PACIFIC JOURNAL OF MATHEMATICS
Vol. 60, No. 1, 1975

NONLINEAR INTEGRAL EQUATIONS AND PRODUCT INTEGRALS

ALVIN J. KAY

B. W. Helton has studied linear equations of the form

$$f(x)=f(a)+(RL)\int_x^a (Kf+Mf); \tag{1}$$

this paper extends some of his results to a nonlinear setting. Let S be a linearly ordered set, $\{G,+,\|\ \|\}$ a complete normed abelian group, H the set of functions from G to G that take 0 to 0, $\mathcal{OA}$ and $\mathcal{OM}$ classes of functions from SXS to H that are order-additive and order-multiplicative respectively and satisfy a Lipschitz-type condition, and $\mathscr{E}$ be J. S. Mac Nerney's reversible mapping from $\mathcal{OA}$ onto $\mathcal{OM}$. If $\{V,W\}$ is in $\mathscr{E}$, we show the collection of all functions that are differentially equivalent to V is the same as the collection of functions that are differentially equivalent to $W-1$. This analysis is used to prove existence theorems for product integrals which we show solve (1).

1. Introduction. In his 1966 paper, *Integral Equations and Product Integrals* [2], B. W. Helton obtained product integral solutions of linear integral equations of the form (1) where the integration is directed along intervals in some linearly ordered system, the functions involved have their values in a complete normed ring, and the right-left integral is of the subdivision-refinement type.

We extend some of these results to the nonlinear setting developed by J. S. Mac Nerney in [7]. As in [7], S denotes some nondegenerate set, with linear ($\leqq$) ordering $\mathcal{O}$; $\{G,+,\|\ \|\}$ denotes a complete normed abelian group with zero element 0, and H denotes the class of all functions from G to G to which $\{0,0\}$ belongs, with identity function 1. $\mathcal{OA}^+$ denotes the class of all $\mathcal{O}$-additive functions from SXS to the set of nonnegative real numbers, and $\mathcal{OM}^+$ denotes the class of all $\mathcal{O}$-multiplicative functions from SXS to the set of real numbers not less than 1.

The class $\mathcal{OA}$ consists of all functions V from SXS to H such that

(i) V is $\mathcal{O}$-additive in the sense that, for each $\{x,z,P\}$ in $SXSXG$, if $\{x,y,z\}$ is an $\mathcal{O}$-subdivision of $\{x,z\}$ then

$$V(x,y)P+V(y,z)P=V(x,z)P, \qquad \text{and}$$

(ii) there is a member α in $\mathcal{OA}^+$ such that if $\{x,y\}$ is in SXS and $\{P,Q\}$ is in GXG then

$$\| V(x,y)P - V(x,y)Q \| \leqq \alpha(x,y) \| P - Q \|.$$

The class $\mathcal{OM}$ consists of all functions W from SXS to H such that

(i) W is $\mathcal{O}$-multiplicative in the sense that, for each $\{x,z\}$ in SXS and P in G, if $\{x,y,z\}$ is an $\mathcal{O}$-subdivision of $\{x,z\}$ then

$$W(x,y)W(y,z)P = W(x,z)P, \qquad \text{and}$$

(ii) there is a member μ of $\mathcal{OM}^+$ such that if $\{x,y\}$ is in SXS and $\{P,Q\}$ is in GXG then

$$\| [W(x,y)-1]P - [W(x,y)-1]Q \| \leqq [\mu(x,y)-1] \| P - Q \|.$$

In **[7]**, Mac Nerney establishes that there is a reversible function $\mathscr{E}$ from $\mathcal{OA}$ onto $\mathcal{OM}$ such that if V is in $\mathcal{OA}$ and $W = \mathscr{E}(V)$ then, for $\{x,y,P\}$ in $SXSXG$,

$$W(x,y)P = {}_x\prod^y [1+V]P \quad \text{and} \quad V(x,y)P = {}_x\sum^y [W-1]P.$$

If $\{V,W\}$ is in $\mathscr{E}$, we show the collection of functions that are differentially equivalent to V is the same as the collection of functions that are differentially equivalent to $W-1$ (i.e., functions M and N from SXS to H are differentially equivalent only in case there is a function k from SXS to the real numbers such that ${}_x\Sigma^y k = 0$ and $\|M(x,y)P - N(x,y)P\| \leqq k(x,y)\|P\|$ for each $\{x,y,P\}$ in $SXSXG$ **[6]**). This analysis is used to prove existence theorems for product integrals of the form

$$W(x,y)P = {}_x\prod^y [1-M]^{-1}[1+K]P,$$

where $\{x,y,P\}$ is in $SXSXG$ and there is a $\{V_1, V_2\}$ in $\mathcal{OA}X\mathcal{OA}$ such that K and M are differentially equivalent to V_1 and V_2 respectively. Product integrals of this form were introduced in the linear case by Helton in **[2]**. In addition, we show that if $V(x,y)P = {}_x\Sigma^y \{[1-M]^{-1}[1+K]-1\}P$ for each $\{x,y,P\}$ in $SXSXG$, then $\{V,W\}$ is in $\mathscr{E}$. Finally we show if f is a function from S to G that is of bounded variation on each $\mathcal{O}$-interval of S then $W(x,a)f(a)$ solves (1) and as in **[7**, §3**]** it is shown that the theory of the seemingly more general equation

$$u(x) = P_1 + (RL)\int_c^x (Ku + Mu) + V(x,c)P_2$$

is subsumed in this treatment.

In [7, p. 624] Professor Mac Nerney defines sum and product integrals in this setting. We indicate the definitions: if g is a function from SXS to G, h is a function from SXS to H and $\{x, y, P\}$ is in $SXSXG$, ${}_x\sum^y g \sim \sum_1^n g(t_{j-1}, t_j)$ and ${}_x\prod^y [h]P \sim \left\{\prod_1^n h(t_{j-1}, t_j)\right\} P$ (functional composite) where $\{t_j\}_0^n$ is an $\mathcal{O}$-subdivision of $\{x, y\}$.

Let Φ denote a function from $\mathcal{O}\mathcal{A}$ such that if V is in $\mathcal{O}\mathcal{A}$ then $\Phi(V)$ is the set to which K belongs only in case K is differentially equivalent to V.

Let ψ denote a function from $\mathcal{O}\mathcal{M}$ such that if W is in $\mathcal{O}\mathcal{M}$ then $\psi(W)$ is the set to which K belongs only in case K is differentially equivalent to $W - 1$.

REMARK. In [2, p. 299] Professor Helton defines function classes OA^0, OM^0 and OB^0. In the linear case, our $\Phi(\mathcal{O}\mathcal{A})$ includes the common part of OA^0 and OB^0 and $\psi(\mathcal{O}\mathcal{M})$ includes the common part of OM^0 and OB^0.

2. $\psi[\mathcal{E}] = \Phi$. In this section we prove two theorems that will be used in the proofs of later theorems. In the first theorem we prove that if K is in $\psi(\mathcal{O}\mathcal{M})$ then the sum and product integrals of K exist and in the second theorem we prove that if $\{V, W\}$ is in $\mathcal{E}$, the collection of functions which are differentially equivalent to V is the same as the collection of functions which are differentially equivalent to $W - 1$.

THEOREM 2.1. *If $\{V, W\}$ is in $\mathcal{E}$ and K is in $\psi(W)$ then*

(1) $W(x, y)P = {}_x\Pi^y [1 + V]P = {}_x\Pi^y [1 + K]P$ *for every* $\{x, y, P\}$ *in* $SXSXG$, *and*

(2) $V(x, y)P = {}_x\Sigma^y [W - 1]P = {}_x\Sigma^y KP$ *for every* $\{x, y, P\}$ *in* $SXSXG$.

Proof. (1) Let W be in $\mathcal{O}\mathcal{M}$ and K in $\psi(W)$, k be a function from SXS to the real numbers such that for $\{x, y, P\}$ in $SXSXG$

$$\| K(x, y)P - [W(x, y) - 1]P \| \leqq k(x, y) \| P \|$$

and ${}_x\Sigma^y k = o$, and μ be a member of $\mathcal{O}\mathcal{M}^+$ such that for each $\{x, y\}$ in SXS and $\{P, Q\}$ in GXG

$$\| [W(x, y) - 1]P - [W(x, y) - 1]Q \| \leqq [\mu(x, y) - 1] \| P - Q \|.$$

Suppose c is a positive number and $\{x, y, P\}$ is in $SXSXG$. There is an $\mathcal{O}$-subdivision s of $\{x, y\}$ such that if $\{t_j\}_0^n$ is a refinement of s then $\Sigma_1^n k(t_{j-1}, t_j) < c/2\mu(x, y)^2$ and $\mathrm{Exp}[\Sigma_1^n k(t_{j-1}, t_j)] < 2$. By Lemma 1.2 [7, p. 623]

$$\left\| {}_{l}\prod^{n} [1+K(t_{j-1},t_j)]P - {}_{l}\prod^{n} W(t_{j-1},t_j)P \right\|$$

$$\leqq \left\{ {}_{1}\prod^{n} [\mu(t_{j-1},t_j)+k(t_{j-1},t_j)] - {}_{1}\prod^{n} \mu(t_{j-1},t_j) \right\} \|P\|$$

$$= \sum_{j=1}^{n} \prod_{i=1}^{j-1} \mu(t_{i-1},t_i) k(t_{j-1},t_j) \prod_{q=j+1}^{n} [\mu(t_{q-1},t_q)+k(t_{q-1},t_q)] \|P\|$$

$$\leqq \mu(x,y)^2 \operatorname{Exp}\left[\sum_{j=1}^{n} k(t_{j-1},t_j) \right] \sum_{j=1}^{n} k(t_{j-1},t_j) \|P\| < c\,\|P\|.$$

(2) For each $\mathcal{O}$-subdivision $\{t_j\}_0^n$ of $\{x,y\}$ in SXS

$$\left\| V(x,y)P - \sum_{t}^{n} K(t_{j-1},t_j)P \right\|$$

$$\leqq \sum_{1}^{n} \|[W(t_{j-1},t_j)-1]P - K(t_{j-1},t_j)P\|$$

$$+ \left\| \sum_{t} [W-1]P - V(x,y)P \right\|$$

$$\leqq \left\{ \sum_{1}^{n} k(t_{j-1},t_j) + \sum_{t} [\mu - 1] - \alpha(x,y) \right\} \|P\|.$$

Since ${}_x\Sigma^y k + {}_x\Sigma^y (\mu - 1) - \alpha(x,y) = 0$ the proof is complete.

REMARK. The proof of the following theorem is similar to the proof of Theorem 3.4 [2, p. 301] of which this theorem is an extension.

THEOREM 2.2. $\psi[\mathscr{E}] = \Phi$.

Proof. Part I. Let V be in $\mathcal{O}\mathscr{A}$ and $\mathscr{E}(V) = W$ and K be in $\psi(W)$; there is a μ in $\mathcal{O}\mathscr{M}^+$ such that for each $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG

$$\|\{W(x,y)-1\}P - \{W(x,y)-1\}Q\| \leqq [\mu(x,y)-1]\|P-Q\|.$$

Also there is a function k from SXS to the real numbers such that ${}_x\Sigma^y k = 0$ and

$$\|\{1+K(x,y)\}P - W(x,y)P\| \leqq k(x,y)\|P\|.$$

By Corollary 1.1 [7, p. 626]

$$\left\{[\mu(x,y)-1]-{}_x\sum^y[\mu-1]\right\}\|P\|$$
$$\geqq\left\|[W(x,y)-1]P-{}_x\sum^y[W-1]P\right\|$$
$$\geqq\left\|K(x,y)P-{}_x\sum^y[W-1]P\right\|-\|[1+K(x,y)]P-W(x,y)P\|;$$

hence

$$\|K(x,y)P-V(x,y)P\|\leqq\left\{[\mu(x,y)-1]-{}_x\sum^y[\mu-1]+k(x,y)\right\}\|P\|$$

so K is in $\Phi(V)$.

Part 2. Let K be in $\Phi(V)$; there is an α in $\mathcal{OA}^+$ such that if $\{x,y\}$ is in SXS and $\{P,Q\}$ is in GXG then

$$\|V(x,y)P-V(x,y)Q\|\leqq\alpha(x,y)\|P-Q\|$$

and there is a function h from SXS to the real numbers such that $\|V(x,y)P-K(x,y)P\|\leqq h(x,y)\|P\|$ and ${}_x\Sigma^y h=0$. By Corollary 1.1 [7, p. 626]

$$\|[1+K(x,y)]P-W(x,y)P\|$$
$$\leqq\left\|[1+V(x,y)]P-{}_x\prod^y[1+V]P\right\|+\|V(x,y)P-K(x,y)P\|$$
$$\leqq\left[{}_x\prod^y[1+\alpha]-\alpha(x,y)-1+h(x,y)\right]\|P\|;\text{ therefore } K \text{ is in } \psi(\mathscr{E}(V)).$$

3. Existence theorems. In this section we will prove that if each of K and M is in $\Phi(\mathcal{OA})$ and $[1-M(x,y)]^{-1}P$ exists and is bounded sufficiently there is a member V of $\mathcal{OA}$ such that

$$[1+K][1-M]^{-1}-1$$

is in $\Phi(V)$; hence ${}_x\Pi^y[1+K][1-M]^{-1}P={}_x\Pi^y[1+V]P$ for every $\{x,y,P\}$ in $SXSXG$. This extends existence theorems proven by J. S. Mac Nerney [7], B. W. Helton [3], J. V. Herod [5] and J. C. Helton [4].

THEOREM 3.0. *If each of α_1 and α_2 is in $\mathcal{OA}^+$ and $\alpha_2(x,y)<1$ for each $\{x,y\}$ in SXS, then*

(1) $\alpha(x,y)={}_x\Sigma^y\{[1+\alpha_1][1-\alpha_2]^{-1}-1\}$ *exists for each $\{x,y\}$ in SXS and α is in $\mathcal{OA}^+$;*

(2) $\mu(x,y) = {}_x\Pi^y\,[1+\alpha_1][1-\alpha_2]^{-1}$ *exists for each* $\{x,y\}$ *in SXS and* μ *is in* $\mathcal{OM}^+$; *and*

(3) $\mu(x,y) = {}_x\Pi^y\,\{1+\alpha\}$ *for each* $\{x,y\}$ *in SXS.*

Proof. Let $\beta = [1+\alpha_1][1-\alpha_2]^{-1} - 1$; if $\{r,s,t\}$ is an $\mathcal{O}$-subdivision of $\{r,t\}$ in SXS,

$$\beta(r,t) \geqq \beta(r,s) + \beta(s,t) \geqq 0.$$

Hence $\alpha(x,y) = {}_x\Sigma^y\beta = G.L.B.\ \Sigma_t\beta \geqq 0$ for all $\mathcal{O}$-subdivisions t of $\{x,y\}$ in SXS. β is in $\Phi(\alpha)$ and from Theorem 2.2 β is in $\psi(\mathscr{E}(\alpha))$. Hence from Theorem 2.1 $\mu(x,y) = {}_x\Pi^y\,[1+\beta] = {}_x\Pi^y\,[1+\alpha]$ for all $\{x,y\}$ in SXS, and the proof is complete.

REMARK. As noted by Herod **[5**, p. 188] and proved by Neuberger **[8**, p. 101], if T is in H and $0 < t < 1$ and $\|TP - TQ\| \leqq t\,\|P-Q\|$ for all $\{P,Q\}$ in GXG then $(1-T)^{-1}$ is in H, $(1-T)^{-1} = 1 + T(1-T)^{-1}$, and for each such $\{P,Q\}$ $\|(1-T)^{-1}P - (1-T)^{-1}Q\| \leqq (1-t)^{-1}\|P-Q\|$. These and closely related inequalities are used in the sequel, usually without explicit reference.

THEOREM 3.1. *If each of* V_1 *and* V_2 *is in* $\mathcal{OA}$ *and each of* α_1 *and* α_2 *is in* $\mathcal{OA}^+$ *such that for* $\{x,y\}$ *in SXS and* $\{P,Q\}$ *in GXG,* $\alpha_2(x,y) < 1$,

$$\|V_1(x,y)P - V_1(x,y)Q\| \leqq \alpha_1(x,y)\,\|P-Q\| \quad \textit{and}$$

$$\|V_2(x,y)P - V_2(x,y)Q\| \leqq \alpha_2(x,y)\,\|P-Q\|,$$

then

(1) $V(x,y)P = {}_x\Sigma^y\,\{[1+V_1][1-V_2]^{-1} - 1\}P$ *exists for each* $\{x,y,P\}$ *in SXSXG and* V *is in* $\mathcal{OA}$;

(2) $W(x,y)P = {}_x\Pi^y\,[1+V_1][1-V_2]^{-1}P$ *exists for each* $\{x,y,P\}$ *in SXSXG and* W *is in* $\mathcal{OM}$; *and*

(3) $\{V,W\}$ *is in* $\mathscr{E}$.

Proof. (1) Note that $[1+V_1][1-V_2]^{-1} - 1 = [V_1+V_2][1-V_2]^{-1}$ and if $\{x,y,P\}$ is in $SXSXG$ and $\{x,s,t,y\}$ is an $\mathcal{O}$-subdivision of $\{x,y\}$ then

$$\begin{aligned}
&\|[1-V_2(x,y)]^{-1}P - [1-V_2(s,t)]^{-1}P\| \\
&\quad = \|V_2(x,y)[1-V_2(x,y)]^{-1}P - V_2(s,t)[1-V_2(s,t)]^{-1}P \\
&\qquad \pm V_2(x,y)[1-V_2(s,t)]^{-1}P\| \\
&\quad \leqq \alpha_2(x,y)\,\|[1-V_2(x,y)]^{-1}P - [1-V_2(s,t)]^{-1}P\| \\
&\qquad + [\alpha_2(x,y) - \alpha_2(s,t)][1-\alpha_2(s,t)]^{-1}\,\|P\| \\
&\quad \leqq \{[1-\alpha_2(x,y)]^{-1} - [1-\alpha_2(s,t)]^{-1}\}\,\|P\|.
\end{aligned}$$

For each $\{x,y,P\}$ in $SXSXG$ and $\mathcal{O}$-subdivision $\{t_j\}_0^n$ of $\{x,y\}$

$$\|[V_1(x,y)+V_2(x,y)][1-V_2(x,y)]^{-1}P-\sum_t[V_1+V_2]\quad\times[1-V_2]^{-1}P\|$$

$$=\|\left\{\sum_t[V_1+V_2]\right\}[1-V_2(x,y)]^{-1}P-\sum_t[V_1+V_2][1-V_2]^{-1}P\|$$

$$\leqq\sum_1^n[\alpha_1(t_{j-1},t_j)+\alpha_2(t_{j-1},t_j)]\{[1-\alpha_2(x,y)]^{-1}$$
$$-[1-\alpha_2(t_{j-1},t_j)]^{-1}\}\|P\|$$
$$=\{[\alpha_1(x,y)+\alpha_2(x,y)][1-\alpha_2(x,y)]^{-1}-\sum_t[\alpha_1+\alpha_2]\times[1-\alpha_2]^{-1}\}\|P\|.$$

If follows that if s is a refinement of t

$$\left\|\sum_s[V_1+V_2][1-V_2]^{-1}P-\sum_t[V_1+V_2][1-V_2]^{-1}P\right\|$$
$$\leqq\left\{\sum_s[\alpha_1+\alpha_2][1-\alpha_2]^{-1}-\sum_t[\alpha_1+\alpha_2][1-\alpha_2]^{-1}\right\}\|P\|.$$

Hence, by the completeness of $\{G,+,\|\ \ \|\}$ and Theorem 3.0 $V(x,y)P={}_x\Sigma^y[V_1+V_2][1-V_2]^{-1}P$ exists. For each $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG $\|V(x,y)P-V(x,y)Q\|\leqq\alpha(x,y)\|P-Q\|$ where α is defined as in Theorem 3.0. Therefore V is in $\mathcal{OA}$ and, with β as in the proof of Theorem 3.0, considerations of $\beta-\alpha$ may be seen to show that $[1+V_1][1-V_2]^{-1}-1$ is in $\Phi(V)$. (2) and (3) follow from (1) and Theorems 2.1 and 2.2.

THEOREM 3.2. *If each of V_1 and V_2 is in $\mathcal{OA}$, and each of α_1 and α_2 is in $\mathcal{OA}^+$ such that for each $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG $\alpha_2(x,y)<1$,*

$$\|V_1(x,y)P-V_1(x,y)Q\|\leqq\alpha_1(x,y)\|P-Q\|,\quad\text{and}$$
$$\|V_2(x,y)P-V_2(x,y)Q\|\leqq\alpha_2(x,y)\|P-Q\|,$$

then

(1) *$V(x,y)P={}_x\Sigma^y\{[1-V_2]^{-1}[1+V_1]-1\}P$ exists for each $\{x,y,P\}$ in $SXSXG$, and V is in $\mathcal{OA}$;*

(2) *$W(x,y)P={}_x\Pi^y[1-V_2]^{-1}[1+V_1]P$ exists for each $\{x,y,P\}$ in $SXSXG$ and W is in $\mathcal{OM}$; and*

(3) *$\{V,W\}$ is in $\mathcal{E}$.*

Proof. (1) Note that

$$[1 - V_2]^{-1}[1 + V_1] - 1 = V_1 + V_2[1 - V_2]^{-1}[1 + V_1] \quad \text{and if} \quad \{x, y, P\}$$

is in $SXSXG$ and $\{x, s, t, y\}$ is an $\mathcal{O}$-subdivision of $\{x, y\}$ then

$$\begin{aligned}
&\|[1 - V_2(x, y)]^{-1}[1 + V_1(x, y)]P - [1 - V_2(s, t)]^{-1}[1 + V_1(s, t)]P\| \\
&\quad = \| V_1(x, y)P - V_1(s, t)P + V_2(x, y)[1 - V_2(x, y)]^{-1}[1 + V_1(x, y)]P \\
&\qquad - V_2(s, t)[1 - V_2(s, t)]^{-1}[1 + V_1(s, t)]P \\
&\qquad \pm V_2(x, y)[1 - V_2(s, t)]^{-1}[1 + V_1(s, t)]\| \\
&\quad \leqq [\alpha_1(x, y) - \alpha_1(s, t)]\|P\| + \alpha_2(x, y)\|[1 - V_2(x, y)]^{-1}[1 + V_1(x, y)]P \\
&\qquad - [1 - V_2(s, t)]^{-1}[1 + V_1(s, t)]P\| \\
&\qquad + [\alpha_2(x, y) - \alpha_2(s, t)][1 - \alpha_2(s, t)]^{-1}[1 + \alpha_1(s, t)]\|P\| \\
&\quad \leqq \{[1 - \alpha_2(x, y)]^{-1}[1 + \alpha_1(x, y)] - [1 - \alpha_2(s, t)]^{-1}[1 + \alpha_1(s, t)]\}\|P\|.
\end{aligned}$$

For each $\{x, y, P\}$ in $SXSXG$ and $\mathcal{O}$-subdivision $\{t_j\}_0^n$ of $\{x, y\}$

$$\begin{aligned}
&\| V_1(x, y)P + V_2(x, y)[1 - V_2(x, y)]^{-1}[1 + V_1(x, y)]P \\
&\qquad - \textstyle\sum_t \{V_1 + V_2[1 - V_2]^{-1}[1 + V_1]\}P\Big\| \\
&\quad = \Big\|\Big\{\textstyle\sum_t V_2\Big\}[1 - V_2(x, y)]^{-1}[1 + V_1(x, y)]P \\
&\qquad\qquad - \textstyle\sum_t V_2[1 - V_2]^{-1}[1 + V_1]P\Big\| \\
&\quad \leqq \textstyle\sum_1^n \alpha_2(t_{j-1}, t_j)\{[1 - \alpha_2(x, y)]^{-1}[1 + \alpha_1(x, y)] \\
&\qquad - [1 - \alpha_2(t_{j-1}, t_j)]^{-1}[1 + \alpha_1(t_{j-1}, t_j)]\}\|P\| \\
&\quad = \{[1 - \alpha_2(x, y)]^{-1}[1 + \alpha_1(x, y)]\} - 1 - \textstyle\sum_t \{[1 - \alpha_2]^{-1}[1 + \alpha_1] - 1\}\|P\|.
\end{aligned}$$

The rest of the proof is identical to the proof of Theorem 3.1.

THEOREM 3.3. *Suppose*

(1) *each of* V_1 *and* V_2 *is in* $\mathcal{O}\mathcal{A}$, *and each of* α_1 *and* α_2 *is in* $\mathcal{O}\mathcal{A}^+$ *such that for each* $\{x, y, P, Q\}$ *in* $SXSXGXG$

$$\| V_1(x, y)P - V_1(x, y)Q\| \leqq \alpha_1(x, y)\|P - Q\| \quad \textit{and}$$
$$\| V_2(x, y)P - V_2(x, y)Q\| \leqq \alpha_2(x, y)\|P - Q\|;$$

(2) K *is in* $\Phi(V_1)$ *and* M *is in* $\Phi(V_2)$ *and each of* h *and* k *is a function from* SXS *to the real numbers such that for each* $\{x, y, P\}$ *in*

SXSXG, ${}_x\Sigma^y k = 0$, $\|K(x,y)P - V_1(x,y)P\| \leqq h(x,y)\|P\|$, ${}_x\Sigma^y h = 0$ *and* $\|M(x,y)P - V_2(x,y)P\| \leqq h(x,y)\|P\|$;

(3) *there is a number* $a < 1$ *such that for each* $\{x,y\}$ *in SXS* $\alpha_2(x,y) + h(x,y) \leqq a$; *and*

(4) $\beta = [1-\alpha_2]^{-1}[1+\alpha_1]$ *and* $\gamma = [1-\alpha_2 - h]^{-1}[1+\alpha_1+k]$.

Conclusion:

(1) $\|[1-V_2(x,y)]^{-1}[1+V_1(x,y)]P - [1-V_2(x,y)]^{-1}[1+V_1(x,y)]Q\| \leqq [1-\alpha_2(x,y)]^{-1}[1+\alpha_1(x,y)]\,\|P-Q\|$ *for every* $\{x,y\}$ *in SXS and* $\{P,Q\}$ *in GXG.*

(2) $\|[1-M(x,y)]^{-1}[1+K(x,y)]P - [1-V_2(x,y)]^{-1}[1+V_1(x,y)]P\| \leqq [\gamma(x,y)-\beta(x,y)]\,\|P\|$ *for every* $\{x,y\}$ *in SXS and P in G.*

(3) ${}_x\Pi^y\,[1-M]^{-1}[1+K]P = {}_x\Pi^y\,[1-V_2]^{-1}[1+V_1]P$ *for every* $\{x,y,P\}$ *in SXSXG.*

Proof. Let $\{x,y\}$ be in *SXS*, $\{P,Q\}$ be in *GXG* and $A = [1-V_2(x,y)]^{-1}[1+V_1(x,y)]$. First note that $A = 1 + V_1(x,y) + V_2(x,y)A$.

$$\|AP - AQ\| \leqq [1+\alpha_1(x,y)]\,\|P-Q\| + \alpha_2(x,y)\,\|AP-AQ\|;$$

and assertion (1) follows. Let $B = [1-M(x,y)]^{-1}[1+K(x,y)]$;

$$\begin{aligned}
\|BP - AP\| &= \\
&\|[1+K(x,y)+M(x,y)B]P - [1+V_1(x,y)+V_2(x,y)A]P \\
&\qquad\qquad\qquad \pm V_2(x,y)BP\| \\
&\leqq k(x,y)\,\|P\| + h(x,y)\,\|BP\| + \alpha_2(x,y)\,\|BP-AP\| \\
&\leqq k(x,y)\,\|P\| + h(x,y)\,\|AP\| + [h(x,y)+\alpha_2(x,y)]\,\|BP-AP\| \\
&\leqq k(x,y)\,\|P\| + h(x,y)\,[1+\alpha_1(x,y)]\,[1-\alpha_2(x,y)]^{-1}\,\|P\| \\
&\quad + [h(x,y)+\alpha_2(x,y)]\,\|BP-AP\|
\end{aligned}$$

which, except for minor algebraic manipulation, establishes (2).

For each $\mathcal{O}$-subdivision t of $\{x,y\}$ it follows from Lemma 1.2 [7, p. 623] that

$$\left\|\prod_t [1-M]^{-1}[1+K]P - \prod_t [1-V_2]^{-1}[1+V_1]P\right\| \leqq \left\{\prod_t \gamma - \prod_t \beta\right\}\|P\|.$$

By Theorem 3.2 [2, p. 300] and hypothesis (3) of this theorem, there is a number b such that

$$\prod_t \gamma - \prod_t \beta \leqq b^2 \sum_t (\gamma - \beta) \leqq b^3 \sum_t k + b^4[1+\alpha_1(x,y)] \sum_t h.$$

Since ${}_x\sum^y k = 0$ and ${}_x\sum^y h = 0$ the proof is complete.

THEOREM 3.4. *If V is in $\mathcal{O}\mathcal{A}$, α is in $\mathcal{O}\mathcal{A}^+$, $\alpha(x,y) < 1$*

$\| V(x,y)P - V(x,y)Q \| \leqq \alpha(x,y) \| P - Q \|$ and ${}_x\Sigma^y \alpha^2 = 0$ for each $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG, then for each $\{x,y,P\}$ in $SXSXG$

(1) ${}_x\prod^y [1 - V]^{-1}P = {}_x\prod^y [1 + V]P$;

(2) ${}_x\prod^y [1 + V]^{-1}P = {}_x\prod^y [1 - V]P$;

(3) ${}_x\prod^y [1 - V^2]P = P$; *and*

(4) ${}_x\prod^y [1 + V][1 - V]^{-1}P = {}_x\prod^y [1 + 2V]P$.

Indication of proof of (1). For each $\{x,y,P\}$ in $SXSXG$

$$\begin{aligned} &\| [1 - V(x,y)]^{-1}P - [1 + V(x,y)]P \| \\ &\quad = \| V(x,y)[1 - V(x,y)]^{-1}P - V(x,y)P \| \\ &\quad \leqq \alpha(x,y) \| V(x,y)[1 - V(x,y)]^{-1}P \| \\ &\quad \leqq \alpha^2(x,y)[1 - \alpha(x,y)]^{-1} \| P \|. \end{aligned}$$

Similar inequalities can be established for (2), (3) and (4).

4. The integral equations.

Let each of R and L denote a function from SXS into S such that $R(x,y) = y$ and $L(x,y) = x$ for each $\{x,y\}$ in SXS.

REMARK. This notation due to W. L. Gibson in **[1]** provides a more precise notation for left and right integral process than that used before. Hence

$$(RL)\int_x^y (Kf + Mf) \text{ becomes } \int_x^y (Kf[R] + Mf[L]).$$

As in **[7]** $\mathcal{F}(c,P)$ denotes the class of all functions f from S to G such that $f(c) = P$ and there is a member β of $\mathcal{O}\mathcal{A}^+$ such that $\|f(y) - f(x)\| \leqq \beta(x,y)$ for each $\{x,y\}$ in SXS (i.e., f is of bounded variation on each $\mathcal{O}$-interval of S).

REMARK. The construction of the proof of the next lemma is similar to that of Lemma 2.2 **[7, p. 623]**.

LEMMA 1. *Suppose*

(1) *each of V_1 and V_2 is in $\mathcal{O}\mathcal{A}$ and α_1 and α_2 is in $\mathcal{O}\mathcal{A}^+$ such that for each $\{x,y\}$ in SXS and $\{P,Q\}$ in GXG*

$$\| V_1(x,y)P - V_1(x,y)Q \| \leqq \alpha_1(x,y) \| P - Q \| \quad \textit{and}$$
$$\| V_2(x,y)P - V_2(x,y)Q \| \leqq \alpha_2(x,y) \| P - Q \|;$$

(2) *f is in* $\mathscr{F}(c,P)$; *and*
(3) *for each* $\{x,y\}$ *in* SXS

$$C(x,y) = \int_x^y \{V_1 f[R] + V_2 f[L]\} - V_1(x,y)f(y) - V_2(x,y)f(x).$$

Conclusion. *For each* $\{x,y\}$ *in* SXS

$$_x\sum^y \| C \| = 0.$$

Proof. Let β be in $\mathcal{O}\mathscr{A}^+$ such that $\| df \| \leqq \beta$, $\{x,y\}$ be in SXS such that $\{x,y,c\}$ is an $\mathcal{O}$-subdivision of $\{x,c\}$ where c is in S, and $\{t_i\}_0^n$ be an $\mathcal{O}$-subdivision of $\{x,y\}$; then

$$\left\| \sum_t \{V_1 f[R] + V_2 f[L]\} - V_1(x,y)f(y) - V_2(x,y)f(x) \right\|$$
$$\leqq \left\| \sum_1^n V_2(t_{i-1},t_i)f(t_{i-1}) - \sum_1^n V_2(t_{i-1},t_i)f(x) \right\|$$
$$+ \left\| \sum_1^n V_1(t_{i-1},t_i)f(t_i) - \sum_1^n V_1(t_{i-1},t_i)f(y) \right\|$$
$$\leqq \sum_1^n \| V_2(t_{i-1},t_i)f(t_{i-1}) - V_2(t_{i-1},t_i)f(x) \|$$
$$+ \sum_1^n \| V_1(t_{i-1},t_i)f(t_i) - V_1(t_{i-1},t_i)f(y) \|$$
$$\leqq \sum_1^n \alpha_2(t_{i-1},t_i)\beta(x,t_{i-1}) + \sum_1^n \alpha_1(t_{i-1},t_i)\beta(t_i,3)$$
$$= \alpha_2(x,y)\beta(x,c) - \sum_t \alpha_2\beta(\quad,c)[L] + \sum_t \alpha_1\beta(\quad,c)[R]$$
$$- \alpha_1(x,y)\beta(y,c).$$

Let $h(x,y) = \alpha_2(x,y)\beta(x,c) - {}_x\sum^y \alpha_2\beta(\quad,c)[L]$

$$+ {}_x\sum^y \alpha_1\beta(\quad,c)[R] - \alpha_1(x,y)\beta(y,c).$$

Since ${}_x\Sigma^y\alpha_2\beta(\quad,c)[L]$ and ${}_x\Sigma^y\alpha_1\beta(\quad,c)[R]$ exist for every $\{x,y\}$ in SXS (as in [7, p. 629]) and each is real valued then by a theorem of Kolmogoroff's [6, p. 668] ${}_a\Sigma^b h = 0$ for all $\{a,b\}$ in SXS and the proof is complete.

LEMMA 2. *Suppose*

(1) *each of* V_1 *and* V_2 *is in* $\mathcal{OA}$, *each of* α_1 *and* α_2 *is in* $\mathcal{OA}^+$ *and there is a number* $a < 1$ *such that for each* $\{x, y\}$ *in* SXS *and* $\{P, Q\}$ *in* GXG

$$\| V_1(x,y)P - V_1(x,y)Q \| \leqq \alpha_1(x,y) \| P - Q \| \quad \text{and}$$
$$\| V_2(x,y)P - V_2(x,y)Q \| \leqq \alpha_2(x,y) \| P - Q \| \leqq a \| P - Q \|;$$

(2) C *is a function from* SXS *to* G *such that for each* $\{x, y\}$ *in* SXS, ${}_x\Sigma^y \| C \| = 0$;

(3) ${}_x\Pi^y [1 - V_2]^{-1}[1 + V_1]P$ *exists for every* $\{x, y, P\}$ *in* $SXSXG$; *and*

(4) $A(x,y)P = [1 - V_2(x,y)]^{-1}\{[1 + V_1(x,y)]P + C(x,y)\}$ *for each* $\{x, y, P\}$ *in* $SXSXG$.

Conclusion. ${}_x\Pi^y [1 - V_2]^{-1}[1 + V_1]P = {}_x\Pi^y AP$ *for every* $\{x, y, P\}$ *in* $SXSXG$.

Proof. First note from Theorem 3.3 that

$$\| [1 - V_2(x,y)]^{-1}\{[1 + V_1(x,y)]P + C(x,y)\} - [1 - V_2(x,y)]^{-1}[1 + V_1(x,y)]P \|$$

$\leqq [1 - \alpha_2(x,y)]^{-1} \| C(x,y) \|$ for each $\{x, y, P\}$ in $SXSXG$. Let $\{x, y, P\}$ be in $SXSXG$ and t be an $\mathcal{O}$-subdivision of $\{x, y\}$. Using Lemma 1.2 **[7**, p. 623**]**, $\| \Pi_t AP - \Pi_t [1 - V_2]^{-1}[1 + V_1]P \| \leqq \Pi_t [1 - \alpha_2]^{-1}\{[1 + \alpha_1] + \| C \|\} - \Pi_t [1 - \alpha_2]^{-1}[1 + \alpha_1]$; from Theorem 3.2 **[2**, p. 300**]** and our hypothesis there is a number b such that the difference between these last two products does not exceed $b \Sigma_t \| C \|$, which completes our proof.

THEOREM 4.1. *Suppose*

(1) *each of* V_1 *and* V_2 *is in* $\mathcal{OA}$,

(2) K *is in* $\Phi(V_1)$ *and* M *is in* $\Phi(V_2)$,

(3) f *is a function from* S *to* G *that is bounded on each* $\mathcal{O}$*-interval of* S, *and*

(4) *for each* $\{x, y\}$ *in* SXS $\int_x^y (V_1 f[R] + V_2 f[L])$ *exists.*

Conclusion. For each $\{x, y\}$ *in* SXS

$$\int_x^y (Kf[R] + Mf[L]) = \int_x^y (V_1 f[R] + V_2 f[L]).$$

Proof. Let each of h and k be a function from SXS to the real numbers such that for each $\{x, y, P\}$ in $SXSXG$, ${}_x\Sigma^y k = 0$, $\| K(x,y)P - V_1(x,y)P \| \leqq k(x,y) \| P \|$, ${}_x\Sigma^y h = 0$, and

$$\| M(x,y)P - V_2(x,y)P \| \leqq h(x,y) \| P \|.$$

Pick $\{x,y\}$ in SXS and a number b such that if $\{x,z,y\}$ is an $\mathcal{O}$-subdivision of $\{x,y\}$ then $\|f(z)\| \leqq b$. Let $\{t_i\}_0^n$ be an $\mathcal{O}$-subdivision of $\{x,y\}$; then

$$\begin{aligned}
&\Big\| \sum_1^n \{K(t_{i-1},t_i)f(t_i) + M(t_{i-1},t_i)f(t_{i-1})\} \\
&\quad - \sum_1^n \{V_1(t_{i-1},t_i)f(t_i) + V_2(t_{i-1},t_i)f(t_{i-1})\} \Big\| \\
&\leqq \sum_1^n \| K(t_{i-1},t_i)f(t_i) - V_1(t_{i-1},t_i)f(t_i) \| \\
&\quad + \sum_1^n \| M(t_{i-1},t_i)f(t_{i-1}) - V_2(t_{i-1},t_i)f(t_{i-1}) \| \\
&\leqq \sum_1^n k(t_{i-1},t_i)\|f(t_i)\| + \sum_1^n h(t_{i-1},t_i)\|f(t_{i-1})\| \\
&\leqq b\Big\{ \sum_1^n k(t_{i-1},t_i) + \sum_1^n h(t_{i-1},t_i) \Big\}.
\end{aligned}$$

REMARK. The construction of the proof of the next theorem is similar to that of Theorem 5.1 [2, p. 310].

THEOREM 4.2. *Suppose*

(1) *each of* V_1 *and* V_2 *is in* $\mathcal{OA}$ *and each of* α_1 *and* α_2 *is in* $\mathcal{OA}^+$ *such that for each* $\{x,y\}$ *in* SXS *and* $\{Q_1,Q_2\}$ *in* GXG

$$\begin{aligned}
&\| V_1(x,y)Q_1 - V_1(x,y)Q_2 \| \leqq \alpha_1(x,y)\| Q_1 - Q_2 \| \quad \textit{and} \\
&\| V_2(x,y)Q_1 - V_2(x,y)Q_2 \| \leqq \alpha_2(x,y)\| Q_1 - Q_2 \|;
\end{aligned}$$

(2) K *is in* $\Phi(V_1)$ *and* M *is in* $\Phi(V_2)$ *and each of* h *and* k *is a function from* SXS *to the real numbers such that for each* $\{x,y,Q\}$ *in* $SXSXG$, ${}_x\Sigma^y k = 0$, $\| V_1(x,y)Q - K(x,y)Q \| \leqq k(x,y)\| Q \|$, ${}_x\Sigma^y h = 0$, $\| V_2(x,y)Q - M(x,y)Q \| \leqq h(x,y)\| Q \|$ *and there is a number* $a < 1$ *such that* $\alpha_2(s,t) + h(s,t) < a$ *for all* $\{s,t\}$ *in* SXS;

(3) $\{c,P\}$ *is in* SXG.

Conclusion. The following statements are equivalent:

(1) f *is in* $\mathcal{F}(x,P)$ *and* $f(x) = P + \int_x^c (Kf[R] + Mf[L])$ *for each* x *in* S;

(2) $f(x) = {}_x\Pi^c [1-M]^{-1}[1+K]P$ *for each* x *in* S; *and*

(3) *if for each* $\{a,b,Q\}$ *in* $SXSXG$

$$V(a,b)Q = {}_a\sum^b \{[1-M]^{-1}[1+K]-1\}Q$$

then $f(x) = x\Pi^c [1+V]P$ *for each* x *in* S.

Proof. $(1 \rightarrow 2)$: If $\{x,y,c\}$ is an $\mathcal{O}$-subdivision of $\{x,c\}$ then by Theorem 4.1

$$f(x) = P + \int_x^c (Kf[R] + Mf[L]) = P + \int_x^c (V_1f[R] + V_2f[L])$$

$$= f(y) + \int_x^y (V_1f[R] + V_2f[L]).$$

Hence if $\{t_j\}_0^n$ is an $\mathcal{O}$-subdivision of $\{x,c\}$ and j is an integer in $[1,n]$ then

$$f(t_{j-1}) - f(t_j) = \int_{t_{j-1}}^{t_j} (V_1f[R] + V_2f[L]) \quad \text{and}$$

$$f(t_{j-1}) = f(t_j) + V_1(t_{j-1},t_j)f(t_j) + V_2(t_{j-1},t_j)f(t_{j-1}) + C(t_{j-1},t_j)$$

$$\text{where } C(t_{j-1},t_j) = \int_{t_{j-1}}^{t_j} (V_1f[R] + V_2f[L]) - V_1(t_{j-1},t_j)f(t_j) - V_2(t_{j-1},t_j)f(t_{j-1}).$$

$$[1 - V_2(t_{j-1},t_j)]f(t_{j-1}) = [1 + V_1(t_{j-1},t_j)]f(t_j) + C(t_{j-1},t_j).$$

$$f(t_{j-1}) = [1 - V_2(t_{j-1},t_j)]^{-1}\{[1 + V_1(t_{j-1},t_j)]f(t_j) + C(t_{j-1},t_j)\}.$$

Let $A(x,y)Q = [1 - V_2(x,y)]^{-1}\{[1 + V_1(x,y)]Q + C(x,y)\}$. By iteration $j = n, n-1, n-2, \cdots, 1$, in order, we obtain

$$f(t_0) = \prod_{j=1}^n A(t_{j-1},t_j)f(t_n).$$

Using our Lemmas 1 and 2 and Theorem 3.3

$$f(x) = {}_x\prod^c [1 - V_2]^{-1}[1 + V_1]P = {}_x\prod^c [1-M]^{-1}[1+K]P.$$

$(2 \rightarrow 1)$: If $\{x,c\}$ is in SXS and $\{t_i\}_0^n$ is an $\mathcal{O}$-subdivision of $\{x,c\}$ and i is an integer in $[1,n]$ then from Theorem 3.3

$$f(t_{i-1}) = {}_{t_{i-1}}\prod^{t_i} [1-M]^{-1}[1+K]f(t_i)$$

$$= {}_{t_{i-1}}\prod^{t_i} [1 - V_2]^{-1}[1 + V_1]f(t_i)$$

$$= [1 - V_2(t_{i-1}, t_i)]^{-1}[1 + V_1(t_{i-1}, t_i)]f(t_i)$$

$$+ D(t_{i-1}, t_i)f(t_i)$$

where $D(t_{i-1}, t_i) = {}_{t_{i-1}}\prod^{t_i} [1 - V_2]^{-1}[1 + V_1]$

$$- [1 - V_2(t_{i-1}, t_i)]^{-1}[1 + V_1(t_{i-1}, t_i)].$$

$$[1 - V_2(t_{i-1}, t_i)]\,[f(t_{i-1}) - D(t_{i-1}, t_i)f(t_i)] = [1 + V_1(t_{i-1}, t_i)]f(t_i);$$

$$f(t_{i-1}) - f(t_i) = V_1(t_{i-1}, t_i)f(t_i) + V_2(t_{i-1}, t_i)\,[f(t_{i-1})$$

$$- D(t_{i-1}, t_i)f(t_i)] + D(t_{i-1}, t_i)f(t_i);$$

$$f(x) = f(c) + \int_x^c \{V_1 f[R] + V_2(f[L] + Df[R])\} + {}_x\sum^c Df[R];$$

but $\int_x^c V_2[f[L] + Df[R]] + {}_x\sum^c Df[R] = \int_x^c V_2 f[L]$ because

$$\Big\| \sum_1^n V_2(t_{i-1}, t_i)\,[f(t_{i-1}) - D(t_{i-1}, t_i)f(t_i)]$$

$$+ \sum_1^n D(t_{i-1}, t_i)f(t_i) - \sum_1^n V_2(t_{i-1}, t_i)f(t_{i-1}) \Big\|$$

$$\leqq \sum_1^n \alpha_2(t_{i-1}, t_i)\,\| D(t_{i-1}, t_i)f(t_i)\| + \sum_1^n \| D(t_{i-1}, t_i)f(t_i)\|$$

$$\leqq \{1 + \alpha_2(x, c)\} \sum_1^n \| D(t_{i-1}, t_i)f(t_i)\|$$

$$\leqq \{1 + \alpha_2(x, c)\} \sum_1^n d(t_{i-1}, t_i)\,\| f(t_i)\|$$

where $d(a, b) = {}_a\Pi^b\, [1 - \alpha_2]^{-1}[1 + \alpha_1] - [1 - \alpha_2(a, b)]^{-1}[1 + \alpha_1(a, b)]$ for each $\{a, b\}$ in SXS. The preceding inequality follows from the proof of Theorem 3.2, and it follows from Theorem 3.0 and [**6**, p. 668] that ${}_a\Sigma^b d = 0$ for each $\{a, b\}$ in SXS.

Hence $f(x) = P + \int_x^c (V_1 f[R] + V_2 f[L]) = P + \int_x^c (Kf[R] + Mf[L])$.

It follows from Theorems 3.2 and 3.3 that (3) is equivalent to (2) and the proof is complete.

REMARK. From the foregoing argument it is evident that each of the following statements is equivalent to those in the conclusion of the preceding theorem:

(4) f is in $\mathscr{F}(c,P)$ and

$$f(x) = P + \int_x^c (V_1 f[R] + V_2 f[L])$$

for each x in S; and

(5) $f(x) = {}_x\Pi^c [1 - V_2]^{-1}[1 + V_1]P$ for each x in S.

THEOREM 4.3. *Suppose*

(1) *each of V_1 and V_2 is in $\mathcal{O}\mathscr{A}$ and each of α_1 and α_2 is in $\mathcal{O}\mathscr{A}^+$ such that for each $\{x,y\}$ in SXS and $\{Q_1,Q_2\}$ in GXG*

$$\| V_1(x,y)Q_1 - V_1(x,y)Q_2 \| \leqq \alpha_1(x,y) \| Q_1 - Q_2 \| \quad \text{and}$$

$$\| V_2(x,y)Q_1 - V_2(x,y)Q_2 \| \leqq \alpha_2(x,y) \| Q_1 - Q_2 \|;$$

(2) *K is in $\Phi(V_1)$ and M is in $\Phi(V_2)$ and each of h and k is a function from SXS to the real numbers such that for each $\{x,y,Q\}$ in $SXSXG$, ${}_x\Sigma^y k = 0$,*

$$\| V_1(x,y)Q - K(x,y)Q \| \leqq k(x,y) \| Q \|, \ {}_x\sum^y h = 0,$$

$$\| V_2(x,y)Q - M(x,y)Q \| \leqq h(x,y) \| Q \|,$$

and there is a number $a < 1$ such that $\alpha_1(s,t) + k(s,t) \leqq a$ for all $\{s,t\}$ in SXS;

(3) *$K'(y,x)Q = K(x,y)Q$, $M'(y,x)Q = M(x,y)Q$ for each $\{x,y,Q\}$ in $SXSXG$;*

(4) *$\{c,P\}$ is in SXG.*

Conclusion. The following statements are equivalent:

(1) *f is in $\mathscr{F}(c,P)$ and $f(x) = P + \int_c^x (Kf[R] + Mf[L])$ for x in S;*

(2) *$f(x) = {}_x\Pi^c [1 - K']^{-1}[1 + M']P$ for each x in S; and*

(3) *if $V(a,b)Q = {}_a\Sigma^b \{[1 - K']^{-1}[1 + M'] - 1\}Q$ for each $\{a,b,Q\}$ in $SXSXG$, then*

$f(x) = {}_x\Pi^c [1 + V]P$ for each x in S.

The proof of this theorem, except for minor algebraic manipulations, is the same as the proof of the previous theorem.

5. A seemingly more general integral equation. In [7, pp. 632–633] Mac Nerney showed that the theory he developed in solving an integral equation of the form $f(x) = P + (R)\int_x^c Vf$ could be used to solve a seemingly more general equation of the form

$$f(x) = P_1 + (R)\int_x^c V_1 f + V_2(x,c)P_2.$$

We repeat that procedure here by using the theory developed in the preceding section to solve an equation of the form

$$f(x) = P_1 + \int_c^x (Kf[R] + Mf[L]) + V(x,c)P_2,$$

and the solution of this equation in the purely linear case is shown to include the solutions Helton obtained in Theorems 5.1–5.4 [2, pp. 310–314].

Let $\{GXG, +, \|\ \ \|\}$, $\mathcal{OA}''$ and $\mathcal{OM}''$ be defined as in [7, p. 632]. Let Φ'' and ψ'' be the mappings corresponding to the mappings Φ and ψ. The following theorem is a reinterpretation of Theorem 4.3. We will not state the corresponding reinterpretation of Theorem 4.2.

THEOREM 5. *Assume the hypothesis of Theorem* 4.3 *with K and M as defined there. Let P be in GXG, V be in $\mathcal{OA}$ and each of K'' and M'' be in $\Phi''(\mathcal{OA}'')$ such that*

$K''(x,y)Q = \{K(y,x)Q_1, 0\}$ *and*

$M''(x,y)Q = \{M(y,x)Q_1 + V(y,x)Q_2, 0\}$

for each $\{x,y\}$ in SXS and Q in GXG. If u is a function from S to G, the following are equivalent:

(1) $\{u(x), P_2\} = {}_x\Pi^c\,[1 - K'']^{-1}[1 + M'']P$ *for each x in S, and*

(2) *u is in $\mathcal{F}(c, P_1)$ such that for each x in S*

$$u(x) = P_1 + \int_c^x (Ku[R] + Mu[L]) + V(x,c)P_2.$$

The next corollary shows that in the purely linear case this theorem includes the solutions Helton obtained in Theorems 5.1–5.4 [2, pp. 310–314].

Let $\{N, +, \cdot, |\ \ |\}$ be a complete normed ring.

COROLLARY. *Suppose*

(1) *each of K and M is a function from SXS to N that is in the common part of* $\mathbf{OA}^0$ *and* $\mathbf{OB}^0$ [2, p. 299];

(2) *there exists a number* $a < 1$ *such that for each* $\{x, y\}$ *in SXS*

$$\left| K(x,y) - {}_x\sum^y K \right| + {}_x\sum^y |K| \leqq a \quad \text{and}$$

$M'(x,y) = M(y,x)$ *and* $K'(x,y) = K(y,x)$; *and*

(3) c *is in* S *and each of* f *and* h *is a function from* S *to* N *such that* $f(c) = h(c)$ *and* dh *is in* OB^0.

Conclusions.

(1) *The following two statements are equivalent*:

(a) df *is in* OB^0 *and*

$$f(x) = h(x) + \int_c^x (f[R]K + f[L]M) \text{ for each } x \text{ in } S; \text{ and}$$

(b) $f(x) = f(c)_c\Pi^x [1 + M][1 - K]^{-1}$

$$+ \int_c^x (dh)_t\Pi^x [1 + M)[1 - K]^{-1}[R]$$

for each x *in* S.

(2) *The following two statements are equivalent*:

(a) df *is in* OB^0 *and*

$$f(x) = h(x) + \int_c^x (Kf[R] + Mf[L]) \text{ for each } x \text{ in } S; \text{ and}$$

(b) $f(x) = {}_x\Pi^c [1 - K']^{-1}[1 + M']f(c)$

$$+ \int_c^x {}_x\Pi^t [1 - K']^{-1}[1 + M'][R]dh$$

for each x *in* S.

(3) *The following two statements are equivalent*:

(a) df *is in* OB^0 *and*

$$f(x) = h(x) + \int_c^x (Kf[R] + f[L]M) \text{ for each } x \text{ in } S; \text{ and}$$

(b) $f(x) = {}_x\Pi^c [1 - K']^{-1} f(c)_c\Pi^x [1 + M]$

$$+ \int_c^x {}_x\Pi^t [1 - K']^{-1}[R](dh)_t\Pi^x [1 + M][R]$$

for each x *in* S.

(4) *The following two statements are equivalent*:

(a) *df is in* OB^0 *and*

$$f(x) = h(x) + \int_c^x (f[R]K + Mf[L]) \text{ for each } x \text{ in } S; \text{ and}$$

(b) $$f(x) = {}_x\Pi^c [1 + M']f(c)_c\Pi^x [1 - K]^{-1} + \int_c^x {}_x\Pi^t [1 + M'][R](dh)_t\Pi^x [1 - K]^{-1}[R]$$

for each x in S.

Indication of proof.

(1) For each $\{x, y\}$ in SXS and Q in NXN

$$K''(x,y)[Q] = \{Q_1 \cdot K(y,x), 0\} \quad \text{and}$$

$$M''(x,y)[Q] = \{Q_1 \cdot M(y,x) - dh(x,y)Q_2, 0\}.$$

Let P be in NXN such that $P_1 = h(c)$ and $P_2 = 1$;

$$h(x) + \int_c^x (f[R]K + f[L]M)$$

$$= P_1 + \int_x^c (K''f[R] + M''f[L]) + (-dh)(c,x) \cdot P_2,$$

and for each $\mathcal{O}$-subdivision $\{t_j\}_0^n$ of $\{x, c\}$

$$\prod_{j=1}^n [1 - K''(t_{j-1}, t_j)]^{-1}[1 + M''(t_{j-1}, t_j)]P$$

$$= \Big\{ f(c) \prod\nolimits_{j=1}^n [1 + M(t_{n-j+1}, t_{n-j})][1 - K(t_{n-j+1}, t_{n-j})]^{-1}$$

$$+ \sum\nolimits_{j=1}^n dh(t_{n-j}, t_{n-j+1}) \prod\nolimits_{q=1}^{n-j} [1 + M(t_{n-q+1}, t_{n-q})]$$

$$\cdot [1 - K(t_{n-q+1}, t_{n-q})]^{-1}, P_2 \Big\}.$$

(2) $$K''(x,y)[Q] = \{K'(x,y) \cdot Q_1, 0\} \quad \text{and}$$

$$M''(x,y)[Q] = \{M'(x,y) \cdot Q_1 - dh(x,y) \cdot Q_2, 0\}$$

for each $\{x, y\}$ in SXS and Q in NXN.

(3) $$K''(x,y)[Q] = \{K'(x,y) \cdot Q_1, 0\} \quad \text{and}$$

$$M''(x,y)[Q] = \{Q_1 \cdot M(y,x) - dh(x,y) \cdot Q_2, 0\}$$

for each $\{x, y\}$ in SXS and Q in NXN.

$$\begin{aligned} (4) \quad & K''(x,y)[Q] = \{Q_1 \cdot K(y,x), 0\} \quad \text{and} \\ & M''(x,y)[Q] = \{M'(x,y) \cdot Q_1 - dh(x,y) \cdot Q_2, 0\} \end{aligned}$$

for each $\{x, y\}$ in $S \times S$ and Q in $N \times N$.

Acknowledgment. The author would like to thank Professor J. S. Mac Nerney for his many helpful suggestions in the writing of this paper.

REFERENCES

1. W. L. Gibson, *Stieltjes and Stieltjes-Volterra integral equations*, Ph. D. Dissertation, University of Houston (1974).
2. B. W. Helton, *Integral equations and product integrals*, Pacific J. of Math., **16** (1966), 297–322.
3. ——————, *A product integral representation for a Gronwall inequality*, Proc. Amer. Math. Soc., **23**, No. 3 (1969), 493–500.
4. J. C. Helton, *Product integrals and inverses in normed rings*, Pacific J. of Math., **51** (1974), 155–166.
5. J. V. Herod, *Coalescence of solutions for nonlinear Stieltjes equations*, J. Reine Angew. Math., **252** (1972), 187–194.
6. A. Kolmogoroff, *Untersuchungen über den Integralbegriff*, Math. Ann., **103** (1930), 654–696.
7. J. S. Mac Nerney, *A nonlinear integral operation*, Illinois J. Math., **8** (1964), 621–638.
8. J. W. Neuberger, *Toward a characterization of the identity component of rings and near-rings of continuous transformations*, J. Reine Angew. Math., **238** (1969), 100–104.

Received May 29, 1974 and in revised form September 9, 1974.

TEXAS A AND I UNIVERSITY