SET APPROXIMATION BY LEMNISCATES AND THE SPECTRUM OF AN OPERATOR ON AN INTERPOLATION SPACE

JAMES D. STAFNEY

Let $B_0, B_1$ be an interpolation pair of Banach spaces and $T$, a bounded linear operator on the corresponding interpolation space $[B_0, B_1]$, $0 \leq s \leq 1$, such that the operators $T_s$ all agree on $B_0 \cap B_1$. In this paper we extend our previous work by giving a general upper bound for the spectrum of $T$, constructed from the spectra of $T_0$ and $T_1$ using a set interpolation formula which we introduce in §1 for compact sets in the plane. In §3 we show that this upper bound is essentially best possible. This requires a theorem about approximating sets with lemniscates, which we prove in §2. Finally, we show in §4 that under certain conditions the operators $T_s$, $0 \leq s \leq 1$, all have the same spectrum.

1. Upper bound for spectrum. In this section we introduce the notion of interpolating between two sets in the complex plane (see 1.3) and then we obtain an upper bound for the spectrum of an operator on an interpolation space by “interpolating its extreme spectra (see 1.9).”

Basic situation 1.1. $B_0, B_1$ are two Banach spaces continuously embedded in a topological vector space $V$ so that $B_0 \cap B_1$ is dense in both $B_0$ and $B_1$. The corresponding spaces $[B_0, B_1]$, $0 \leq s \leq 1$, are the spaces defined in [2, p. 114]. By [2, §9.3], $B_j = [B_0, B_1]$, $j = 0, 1$. $T_s$, $0 \leq s \leq 1$, is a bounded linear operator on $[B_0, B_1]$, and all the operators $T_s$ agree on $B_0 \cap B_1$.

General notation. For a normed linear space $B$, $\| B \|$ will denote the norm of $B$. We will sometimes use $\| B \|$ to denote $\| B \|$ if no confusion results. The space of bounded linear operators on a normed linear space $B$ with the operator norm will be denoted $O(B)$. We will call bounded linear operators, operators. If $T$ is an operator, $sp(T)$ will denote the spectrum of $T$ and $r(T)$ the spectral radius of $T$, the smallest number $r$ such that $sp(T) \subset \{ z : |z| \leq r \}$. If $F$ is analytic on $sp(T)$, then $F(T)$ will denote the operator $(2\pi i)^{-1} \int_{\gamma} (\lambda I - T)^{-1} d\lambda$ where $\gamma$ is an envelope of $sp(T)$ in the domain of $F$. If $E$ is a subset of the complex plane, $R(E)$ will denote the rational functions with poles in the complement of $E$. The following is proved in [2].
Lemma 1.2. If \( T_s, 0 \leq s \leq 1 \), is as in 1.1 then

\[ \| T_s \| \leq \| T_0 \|^{1-s} \| T_1 \|^s. \]

If \( B_0 \) and \( B_1 \) are as in 1.1 and \( T_i \) is an operator on \( B_j \), \( j = 0, 1 \), such that \( T_0 \) and \( T_1 \) agree on \( B_0 \cap B_1 \), then there is an operator \( T_s \) on \([B_0, B_1]\), \( 0 \leq s \leq 1 \), such that all the operators \( T_s \) agree on \( B_0 \cap B_1 \).

We next introduce the notion of interpolation sets \( I_s(E_0, E_1) \) corresponding to two given compact sets \( E_0, E_1 \) in the plane. These interpolation sets will be used to provide an upper bound for the spectrum of operators \( T_s \).

If \( F \) is a complex valued function and \( t \) is a positive real number, let \( L(t, F) \) denote the set of elements \( x \) in the domain of \( F \) such that \( |F(x)| \leq t \).

Definition 1.3. Let \( E_0, E_1 \) be compact nonempty sets in the plane such that \( E_0 \subset E_1 \). For \( 0 \leq s \leq 1 \) we define the set \( I_s(E_0, E_1) \) to be \( \cap L(r^s, F) \) where the intersection is taken over all pairs \( (r, F) \) such that: (a) \( r \) is a real number \( > 1 \), (b) \( F \) is an analytic function on an open set in the plane, (c) \( L(r, F) \) is a compact subset of the domain of \( F \) and (d) \( E_i \) is a subset of the interior of \( L(r^i, F) \), \( j = 0, 1 \).

The following lemma is an immediate consequence of the preceding definition.

Lemma 1.4. Suppose that \( E_0, E_1, F_0 \) and \( F_1 \) are compact subsets of the plane and that \( E_0 \subset E_1, F_0 \subset F_1, E_0 \subset F_0 \) and \( E_1 \subset F_1 \). Then,

\[ I_s(E_0, E_1) \subset I_s(F_0, F_1), \quad 0 \leq s \leq 1. \]

For a complex number \( z \) and a positive real number \( t \) let \( D(z, t) \) denote the set of complex numbers \( w \) such that \( |z - w| < t \). Let \( D(z, t] \) denote the closure of \( D(z, t) \). For a set \( A \) in a topological space \( \text{int} A \) will denote the interior of \( A \).

Lemma 1.5. Let \( K \) be a compact subset of \( D(0, 1) \). To each \( \epsilon > 0 \) and \( s, 0 < s < 1 \), there corresponds a \( \delta > 0 \) such that for each \( \zeta \) in \( K, D(\zeta, \delta) \subset D(0, 1) \) and

\[ I_s(D(\zeta, \delta], D(0, 1]) \subset K + D(0, \epsilon). \]

Proof. For \( \zeta \) in \( K \), let \( \lambda_\zeta \) denote the analytic function defined by \( \lambda_\zeta(z) = (z - \zeta)/(1 - z\zeta) \) on the complement of \( \{z^{-1}\} \). Choose \( u \) such that
\(1 < u < \left| \frac{1}{\zeta} \right| \) for each \(\zeta\) in \(K\). Since \(K\) is compact in \(D(0, 1)\) it is clear that

(1) corresponding to \(K, \epsilon, s\) there is an \(r > 1\) such that if 
\[ \zeta \in K \text{ and } |(r/u)\lambda_{\zeta}(z)| \leq r', \text{ then } |z - \zeta| < \epsilon. \]

It is clear that we can choose \(\delta\) so that 
\[ D(\zeta, \delta) \subseteq L(1, (r/u)\lambda_{\zeta}) \text{ and } D(\zeta, \delta) \subseteq D(0, 1) \text{ for each } \zeta \text{ in } K. \]

Let \(\zeta\) be in \(K\). Let 
\[ F(z) = (r/u)\lambda_{\zeta}(z). \]

From what we have observed and Definition 1.3 it follows that

(2) 
\[ I_{s}(D(\zeta, \delta], D(0, 1]) \subseteq L(r', F). \]

It follows from (1) that 
\[ L(r', F) \subseteq K + D(0, \epsilon); \]

and, this together with (2) proves the lemma.

**Lemma 1.6.** Let \(B_0, B_1\) and \(T_s, 0 \leq s \leq 1\), be as in 1.1. Suppose that 
\(E\) is a compact set in the plane such that \(\text{sp}(T_0), \text{sp}(T_1) \subseteq E\) and for each 
\(\lambda \notin E, (\lambda - T_0)^{-1}\) and \((\lambda - T_1)^{-1}\) agree on \(B_0 \cap B_1\). Then \(\text{sp}(T_s) \subseteq E, 0 \leq s \leq 1\); and, if \(F\) is analytic on \(E\), then the operators \(F(T_s)\) all agree on 
\(B_0 \cap B_1\).

**Proof.** Fix an \(s\) in the interval \([0, 1]\). We will first show that

(3) 
\[ \text{sp}(T_s) \subseteq E. \]

If \(\lambda \notin E\), then by the restrictions on \(E\) the operators \((\lambda - T_0)^{-1}\) and 
\((\lambda - T_1)^{-1}\) agree on the space \(B_0 \cap B_1\). From 1.2 it follows that there is an operator \(Q\) on \(B\), such that \(Q\) and \((\lambda - T_0)^{-1}\) agree on \(B_0 \cap B_1\); in particular, \(Q(\lambda - T_1)\) and \((\lambda - T_1)Q\) are equal to the identity on \(B_0 \cap B_1\). Since both \(Q\) and \(T_1\) are bounded on \(B_s\), we see that \(Q\) is the inverse of \(\lambda - T_1\); hence, \(\lambda \notin \text{sp}(T_s)\). Our argument establishes (3).

We next observe that if \(\lambda \notin E\), then the operators \((\lambda - T_s)^{-1}, 0 \leq s \leq 1\), and agree on \(B_0 \cap B_1\). To see this we first note that since \((\lambda - T_0)^{-1}\) and 
\((\lambda - T_1)^{-1}\) agree on \(B_0 \cap B_1\), these two operators leave \(B_0 \cap B_1\) invariant. If \(y \in B_0 \cap B_1\), then 
\[ y = (\lambda - T_0)x \text{ where } x = (\lambda - T_0)^{-1}y, \]

which is an element in \(B_0 \cap B_1\). Thus,

\[ (\lambda - T_s)^{-1}y = (\lambda - T_s)^{-1}(\lambda - T_0)x = (\lambda - T_1)^{-1}(\lambda - T_s)x = x. \]

Since \(y\) was chosen arbitrarily in \(B_0 \cap B_1\) and \(x\) does not depend on \(s\), we have shown that the operators \((\lambda - T_s), 0 \leq s \leq 1\), all agree on 
\(B_0 \cap B_1\). Consequently, the operators \(G(T_s), 0 \leq s \leq 1\), all agree on 
\(B_0 \cap B_1\) if \(G \in R(E)\). Let \(G_n\) be a sequence of elements in \(R(E)\) such
that $G_n$ converges uniformly to $F$ on some neighborhood of $E$. Then $G_n(T_0)$ converges to $F(T_0)$ in $O(B_0)$ and $G_n(T_i)$ converges to $F(T_i)$ in $O(B_i)$. If $x \in B_0 \cap B_1$, then $G_n(T_0)x = G_n(T_i)x$ for $n = 1, 2, \cdots$. Since $G_n(T_0)x$ converges to $F(T_0)x$ in $B_0$ and $F(T_i)x$ in $B_i$, and since $B_0$ and $B_1$ are continuously embedded in $V$, it follows that $F(T_0)x = F(T_i)x$. This completes the proof.

**Comment.** In view of Lemma 1.6 it is natural to ask whether the set $E$ in 1.6 can always be taken to be $\text{sp}(T_0) \cup \text{sp}(T_i)$. We have not been able to answer this question. The best that we have done is the content of the next lemma.

**Lemma 1.7.** Let $B_0, B_1, T_0$ and $T_1$ be as in 1.1. Then the set $H = \{ \lambda : \lambda \not\in \text{sp}(T_0) \cup \text{sp}(T_i) \text{ and } (\lambda - T_0)^{-1}, (\lambda - T_i)^{-1} \text{ agree on } B_0 \cap B_1 \}$ contains the unbounded component of the complement of $\text{sp}(T_0) \cup \text{sp}(T_i)$; and, the set $H$ is a union of components of the complement of $\text{sp}(T_0) \cup \text{sp}(T_i)$.

**Proof.** Suppose $\lambda$ is in the unbounded component of the complement of $E = \text{sp}(T_0) \cup \text{sp}(T_i)$. There is a sequence $p_n$ of polynomials which converges uniformly to $(\lambda - z)^{-1}$ for $z$ in a neighborhood of $E$. It follows that $p_n(T_j)$ converges to $(\lambda - T_j)^{-1}$ in the operator norm of $B_j$, $j = 0, 1$. If $x \in B_0 \cap B_1$, then $p_n(T_j)x$ converges to $(\lambda - T_j)^{-1}x$ in $B_j$. Therefore, $p_n(T_j)x$ converges to $(\lambda - T_j)^{-1}x$ in the topology of $V$, $j = 0, 1$. Since $T_0$ and $T_i$ agree on $B_0 \cap B_1$, $p_n(T_0)x = p_n(T_i)x$ for each $n$. Thus, $(\lambda - T_0)^{-1}x = (\lambda - T_i)^{-1}x$. If $\lambda$ is a point in some bounded component $V$ of the complement of $E$ such that $(\lambda - T_0)^{-1}$ and $(\lambda - T_i)^{-1}$ agree on $B_0 \cap B_1$, and if $\lambda_1$ is also in $V$, then there is a sequence of polynomials $p_n(n = 1, 2, \cdots)$ such that $p_n(1/(\lambda - z))$ converges uniformly to $1/(\lambda_1 - z)$ for $z$ in some neighborhood of $E$. The same reasoning as above shows that $(\lambda_1 - T_0)^{-1}$ and $(\lambda_1 - T_i)^{-1}$ agree on $B_0 \cap B_1$. This completes the proof.

**Lemma 1.8.** Let $B_0, B_1$ and $T_s$, $0 \leq s \leq 1$, be as in 1.1. Then

$$r(T_s) \leq r(T_0)^{s}r(T_i)^{1-s}, \quad 0 \leq s \leq 1.$$ 

**Proof.** If $\epsilon > 0$, then there corresponds on $n_0$ such that

$$\| (T_j)^n \| \leq (r(T_j) + \epsilon)^n \quad j = 0, 1; \quad n \geq n_0.$$

Since the operators $(T_j)^n$, $0 \leq s \leq 1$, all agree on $B_0 \cap B_1$, it follows from 1.2 that

$\| (T_j)^n \| \leq (r(T_j) + \epsilon)^n$
Since $\epsilon$ was chosen arbitrarily, the conclusion of the lemma follows from our observations and the fact that $r(T) = \lim \| (T)^n \|^{1/n}$.

**Theorem 1.9.** Suppose that $B_0, B_1$ and $T_n, 0 \leq s \leq 1$, are as in 1.1 and that $sp(T_0) \subset sp(T_1)$. Let $E$ denote the complement of $\{ \lambda : \lambda \notin sp(T_1) \text{ and } (\lambda - T_0)^{-1}, (\lambda - T_1)^{-1} \text{ agree on } B_0 \cap B_1 \}$. Then

$$sp(T) \subset I_1(sp(T_0), E).$$

**Proof.** Suppose that the theorem is false; that is, suppose that there is a $\lambda$ in $sp(T)$ which is not in $I_1(sp(T_0), E)$. In particular, there is a pair $(r, F)$ corresponding to the sets $sp(T_0)$ and $E$ as in 1.3 such that $\lambda \notin L(r, F)$. However, since $E \subset \text{domain of } F$ and 1.6 implies that $\lambda \in E$, we conclude that

$$(4) \quad |F(\lambda)| > r.$$

Define the operators $U_t = F(T_t), 0 \leq t \leq 1$, in the usual manner of spectral theory. Since $\lambda \in sp(T)$ it follows from the spectral mapping theorem that

$$(5) \quad F(\lambda) \in sp(U).$$

From 1.6 it follows that the operators $U_t, 0 \leq t \leq 1$, all agree on $B_0 \cap B_1$; consequently, we conclude from 1.8 that

$$(6) \quad r(U_t) \subset r(U_0)^{1-r} r(U_1)^r.$$

By the choice of $F$, $sp(T) \subset \{ z : |F(z)| \leq r \}$, $j = 0, 1$. From this and the spectral mapping theorem we obtain

$$(7) \quad sp(U_j) \subset \{ z : |z| \leq r \}, \quad j = 0, 1.$$

If we combine (5) thru (7) we get $|F(\lambda)| \leq r$, which contradicts (4). This completes the proof.

2. **Set approximation by lemniscates.** The main purpose of this section is the proof of the approximation theorem, Theorem 2.6.

By a simple set we mean a compact set with dense interior and with a boundary consisting of a finite number of disjoint simple closed curves which are also regular analytic curves (see [1, p. 226]). For notation we
use $\partial E$ and $\text{int } E$ to denote the boundary and the interior, respectively, of a set $E$ in the plane.

**Remark 2.1.** By a harmonic triple in this paper we mean what is defined in [4, 5.1] except that we do not require that the interior of $E_i$ be connected. The main result about harmonic triples that we will need is [4, 5.3]; and this result is still valid because it can be applied to the closure of each component of the interior of $E_i$.

**Lemma 2.2.** Let $E_0$ and $E_1$ be compact subsets of the plane such that $E_0 \subset E_1$ and each component of the complement of $E_0$ intersects the complement of $E_1$. To each $\epsilon > 0$ there corresponds a harmonic triple $(D_0, D_1, \omega)$ such that:

(a) $D_0$ and $D_1$ are simple sets and

(b) $E_i \subset \text{int } D_i \subset E_i + D(0, \epsilon)$, $j = 0, 1$.

In order to prove this lemma we will use the following topological lemma.

**Lemma 2.3.** Let $E_0$ and $E_1$ be sets which satisfy the conditions of Lemma 2.2. To each $\epsilon > 0$ there corresponds a pair of sets $D_0, D_1$ which satisfy (a) and (b) of Lemma 2.2, $D_0 \subset \text{int } D_1$ and each component of the complement of $D_0$ intersects the complement of $D_1$.

Given sets $E_0, E_1$ as in Lemma 2.1 let $D_0, D_1$ be the sets corresponding to $\epsilon$ in Lemma 2.3. To complete the proof of Lemma 2.2 we need only show that there is an $\omega$ such that $(D_0, D_1, \omega)$ is a harmonic triple. Let $U$ be a component of the set $\text{int } D_1 \setminus D_0$. The set $U$ is a subset of a component $V$ of the complement of $D_0$. If $\partial U \subset D_0$, then $\partial U \subset \partial V$; however, this would imply that $U = V$, since $V$ is connected. But, $V \neq U$ because $V$ contains a point in the complement of $D_1$. Let $x$ be a point in $\partial U$ that is not in $D_0$. In particular, $x \in V$. Since $U$ is a component of $\text{int } D_1 \setminus D_0$, it follows that $x$ is not in $\text{int } D_1$. Thus $x \in \partial D_1$. The boundary of $U$ is the union of two disjoint subsets $A$ and $B$ where $A \subset \partial D_0$ and $B \subset \partial D_1$. We have argued above that $B$ is nonempty; however, it is possible that $A = \emptyset$. If $y$ is in $A$ or $B$, then $y$ must be contained in an arc that lies in $A$ or $B$ since $U$ is a component of $\text{int } D_1 \setminus D_0$. It follows from the general theory on the Dirichlet problem that there is a continuous function $\omega_0$ on the closure of $U$ which is identically 1 on $B$, identically 0 on $A$ and harmonic on $U$. We will now define $\omega$ on $D_1$: Let $\omega$ be defined on the closure of each component of $\text{int } D_1 \setminus D_0$ in the same way that $\omega_0$ was defined on the closure of $U$ and define $\omega$ on $D_0$ to be identically 0. We claim that $(D_0, D_1, \omega)$ is harmonic triple (see 2.1); in fact, this is clear except, possibly for (iii) of [4, 5.1]. Suppose $x \in D_1 \setminus D_0$. If $x \in \partial D_1$, then
\( \omega(x) = 1 \). Since each component of \( \text{int} D \setminus D_0 \) has a boundary point in \( \partial D \), (We showed this above.) It follows from the minimum principle that \( \omega \) does not assume the value 0 on \( \text{int} D \setminus D_0 \). These observations verify that condition (iii) of [4, 5.1] is met.

**Lemma 2.4.** Suppose that \((D_0, D_1, \omega)\) is a harmonic triple and that \(D_0, D_1\) are simple sets. Then to each \( \varepsilon, \eta > 0 \), there corresponds an \( f \) in \( R(D_1) \) and a positive real number \( b \) such that

\[
|\omega(z) - b \log |f(z)|| < \varepsilon, \quad \text{if} \quad \omega(z) \geq \eta.
\]

**Proof.** Because of Runge's theorem it clearly suffices to prove Lemma 2.4 for the case where \( D_1 \) is a connected simple set, so we will assume that \( D_1 \) is a connected simple set. Let \( \omega_n, n = 1, 2, \ldots \), be the function defined on the domain of \( \omega \) as follows: \( \omega_n(z) = 1/n \) if \( \omega(z) \leq 1/n \) and \( \omega_n(z) = \omega(z) \) if \( \omega(z) > 1/n \). Locally, each \( \omega_n \) is either a harmonic function or the maximum of two harmonic functions. Thus, each \( \omega_n \) is subharmonic; and, since the sequence \( \omega_n \) decreases pointwise to \( \omega \), we see that \( \omega \) is also subharmonic. Let \( U \) denote the interior of \( E \). Since \( \omega \) is subharmonic on the domain \( U \), the Riesz theorem [5, p. 48] implies that

\[
\omega(z) = u(z) + \int \log |z - a| \, d\mu(a), \quad z \in U
\]

where \( u \) is harmonic on \( U \) and \( \mu \) is a positive Borel measure, supported on \( D_0 \). One can easily show that there is a positive integer \( n \) and a finite number of points \( a_i \) in \( D_0 \) such that

\[
\left| \int_{D_0} \log |z - a| \, d\mu(a) - n^{-1} \sum_j \log |z - a_j| \right| < \varepsilon, \quad \text{for} \quad \omega(z) \geq \eta.
\]

We will now establish the following.

2.5. There is a continuous function \( u_1 \) on \( D_1 \) such that: \( u_1 \) is harmonic on \( U \), \( |u(z) - u_1(z)| < \varepsilon \) for \( z \) in \( D_1 \) and all the periods of the harmonic conjugate of \( u \), are rational multiples of \( 2\pi \).

We may assume that the boundary of \( D_1 \) has at least two components. Let \( \gamma_n+1 \) denote the boundary component of \( D_1 \) which also forms the boundary of the unbounded component of the complement of \( D_1 \) and let \( \gamma, \gamma_1, \ldots, \gamma_n \) denote the other boundary components of \( D_1 \). Let \( \omega_j \) denote the harmonic measure of \( \gamma_j, j = 1, \ldots, n \), with respect to the region \( U \) [1, p. 245]. Let \( a_{k,j} \) denote the period of the harmonic conjugate of \( \omega_j \) corresponding to the curve \( \gamma_k \) (see [1, p. 162]). We define \( u_1 \) by:
for an appropriate choice of the numbers $\epsilon_j$. Since the matrix $(a_{k,j})$, $0 \leq k, j \leq n$, is a nonsingular matrix [1, p. 246] and since the period of the harmonic conjugate of $u_j$ corresponding to the curve $y_k$ is $p_k + \sum_j \epsilon_j a_{k,j}$, where $p_k$ is the period of the harmonic conjugate of $u$ corresponding to $y_k$, it follows that we can choose the numbers $\epsilon_j$ in such a way that $-\epsilon < \epsilon_j < \epsilon$ for each $j$ and the numbers $p_k + \sum_j \epsilon_j a_{k,j}$, $k = 1, \cdots, n$, are all rational multiples of $2\pi$. The conclusion of (2.5) follows from the above remarks and the maximum principle.

We now return to the proof of Lemma 2.4. Since the boundary curves of $D_1$ are analytic curves and $\omega$ is constant on each component of the boundary of $D_1$ it follows from the reflection principle that $\omega$ has an extension to a function which is harmonic on a neighborhood of $D_1$. Since the integral in (1) is clearly harmonic on a neighborhood of the boundary of $D_1$, it follows that $u$ has an extension which is harmonic on a neighborhood of $D_1$. Consequently, $u_j$ also has an extension which is harmonic on a neighborhood of $D_1$. By 2.5 we can choose an integer $m$ such that the periods of the harmonic conjugate $h$ of $mu_j$ are all multiples of $2\pi$. Let $g = e^{mu_j + ih}$. From what we have noted above we conclude that $g$ has an extension which is analytic on a neighborhood of $D_1$. Since $g$ has no zeros in $D_1$ we can use Runge's theorem to choose an element $g_i$ in $R(D_1)$ such that

$$|m^{-1} \log |g_i(z)| - u_i(z)| < \epsilon, \ z \in D_1.$$  (3)

Let $p(z) = \prod_j (z - a_j)$ where the $a_j$ are the numbers that appear in (2). From (1), (2), 2.5, and (3) we obtain

$$|\omega(z) - (mn)^{-1} \log |p(z)g_i(z)| | < 3\epsilon, \ \omega(z) \equiv \eta.$$  

This completes the proof of Lemma 2.4.

**Theorem 2.6.** Suppose that $E_0, E_1$ are two compact sets in the plane, that $E_0 \subset E_1$, and that each component of the complement of $E_0$ intersects the complement of $E_1$. To each $\epsilon_i > 0$ there corresponds an $F$ in $R(E_i)$ and an $r > 1$ such that

$$E_i \subset \text{int} L(r, F) \subset E_i + D(0, \epsilon_i), \quad j = 0, 1.$$  (4)

**Proof.** First choose a harmonic triple $(D_0, D_1, \omega)$ corresponding to $\epsilon_i$ as in Lemma 2.2. In particular,
Choose \( \epsilon, \eta > 0 \) so that

\[
(5) \quad \{ z : \omega(z) \leq 2\epsilon + \eta \} \subset E_0 + D(0, \epsilon_i),
\]

\[
(6) \quad \epsilon + \eta < 1 - \epsilon \quad \text{and}
\]

\[
(7) \quad E_1 \subset \text{int}\{ z : \omega(z) \leq 1 - 2\epsilon \}.
\]

Corresponding to \( \epsilon \) and \( \eta \) we choose \( b \) and \( \ell \) as in Lemma 2.4. For convenience, let \( h = b \log |f| \). Let \( H_0 = \{ z : h(z) \leq \epsilon + \eta \} \). Since \( h \) is subharmonic on \( \{ z : \omega(z) < \eta \} \) and \( h \leq \epsilon + \eta \) on the set \( \{ z : \omega(z) = \eta \} \), it follows that \( h \leq \epsilon + \eta \) on \( \{ z : \omega(z) \leq \eta \} \). In particular, \( D_0 \subset H_0 \). Thus,

\[
(8) \quad E_0 \subset \text{int} H_0.
\]

Since \( h \) is superharmonic on the complement of \( D_1 \) and \( h \geq 1 - \epsilon \) on \( \{ z : \omega(z) = 1 \} \), we have

\[
(9) \quad h(z) \geq 1 - \epsilon, \quad z \not\in D_1.
\]

Suppose \( z \in H_0 \), so \( h(z) \leq \epsilon + \eta \). Since \( \epsilon + \eta < 1 \) least. Thus either \( \omega(z) \leq \eta \) or \( \omega(z) \geq 2\epsilon + \eta \). Consequently, from (7) we obtain

\[
(10) \quad H_0 \subset E_0 + D(0, \epsilon_i).
\]

Let \( H_1 = \{ z : H(z) \leq 1 - \epsilon \} \). We have already noted that if \( \omega(z) \leq \eta \), then \( h(z) \leq \epsilon + \eta \). From this and (6) it follows that \( h(z) \leq 1 - \epsilon \) if \( \omega(z) \leq 1 - 2\epsilon \). Thus

\[
H_1 \supset \{ z : \omega(z) \leq 1 - 2\epsilon \}.
\]

We conclude from (7) that

\[
(11) \quad E_1 \subset \text{int} H_1.
\]

From (9) and the definition of \( H_1 \) we have \( H_1 \subset D_1 \); consequently,

\[
(12) \quad H_1 \subset E_1 + D(0, \epsilon_i).
\]

We define \( F \) and \( r \) as follows: \( F = e^{-(\ell + \eta)b}f \) and \( r = e^{(1-\epsilon(1+\eta))b} \). Evidently, \( r > 1, \quad F \in R(E_i) \) and \( H_j = L(r^j, F) \) for \( j = 0, 1 \). The conclusion of the theorem now follows from (8), (10), (11), and (12).
3. The upper bound is best possible. The purpose of this section is to establish Theorem 3.1. This theorem gives a class of operators for which the upper bound given in Theorem 1.9 is attained; therefore, it shows, in particular, that an upper bound for \( \text{sp}(T_s) \) which depends only on \( \text{sp}(T_0) \) and \( \text{sp}(T_i) \) can be no better than the one given in Theorem 1.9 provided that \( \text{sp}(T_0) \) and \( \text{sp}(T_i) \) satisfy the one further condition imposed in Theorem 3.1.

If \( E \) is a compact set in the plane, then \( C_R(\mathbb{E}) \) will denote all functions which are the uniform limit of sequences in \( R(\mathbb{E}) \).

**Theorem 3.1.** Suppose that \( E_0, E_1 \) are compact subsets of the plane, \( E_0 \subseteq E_1 \), the complement of \( E_1 \) meets each component of the complement of \( E_0 \) and that the zero function is the only function in \( C_R(\mathbb{E}) \) which vanishes identically on \( E_0 \). Let \( B_k = C_k(E_k), k = 0, 1, V = B_0 \) and identify \( B_0, B_1 \) with their natural embeddings in \( V \). Let \( T_s \) be the bounded linear operators on \( [B_0, B_1], 0 \leq s \leq 1 \), such that \( T_s f(z) = z/(z) \), \( z \in E_0, f \in B_1 \). Then

\[
\text{sp}(T_s) = I_s(E_0, E_1), \quad 0 \leq s \leq 1.
\]

**Proof.** It is clear in this case that the set \( E \) of Theorem 1.9 is \( E_1 \) and that \( E_0 = \text{sp}(T_0) \). Thus, \( \text{sp}(T_s) \subseteq I_s(E_0, E_1) \).

Now suppose that \( z \in I_s(E_0, E_1) \). Since the conclusion of the theorem is obvious in case \( s = 0 \) or \( 1 \), we assume \( 0 < s < 1 \). To show that \( z \in \text{sp}(T_s) \) we first show that

\[
| x(z) | \leq \| x \|_{B_s}, \quad x \in R(\mathbb{E}_1).
\]

Let \( \varepsilon > 0 \). Since \( B_0 \cap B_1 = B_1 \), it follows from [4, 2.5] that there is an \( f \) in \( \mathcal{F} = \mathcal{F}(B_0, B_1) \) (see [2] for definition of \( \mathcal{F} \)) such that \( f(s) = x \),

\[
\| x \|_{B_s} + \varepsilon \geq \| f \|_{x}
\]

and \( f(\xi) = \sum c_j(\xi)x_j \) where the sum is finite, each \( x_j \in B_1 \) and each \( c_j \) is continuous on \( 0 \leq \text{Re} \xi \leq 1 \), vanishes at \( \infty \) and is analytic on \( 0 < \text{Re} \xi < 1 \). It is clear from the definition of \( C_R(\mathbb{E}_1) \) that each \( x_j \) can be replaced by a \( y_j \) in \( R(\mathbb{E}_1) \) such that for the corresponding function \( f_j(\xi) = \sum c_j(\xi)y_j \) we have

\[
\| f(\xi) - f_j(\xi) \|_{B_s} < \varepsilon, \quad 0 \leq \text{Re} \xi \leq 1.
\]

From (3) and the fact that the norm of \( B_1 \) dominates the norm of \( B_0 \) on \( B_1 \) we have
(4) \[ \| f - f_i \|_s < \epsilon. \]

Given a pair \( (r, F) \) which satisfies (a) through (c) of 1.3 let \( \mathcal{F}_i \) denote the space

\[ \mathcal{F}(C_R(L(1, F)), C_R(L(r, F))). \]

From Theorem 2.6 and the fact that each of the functions \( y_j \) (there are only a finite number of functions \( y_j \)) is in \( R(E_i) \) it is clear that \( (r, F) \) can be chosen so that \( f_i \in \mathcal{F}_i \) and

\[ (1 + \epsilon) \| f_i \|_s \geq \| f_i \|_s. \]

If we let \( E_k \) in [4, 5.3] (see 2.1 above) be \( L(r^k, F) \), \( k = 0, 1 \) and define \( \omega \) by: \( \omega(z) = 0 \), \( z \in E_0 \) and \( \omega(z) = \log |F(z)| \) for \( z \in E_1 \setminus E_0 \); then we can conclude from [4, 5.3] that

\[ \| f_i \|_s \geq | f_i(z, s) |. \]

From Theorem 2.6 it is clear that \( E_1 \supset I_1(E_0, E_1) \). In particular, \( z \in E_1 \). From this and (3) we obtain

\[ | x(z) - f_i(z, s) | \leq \| x - f_i(s) \|_{B_0} < \epsilon. \]

Since \( \epsilon \) is arbitrary, (1) now follows by combining (2), (4), (5), (6) and (7).

Now suppose that \( x \in R(E_i) \) such that \( x(z) \neq 0 \). From (1) we obtain

\[ | x(z) | = | x(z) - (z - T_i)y(z) | \leq \| x - (z - T_i)y \|_{B_0}, \quad y \in R(E_i). \]

Since \( R(E_i) \) is dense in \( B_0 \), this last inequality shows that the range of \( (z - T_i) \) is \( \neq B_0 \). Thus, \( x \in \text{sp}(T_i) \), which completes the proof.

4. A criterion for constant spectra. This section is devoted to Theorem 4.1, which gives a criterion for \( \text{sp}(T_i) = \text{sp}(T_0) \), \( 0 \leq s < 1 \).

**Theorem 4.1.** Suppose that \( T_i \) is a bounded linear operator on \( B^j \), \( j = 0, 1 \), that \( T_0 \) and \( T_i \) agree on \( B_0 \cap B_1 \) and that to each \( \epsilon > 0 \) there correspond subspaces \( Q_{k, j_0}, \ldots, Q_{k, j_1} \) of \( B_{j_0} \), \( j = 0, 1 \) with the following properties: (a) each space \( Q_{k, j} \) is \( T_j \) invariant; (b) \( B_j \) is the direct sum of the spaces \( Q_{k, j} \); (c) \( Q_{k, j} \cap Q_{k, j_1} \) is dense in both \( Q_{k, j} \) and \( Q_{k, j_1} \) for all \( j \); (d) \( \text{sp}(T_0|Q_{k_0}) \subset \text{sp}(T_j|Q_{k_1}) \) for each \( k \); (e) \( \text{sp}(T_0|Q_{k_0}) \) is contained in \( D(z, \epsilon) \) for some \( z \) in \( \text{sp}(T_0) \) depending on \( k \), \( k = 1, 2, \ldots, n \). Then
\[
\text{Proof.} \quad \text{Since } \text{sp}(aT_0) = \text{sp}(T_0) \text{ for any complex number } a, \text{ it clearly suffices to prove the theorem assuming } \|T_0\| \leq 1/2 \text{ and } \|T_i\| \leq 1/2. \text{ Fix } s \text{ and let } \alpha > 0. \text{ Choose } \delta \text{ corresponding to } \epsilon, s \text{ and } K = \text{sp}(T_0)(\text{sp}(T_0) \subset D(0, 1/2)] \text{ in Lemma 1.5. Let } Q_{k,i} \text{ be the spaces corresponding to } \delta \text{ as in the theorem. Let } U_{k,i} \text{ be the restriction of the operator } T_i \text{ to the space } Q_{k,i}. \text{ Let } Q_{k,s} \text{ denote the space } [Q_{k,0}, Q_{k,1}]_s, \text{ where } Q_{k,0} \text{ and } Q_{k,1} \text{ are regarded as embedded in } V \text{ (see 1.1) in the natural manner. If } x \in Q_{k,0} \cap Q_{k,1}, \text{ then } T_ix = T_0x = T_ix; \text{ consequently, } T_ix \in Q_{k,0} \cap Q_{k,1}. \text{ This shows that } Q_{k,s} \text{ is an invariant subspace of } T_i. \text{ Let } U_{k,s} \text{ denote the restriction of } U_{k,i} \text{ to } Q_{k,s}. \text{ For each } k \text{ it is clear from (d) that the hypotheses of Theorem 1.9 are satisfied in the case of the operators } U_{k,0}, j = 0, 1; \text{ thus, we conclude that }
\]

\[
\text{sp}(U_{k,s}) \subset I_s(\text{sp}(U_{k,0}), E_k)
\]

where \(E_k\) is the set corresponding to \(U_{k,i}\) as in Theorem 1.9. Because of the choice of subspaces \(Q_{k,j}\) we have \(\text{sp}(U_{k,0}) \subset D(z_k, \delta)\) for some \(z_k \in \text{sp}(T_0)\). Since \(\|T_i\| \leq 1/2, \text{ sp}(T_i) \subset D(0, 1]\); this together with Theorem 1.7 implies that \(E_k \subset D(0, 1]\). From these observations, Lemma 1.4, Lemma 1.5 and the way in which \(\delta\) was chosen it follows that

\[
\text{sp}(U_{k,s}) \subset \text{sp}(T_0) + D(0, \epsilon).
\]

Since \(\text{sp}(T_i) = \bigcup \text{sp}(U_{k,s}), k = 1, 2, \cdots, n\), it follows that

\[
\text{sp}(T_i) \subset \text{sp}(T_0) + D(0, \epsilon).
\]

The conclusion of the theorem follows from this and the fact that \(\epsilon\) was chosen arbitrarily.

\textbf{Remark.} A special case of Theorem 4.1 can be proved by slightly extending a method used by Hirschman [3] for determining the spectrum of certain multipliers. The special case occurs if, in addition to the assumptions in Theorem 4.1, we assume that each operator \(U_k\), which is obtained by restricting \(T_0\) to \(Q_{k,0}\), has the property that the norm of \(\beta I - U_k\) is equal to its spectral radius for each complex number \(\beta\). Suppose \(\lambda \notin \text{sp}(T_0)\). Let \(d\) be the distance from \(\lambda\) to \(\text{sp}(T_0)\). Let \(0 < s < 1\) and choose \(\epsilon\) so that \(\epsilon^{1+s}(2\|T_i\|/d)^s \leq 1/2\). Choose the subspaces \(Q_{k,i}\) corresponding to \(\epsilon\). Fix \(k\) and let \(S_i\) be the operator obtained by restricting \(T_i\) to \(Q_{k,i}\) and let \(S_k\) be the corresponding operators on the interpolation spaces. Let \(\beta \in \text{sp}(S_0)\). Then \(\lambda I - S_k = (\lambda - \beta)I + \beta I - S = (\lambda - \beta)(I - A_i)\) where, in general, \(A_i = \)
$(\beta - \lambda)^{-1}(\beta I - S)$. Now \(\|A_i\|_s \leq \|A_0\|^1-t \|A_i\|' \leq 1/2\). Thus, \(\lambda I - S\) has an inverse. Since \(k\) was arbitrary, we see that \(\lambda \not\in \text{sp}(T)\).

**REFERENCES**


Received January 14, 1974.

UNIVERSITY OF CALIFORNIA, RIVERSIDE