z' \in B'^+$.

4. Examples.

EXAMPLE 1. Lex X be a locally convex Hausdorff space with countable base and ξ a positive Radon measure on X. Let $\mathscr{X} = \mathscr{X}(x, \xi)$ be a reflexive Banach functional space as in [11, p. 744]. For $1 < m < \infty$ let Φ be a strictly convex real valued function on $\mathscr{X}$ satisfying

$$\Phi(0) = 0$$
$$\Phi(u) \geqq k \| u \|^m \quad \text{for all } u \in \mathscr{X},$$

k a positive constant. Also, let Φ be bounded on bounded sets and the gradient $\nabla\Phi$ exist at each $u \in \mathscr{X}$:

$$\langle v, \nabla\Phi(u) \rangle = \lim_{t \to 0+} \frac{\Phi(u + tv) - \Phi(u)}{t} .$$

Then $\nabla\Phi\colon \mathscr{X} \to \mathscr{X}'$ is a strictly monotone, hemi-continuous and coercive operator. See [11, 12]. Necessary conditions for the infimum and domination principles to hold are therein discussed at length.

EXAMPLE 2. Let $\Omega \subset R^n$ be open. For $1 < p < \infty$ the space $W_0^{1,p}(\Omega)$ is the closure of $\mathscr{D}(\Omega)$, the test space for distributions, in the Sobolev space $W^{1,p}(\Omega)$ in norm

$$\| u \| = \Big(\sum_{|j| \leqq 1} \| D^j u \|_{L^p}^p \Big)^{1/p} .$$

It was shown in [9, p. 322] that the spaces $W_0^{1,p}$ and $W^{1,p}$ are uniformly convex, smooth functional spaces and vector lattices and thus satisfy the requirements of Example 1 above.

EXAMPLE 3. For $p > 1, 0 < \alpha < 1$ and $2n/(n + 2\alpha) < p$, the fractional Sobolev space $L_\alpha^p(R^n)$ is the set of $f \in L^p(R^n)$ such that $\mathscr{D}_\alpha f \in L^p(R^n)$, where

$$\mathscr{D}_\alpha f(x) = \int_{R^n} | f(x) - f(y) |^2 \frac{dy}{| x - y |^{n+2\alpha}} .$$

The norm is $\| f \|_\alpha = (\| f \|_{L^p}^p + \| \mathscr{D}_\alpha \|_{L^p}^p)^{1/p}$. In [9, p. 324] it is shown that L_α^p is a smooth, uniformly convex Banach-Dirichlet space [9, p. 311] and thus a vector lattice satisfying the requirements of Example 1.

EXAMPLE 4. Let B be a reflexive Banach space with continuous dual B'. Let $T: B \to B'$ be a duality map, i.e.,

$$\langle u, T(u)\rangle = \| u \| \, \| Tu \| = \| u \| \, \varphi(\| u \|) \text{ for all } u \in B$$

where $\varphi: R^1 \to R^1$ is a continuous, strictly increasing function with $\varphi(0) = 0$ and $\lim_{r\to+\infty} \varphi(r) = +\infty$. Then T is coercive because for fixed $w \in B$,

$$\begin{aligned}\langle v - w, Tv\rangle/\| v \| &\geqq \| Tv \| - \frac{\| w \| \, \| Tv \|}{\| v \|} \\ &= \varphi(\| v \|)(1 - \| w \|/\| v \|) \to +\infty\end{aligned}$$

as $\| v \| \to +\infty$. It is shown in [3, p. 368] that T is monotone, and if B is strictly convex then T is strictly monotone. Further, if B is smooth, then corresponding to each φ in the definition of duality map, there is a unique, hemi-continuous duality map T.[1]

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[1] In the author's opinion, it was incorrectly reported in [**3**, p. 371] that T corresponding to φ is unique and continuous from B to the weak topology of B' if B is strictly convex. A note indicating this and that Theorem 4 of [**3**] is incorrect, together with counterexample and correction, has been submitted elsewhere.

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