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OPERATOR-VALUED INNER FUNCTIONS ANALYTIC ON THE CLOSED DISC II

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An operator-valued inner function V is called scalar if $\{V(w): |w| < 1\}$ is a commuting family of normal operators. Suppose that T is a bounded linear operator with $\|T\| \leqq 1$ and spectral radius strictly less than one. Let V_T be its Potapov inner function and define $U_T = V_T V_T^*(1)$. The structure of nonnormal T for which U_T is scalar is discussed. An explicit characterization is given if the underlying Hilbert space is finite dimensional. Examples are given for the infinite dimensional case. The relationship between scalar inner functions and operators for which T^*T and $T^* + T$ commute is examined.

1. Introduction. Sherman [9] introduced the concept of an inner function of scalar type. He observed that the Potapov inner function V_T of a normal operator T, $\|T\| < 1$, was of scalar type. On the other hand Campbell [1] has shown that if $\|T\phi\| < \|\phi\|$ for all nonzero vectors ϕ, then V_T is of scalar type if and only if T is normal. There are, however, non-normal operators associated with scalar inner functions. It will be shown that if T^*T and $T + T^*$ commute, then T is the restriction of an operator $\tilde{T}$ such that $V_{\tilde{T}}$ differs from an inner function of scalar type by a constant unitary operator on the right. Thus, when studying the operators such that T^*T and $T + T^*$ commute, it would be helpful to have information on operators with scalar inner functions and their invariant subspaces. We shall develop some of the needed information.

2. Terminology. We assume that the reader is familiar with the basic definitions of our terms. They may be found in [6]. Our notation is that of [1]. We review it briefly. Throughout this paper $\mathfrak{k}$ will be a fixed separable Hilbert space. $H^2_{\mathfrak{k}}$ will denote the $\mathfrak{k}$-valued Hardy space of the circle, $|w| = 1$. S is multiplication by w in $H^2_{\mathfrak{k}}$. T will always be a bounded linear operator from $\mathfrak{k}$ into $\mathfrak{k}$. If $\|T\| \leqq 1$, $T^n \to 0$ strongly, and T^* is not an isometry, then the Potapov inner function V_T of T is defined by

$$V_T(w) = -T^* + w \sum_{n=0}^{\infty} (I - T^*T)^{1/2} w^n T^n (I - TT^*)^{1/2}. \tag{1}$$

If the spectral radius of T, $r(T)$, is less than one, then (1) takes the form

$$V_T(w) = -T^* + w(I - T^*T)^{1/2}(I - wT)^{-1}(I - TT^*)^{1/2} \tag{2}$$

and V_T is analytic on the closed disc $|w| \leqq 1$. If $\|T\| < 1$, then (1) becomes

$$V_T(w) = (I - T^*T)^{-1/2}(wI - T^*)(I - wT)^{-1}(I - TT^*)^{1/2}. \tag{3}$$

For an inner function U, we say that U is an analytic inner function, $U \in (AI)$, if U is analytic on the closed disc $|w| \leqq 1$. If $U \in (AI)$ and $U(1) = I$, the identity on $\mathfrak{K}$, we say U is normalized. For any $U \in (AI)$, $UU^*(1)$ is called its normalized form. An inner function will be called scalar **[9]** if $U(w)$ commutes with $U(u)$ for almost all $|w| = |u| = 1$. This is equivalent to having all the coefficients in the power series for U commute.

If $U \in (AI)$, let $z = i(1-w)/(1+w)$ and define $\tilde{U}(z) = U(w)$. Then $\tilde{U}$ is an inner function on the upper half-plane. Let x, y denote real values of z. The variables x, y, z, w will always be used in this manner. Now $\tilde{U}$ satisfies the differential equation $\tilde{U}'(x) = iA(x)\tilde{U}(x)$ where $A(x) \geqq 0$ and A is analytic on a neighborhood of the real axis **[7]**. If $U = V_T$ for some T, $r(T) < 1$, then we write A_T for A. A straightforward calculation gives

$$A_T(x) = \rho(x)(I - T^*T)^{1/2}(I - wT)^{-1}(wI - T^*)^{-1}(I - T^*T)^{1/2} \tag{4}$$

where $\rho(x)$ is a scalar valued function which is always nonzero.

For bounded linear operators X, Y, let $[X, Y] = XY - YX$.

Finally, we let $\Theta = \{T\colon [T^*T, T + T^*] = 0\}$.

3. Preliminary results. It is clear from (1) that if V_T is scalar, then $[T, T^*] = 0$ and T is normal. Given a $T \in \Theta$, we shall construct in Section 8 a non-trivial extension $\tilde{T}$ of T such that $V_{\tilde{T}}V_{\tilde{T}}^*(1)$ is scalar.

In order to try and understand the structure of these T, we will study those V_T with scalar normalized form. But first, we shall show how our results relate to the structure of a general analytic inner function. Note that if $U \in (AI)$, the normalized form of U is scalar if and only if there exists a constant unitary operator U_0 such that UU_0 is scalar.

PROPOSITION 1. *Suppose that $U \in (AI)$. The normalized form of U is scalar if and only if $[A(x), A(y)] = 0$ for every real x, y.*

Proof. The only if part follows by twice differentiating $U(x)U(y) =$

$U(y)U(x)$. On the other hand if $[A(x), A(y)] = 0$ for every real x, y, then $W(x) = \exp\left(i\int_0^x A(s)ds\right)$ and the normalized form of U both satisfy the differential equation $X' = iAX$, $X(0) = I$, and hence are equal. But W is scalar, hence the normalized form of U is.

It was shown in [**1**, p. 58] that if $U \in (AI)$, then

$$U = cV_T\tau U_1 \oplus U_2 \quad \text{where} \quad \mathscr{k} = \mathscr{k}_1 \oplus \mathscr{k}_2. \tag{5}$$

Here U_i is a constant unitary operator on $\mathscr{k}_i$, T is an operator from $\mathscr{k}_1$ into $\mathscr{k}_1$ such that $\|T\| \leqq 1$, $r(T) < 1$, τ is an isometry from $\mathscr{k}_1$ onto $R([I - TT^*]^{1/2})$, and c^* is an isometry from $\mathscr{k}_1$ onto $R([I - T^*T]^{1/2})$. Furthermore $A(x) = cA_T(x)c^* \oplus 0$. Using (4) and the definition of c we see that $[A(x), A(y)] = 0$ for all real x and y if and only if $[A_T(x), A_T(y)] = 0$ for all real x and y.

Thus as a consequence of Proposition 1 we have:

PROPOSITION 2. *If U and V_T are related as in* (5), *then the normalized form of U and the normalized form of V_T are either both scalar or both not scalar.*

The determination of which normalized analytic inner functions are scalar reduces then to determining which analytic Potapov inner functions have a scalar normalized form. If $\|T\| \leqq 1$, $r(T) < 1$, T is not normal, and V_T has scalar normalized form, then 1 is an eigenvalue of T^*T [**3**, Theorem 6].

We now turn to describing those T for which the normalized form of V_T is scalar. For notational convenience let $U_T(w) = V_T(w)V_T^*(1)$.

The next Proposition will be useful in what follows.

PROPOSITION 3. *Suppose that $\|T\| \leqq 1$, $r(T) < 1$. Then T has a reducing subspace $M \subseteq \mathscr{k}$ if and only if $U_T = U_1 \oplus U_2$ where U_i is an inner function on $H^2_{\mathscr{k}_i}$, $\mathscr{k} = \mathscr{k}_1 \oplus \mathscr{k}_2$.*

Proof. If T has a reducing subspace M, the result is obvious. Suppose then that $U_T = U_1 \oplus U_2$ on $H^2_{\mathscr{k}} = H^2_{\mathscr{k}_1} \oplus H^2_{\mathscr{k}_2}$. Now

$$H^2_{\mathscr{k}} \ominus (U_T H^2_{\mathscr{k}}) = (H^2_{\mathscr{k}_1} \ominus U_1 H^2_{\mathscr{k}_1}) \oplus (H^2_{\mathscr{k}_2} \ominus U_2 H^2_{\mathscr{k}_2}).$$

But $H^2_{\mathscr{k}_i} \ominus U_i H^2_{\mathscr{k}_i}$, $i = 1, 2$, is an invariant subspace for S^* on $H^2_{\mathscr{k}} \ominus U_T H^2_{\mathscr{k}}$. Thus they are reducing subspaces. But S^* restricted to $H^2_{\mathscr{k}} \ominus U_T H^2_{\mathscr{k}}$ is unitarily equivalent to T. The unitary map B from $\mathscr{k}$ to $H^2_{\mathscr{k}} \ominus U_T H^2_{\mathscr{k}}$ is given by $B\phi = (I - T^*T)^{1/2}(I - wT)^{-1}\phi$. That is, $S^*B\phi = BT\phi$ for every $\phi \in \mathscr{k}$. Thus $M = B^{-1}(H^2_{\mathscr{k}_1} \ominus UH^2_{\mathscr{k}_1})$ would be reducing for T.

It is important to note that the $\mathcal{k}_1$ of Proposition 3 may not be a reducing subspace of T. However, as observed in the proof of Proposition 3, $M_i = B^{-1}(H^2_{\mathcal{k}_i} \ominus U_i H^2_{\mathcal{k}_i})$ is reducing for T. Thus $T = T_1 \oplus T_2$ on $M_1 \oplus M_2$. But then $U_T = U_{T_1} \oplus U_{T_2}$, and

$$BM_i = H^2_{M_i} \ominus U_{T_i} H^2_{M_i} = H^2_{\mathcal{k}_i} \ominus U_i H^2_{\mathcal{k}_i}.$$

If one assumes that $U_1'(x = 0)$ is one to one on $\mathcal{k}_1$, then $\mathcal{k}_1 \subseteq M_1$ and U_{T_1} is the identity on $M_1 \ominus \mathcal{k}_1$ since $U_1'(x = 0)$ being one-to-one implies that U_1 does not have a constant summand **[2]**. But $U_{T_1}\phi = \phi$ implies that $(I - T_1^* T_1)\phi = 0$. We summarize these observations in the next theorem.

THEOREM 1. *Suppose that* $U_T = U_1 \oplus U_2$ *on* $H^2_{\mathcal{k}_1} \oplus H^2_{\mathcal{k}_2}$, $\mathcal{k} = \mathcal{k}_1 \oplus \mathcal{k}_2$. *Suppose further that* $U_1'(x = 0)$ *is one-to-one. Then* T *has a reducing subspace* M *such that* $\mathcal{k}_1 \subseteq M$ *and* $M \ominus \mathcal{k}_1 \subseteq N([I - T^*T])$.

4. The finite dimensional case. Suppose that $r(T) < 1$. If U_T is scalar and $\dim \mathcal{k} = n < \infty$, then U_T is unitarily equivalent to an $n \times n$ diagonal matrix, $\text{Diag}\{b_1, \cdots, b_r, 1, \cdots, 1\}$, where b_i is a finite Blaschke product times a complex number of modulus one. We can now apply Theorem 1 to get:

THEOREM 2. *Suppose that* $\|T\| < 1$, $r(T) < 1$, *and* $\dim \mathcal{k} < \infty$. *Then* U_T *is scalar if and only if* $T = \Sigma_{i=1}^r \oplus T_i$ *on* $\mathcal{k} = \Sigma_{i=1}^r \oplus \mathcal{k}_i$ *where* $\text{rank}(I - T_i^* T_i) = 1$.

Note that the T_i of Theorem 2 are unitarily equivalent to S^* restricted to $H^2 \ominus bH^2$ where b is a scalar inner function. A related study may be found in **[10]**. Sickler, however, is primarily interested in the case $r(T) = 1$ and his results do not overlap ours. The following characterization of the T_i in Theorem 2 follows fom the observation that T is an isometry on $N(I - T^*T)$.

PROPOSITION 4. *Suppose that* $\dim \mathcal{k} = n < \infty$ *and* $\|T\| \leqq 1$. *Then* $\text{rank}(I - T^*T) = 1$ *if and only if* $T = \delta V + W$ *where* δ *is a scalar,* $0 \leqq |\delta| < 1$, *and* V, W *are partial isometries such that*; $\text{rank } V = 1$, $\text{rank } W = n - 1$, $V^*W = 0$, *and* $VW^* = 0$.

5. The general case. In **[9]** Sherman showed that if U is scalar, then $U(w) = \int_0^{2\pi} f(w, \lambda)\, dE(\lambda)$, $|w| \leqq 1$, for a particular spectral measure $E(\cdot)$ on the unit circle. An easy modification of his arguments shows that if $U \in (AI)$ and U is scalar, then $f(w, \lambda)$ is a finite Blaschke product for almost all λ. One can also conclude that the zeros of $f(w, \lambda)$ for different λ are essentially bounded away from $|w| = 1$. However, the order and number of the zeros may be arbitrarily large. (Just take T to be the orthogonal sum of the appropriate operators acting in finite dimensional spaces.) One could then argue that if $\|T\| \leqq 1$, $r(T) < 1$, and U_T is scalar, then T is an integral (or orthogonal sum if $E(\cdot)$ is a discrete measure) of operators T_λ where T_λ acts in a finite dimensional space and is of the form described in the previous section.

There is one weakness with this approach. Using $T \in \Theta$ it is possible to construct operators $\tilde{T}$ such that $\|\tilde{T}\| \leqq 1$, $r(\tilde{T}) < 1$, and $U_{\tilde{T}}$ is scalar, but $\tilde{T}$ does not appear to be like the finite dimensional case.

First, we wish to give some additional examples of scalar U_T. A way of testing a T to see whether U_T is scalar is needed. We begin by calculating the coefficients of U_T. From (1) and (2) we have that the constant term is

$$
\begin{aligned}
&T^*T - T^*(I - TT^*)^{1/2}(I - T^*)^{-1}(I - T^*T)^{1/2} \\
(6) \quad &= T^*T - (I - T^*T)^{1/2}T^*(I - T^*)^{-1}(I - T^*T)^{1/2} \\
&= T^*T - (I - T^*T)^{1/2}\{(I - T^*)^{-1} - I\}(I - T^*T)^{1/2} \\
&\qquad = I - (I - T^*T)^{1/2}(I - T^*)^{-1}(I - T^*T)^{1/2}.
\end{aligned}
$$

The $(n+1)$-th term, $n \geqq 0$, is

$$
\begin{aligned}
&-(I - T^*T)^{1/2}T^n(I - TT^*)^{1/2}T + (I - T^*T)^{1/2}T^n(I - TT^*) \\
&\qquad \times (I - T^*)^{-1}(I - T^*T)^{1/2} \\
&= -(I - T^*T)^{1/2}T^{n+1}(I - T^*T)^{1/2} \\
&\qquad + (I - T^*T)^{1/2}T^n(I - T^*)^{-1}(I - T^*T)^{1/2} \\
&\quad -(I - T^*T)^{1/2}T^{n+1}T^*(I - T^*)^{-1}(I - T^*T)^{1/2} \\
(7) &= -(I - T^*T)^{1/2}T^{n+1}(I - T^*T)^{1/2} \\
&\qquad + (I - T^*T)^{1/2}T^n(I - T^*)^{-1}(I - T^*T)^{1/2} \\
&\quad -(I - T^*T)^{1/2}T^{n+1}(I - T^*)^{-1}(I - T^*T)^{1/2} \\
&\qquad + (I - T^*T)^{1/2}T^{n+1}(I - T^*T)^{1/2} \\
&= (I - T^*T)^{1/2}T^n(I - T^*)^{-1}(I - T^*T)^{1/2} \\
&\qquad -(I - T^*T)^{1/2}T^{n+1}(I - T^*)^{-1}(I - T^*T)^{1/2}.
\end{aligned}
$$

Thus we have from (6) and (7) that:

PROPOSITION 5. *Suppose that* $\|T\| \leqq 1$, $r(T) < 1$. *Then* U_T *is scalar* *if and only if* $\{(I - T^*T)^{1/2}T^n(I - T^*)^{-1}(I - T^*T)^{1/2}\colon n \geqq 0\}$ *is a commuting family of normal operators.*

A useful sufficient set of conditions is then immediate.

PROPOSITION 6. *Suppose that* $\|T\| \leqq 1$, $r(T) < 1$, *and*

$$\{(I - T^*T)^{1/2}T^nT^{*m}(I - T^*T)^{1/2}\colon n \geqq 0,\ m \geqq 0\}$$

is a commuting family of normal operators. Then U_T *is scalar.*

The conditions of Proposition 6 are somewhat easier to work with than those of 5. It seems resonable to conjecture that Proposition 6 gives necessary as well as sufficient conditions for U_T to be scalar, but to date we have not been able to prove it.

6. Examples. In this section we will explicitly construct some operators T such that U_T is scalar. The construction using operators in Θ will be given later.

Recall that an operator T on $\mathcal{h}$ is called n-normal if $\mathcal{h}$ can be considered as an orthogonal sum of n-copies of a Hilbert space $\mathcal{h}_0$ and relative to this decomposition T can be written as an $n \times n$ matrix whose entries A_{ij} are all normal operators which commute with each other.

We will also need the following lemma. We omit its proof.

LEMMA 1. *If* X, XYX *are commuting normal operators and* X *is one-to-one, then* Y *is normal and* $[X, Y] = 0$.

Suppose that $\|T\| \leqq 1$, $r(T) < 1$, and U_T is scalar. Relative to the decomposition $\mathcal{h} = R(I - T^*T) \oplus N(I - T^*T)$, T has the operator matrix

$$T = \begin{bmatrix} A & B \\ C & E \end{bmatrix}, \quad \text{where} \tag{8}$$

$$B^*B + E^*E = I,$$

$$A^*B + C^*E = 0,$$

and $I - A^*A - C^*C = D_0$, $D_0 \geqq 0$, is one-to-one. Note that

$$(I - T^*T) = \begin{bmatrix} D_0 & 0 \\ 0 & 0 \end{bmatrix}.$$

From Proposition 6 we have

PROPOSITION 7. *If the A, B, C, E in* (8) *are commuting normal operators, then* $T=\begin{bmatrix} A & B \\ C & E\end{bmatrix}$ *is such that* U_T *is scalar provided that* $r(T)<1$.

In building an example, T will be nonnormal if $BB^*\neq C^*C$ or $AC^*+BE^*\neq 0$ or $CC^*+EE^*\neq I$. One way to get $r(T)<1$ is to have $\|A+E\|+\|A\|+\|C\|<1$. It is an immediate consequence of Lemma 1 that:

PROPOSITION 8. *The A in the block form* (8) *of T is normal if* U_T *is scalar.*

However, it is easy to construct T such that U_T is scalar and T in the block form (8) is not n-normal.

EXAMPLE 1. Let $\mathscr{h}=\mathscr{h}_0\oplus\mathscr{h}_0$ and let S be a unilateral shift on $\mathscr{h}_0$. Define T_δ on $\mathscr{h}$ by

$$T_\delta=\begin{bmatrix} \delta I & 1/\sqrt{2}\,S \\ -\delta S^* & 1/\sqrt{2}\,I\end{bmatrix},\quad 0\leqq\delta<1.$$

Then T_δ is in the block form (8). $T_\delta\to T_0$ uniformly as $\delta\to 0$. But $r(T_0)=1/\sqrt{2}<1$ so $r(T_\delta)<1$ for sufficiently small δ. That U_{T_δ} is scalar for $r(T_\delta)<1$ follows from Proposition 6.

Finally we note that if dim $R(I-T^*T)>1$ and U_T is scalar, then T has reducing subspaces by Theorem 1 and Proposition 3.

7. The case $r(T)=1$. If $r(T)=1$, $T^n\to 0$ strongly, and T^* is not an isometry, then V_T exists and has the formula (1). Suppose V_TU_0 is scalar for some constant unitary U_0. Proposition 2 does not necessarily apply since $\{V_T(w)U_0\colon |w|=1\}$ is only a commuting family of unitary operators for almost all w. However, this difficulty is easily avoided. Suppose $|\alpha|=1$. Then $V_{\alpha T}(w)=V_T(\alpha w)\bar{\alpha}$. Thus by replacing T by αT for a suitable α, we get that Proposition 2 holds for αT.

In trying to duplicate the results of §5 one has to use the series (1) since $(I-wT)$ does not have a bounded inverse for all $|w|=1$. This presents difficulties in calculating U_T. The proof of Proposition 5 relies on the fact that $(I-TT^*)^{1/2}$ can be factored out of $\Sigma_{n=0}^{\infty}(I-TT^*)^{1/2}T^{*n}$. If α can be chosen so that $|\alpha|=1$, $\alpha\notin\sigma(T)$, then one can get Proposition 5 to hold for $\bar{\alpha}T$. If $\{w : |w| = 1\} \subseteq \sigma(T)$, then the calculations would have to be modified. We will not attempt them here.

8. An application. There is another way to "build" scalar inner functions. It was part of our original reason for undertaking this study. Recall that we let Θ denote $\{T: [T^*T, T + T^*] = 0\}$. If Q is quasinormal ($[Q, Q^*Q] = 0$), A is self-adjoint, and $[A, Q] = 0$, then $A + Q \in \Theta$. Whether there exists other types of operators in Θ is an open question. The defining condition of Θ occurs, for example, in **[5]**. It would be of interest to characterize the operators in Θ.

Suppose that $\|T\| < 1$ and form the inner function $V = V_T V_{T^*}$. $V \in (AI)$ since $V_T, V_{T^*} \in (AI)$. From (3) we have that

$$
\begin{aligned}
V(w) &= (I - T^*T)^{-1/2}(wI - T^*)(I - wT)^{-1}(wI - T) \\
&\qquad\qquad \times (I - wT^*)^{-1}(I - T^*T)^{1/2} \\
&= (I - T^*T)^{-1/2}(wI - T^*)(wI - T) \\
&\qquad\qquad \times (I - wT)^{-1}(I - wT^*)^{-1}(I - T^*T)^{1/2} \\
&= (I - T^*T)^{-1/2}(T^*T - w(T^* + T) + w^2 I) \\
&\qquad\qquad \times (I - w(T^* + T) + w^2 T^*T)^{-1}(I - T^*T)^{1/2}.
\end{aligned}
$$

If $T \in \Theta$, then V would be scalar. Now $H^2_{\mathfrak{k}} \ominus V_T H^2_{\mathfrak{k}} \subseteq H^2_{\mathfrak{k}} \ominus V H^2_{\mathfrak{k}}$ since $V = V_T V_{T^*}$. We have then that T is unitarily equivalent to the restriction of an operator with scalar inner function to an invariant subspace. Thus information about operators associated with scalar inner functions might prove useful in analyzing the operators in θ. Conversely, finding additional operators in Θ allows us to construct additional examples of scalar inner functions.

We will not go into a detailed study of $T \in \Theta$, but will make a few basic observations. The proofs of the first two are trivial, but the propositions are often useful when working with operators in Θ.

PROPOSITION 9. *$T \in \Theta$ if and only if $T^*[T^*, T] = [T^*, T]T$.*

PROPOSITION 10. *If $T \in \Theta$, then $N(T)$ is reducing.*

THEOREM 3. *If $T \in \Theta$ and T is a trace class compact operator, then T is normal.*

Proof. Suppose that $T \in \Theta$ and T is a trace class compact operator. Then

$$[T^*, T]^2 = T^*TT^*T - T^*T^2T^* - TT^{*2}T + TT^*TT^*.$$

Thus,

$$\begin{aligned}\tfrac{1}{2}\operatorname{trace}([T^*, T]^2) &= \operatorname{trace}((T^*T)^2) - \operatorname{trace}(T^{*2}T^2)\\ &= \operatorname{trace}((T^*T)^2 - T^{*2}T^2)\\ &= -\operatorname{trace}(T^*[T^*, T]T)\\ &= -\operatorname{trace}([T^*, T]T^2)\\ &= -\operatorname{trace}(T^*T^3 - TT^*T^2)\\ &= -\operatorname{trace}(T^*T^3) + \operatorname{trace}(T^*T^3) = 0.\end{aligned}$$

But $[T^*T]^2 \geqq 0$, thus $[T^*T] = 0$ and T is normal.

A discussion of trace class operators may be found in [**4**, pp. 1088–1119].

Note that if $\dim \mathfrak{k} < \infty$, then Theorem 3 says that $T \in \Theta$ if and only if T is normal.

We shall now examine the relationship between V and T more closely. Our first two results, while dealing with basic uniqueness properties of Potapov inner functions, are apparently new.

Assume for the remainder of this paper that $T \in \Theta$ and $\|T\| < 1$. Then $V = V_T V_{T^*}$ is scalar. Let $\tilde{T}$ denote S^* restricted to $H^2_{\mathfrak{k}} \ominus VH^2_{\mathfrak{k}}$. Then $\|\tilde{T}\| \leqq 1$, $r(\tilde{T}) < 1$ since $V \in (AI)$. V is scalar so that $V(w) = \int b(w, \lambda)E(d\lambda)$. If $c(w, \lambda)$, $|w| = 1$, is an inner function that divides $b(w, \lambda)$ almost everywhere and $c(\cdot, \cdot)$ is a borel measurable function, then $V_1(w) = \int c(w, \lambda)E(d\lambda)$ defines a scalar inner function that divides V [**9**]. Inner functions like V_1 are the obvious factors for a scalar V. Note that $[V, V_1] = 0$. V_T is also a factor of V but neither V_T nor U_T are scalar if T is nonnormal. Thus V_T is a nontrivial example of a nonscalar factor of a scalar inner function.

Now if $\|\tilde{T}f\| < \|f\|$ for all nonzero $f \in H^2_{\mathfrak{k}} \ominus VH^2_{\mathfrak{k}}$, then $V = V_{\tilde{T}}U_0$ for some $\tilde{T}_0$ acting in $\mathfrak{k}$ and a unitary U_0[**1**]. $\tilde{T}_0$ is unitarily equivalent to $\tilde{T}$. $\tilde{T}_0$ would be normal since V is scalar [**3**], and T would have to be subnormal. Since it is unknown if all the operators in Θ are subnormal, it becomes important to determine if $\|\tilde{T}f\| < \|f\|$ for all nonzero f.

It follows from a result of Virot [**11**] about Rota inner functions that:

THEOREM 4. *If T_1, T_2 are operators of norm less than one, and $V_{T_1}H^2_{\mathfrak{k}} \subseteq V_{T_2}H^2_{\mathfrak{k}}$, then $T_1 = T_2$.*

Proof. Since $\|T_1\| < 1$, $\|T_2\| < 1$, there exists T_3, T_4 such that $r(T_3)$, $r(T_4) < 1$ and $V_{T_1} = R_{T_3}U_1$, $V_{T_2} = R_{T_4}U_2$ where R_{T_i} is the Rota inner function of T_i and U_i is a constant unitary operator [8, p. 29]. But Virot has shown that $T_3 = T_4$ if $R_{T_3}H^2_{\mathfrak{K}} \subseteq R_{T_4}H^2_{\mathfrak{K}}$. Hence $V_{T_1}H^2_{\mathfrak{K}} = V_{T_2}H^2_{\mathfrak{K}}$. Thus $V_{T_1} = V_{T_2}U_0$ for some constant unitary U_0. Evaluation at $w = 0$ gives $T_1^* = T_2^*U_0$, so that $(I - T_1^*T_1) = (I - T_2^*T_2)$, and $(I - T_1T_1^*) = U_0^*(I - T_2T_2^*)U_0$. But the w terms of V_{T_1} and $V_{T_2}U_0$ must be equal. That is,

$$(I - T_1^*T_1)^{1/2}(I - T_1T_1^*)^{1/2} = (I - T_2^*T_2)^{1/2}(I - T_2T_2^*)^{1/2}U_0.$$

But $(I - T_i^*T_i)^{1/2}$, $(I - T_iT_i^*)^{1/2}$ are one-to-one. Thus

$$U_0^*(I - T_2T_2^*)^{1/2}U_0 = (I - T_1T_1^*)^{1/2} = (I - T_2T_2^*)^{1/2}U_0$$

and $U_0 = I$. Hence $T_1 = T_2$.

The last half of the proof of Theorem 4 is of some interest in its own right. It shows that:

PROPOSITION 11. *If* $\|T_1\phi\| < \|\phi\|$ *for all nonzero* $\phi \in \mathfrak{K}$ *and if* $V_{T_1}H^2_{\mathfrak{K}} = V_{T_2}H^2_{\mathfrak{K}}$, *then* $T_1 = T_2$.

The assumption on T_1 in Proposition 11 is needed.

EXAMPLE 2. Let

$$T_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad U_0 = \begin{bmatrix} \bar{\alpha} & 0 \\ 0 & \alpha \end{bmatrix}$$

where $|\alpha| = 1$, $\alpha \neq 1$. Then

$$V_{T_1}(w) = \begin{bmatrix} 0 & w^2 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad V_{T_2}(w) = \begin{bmatrix} 0 & \alpha w^2 \\ -\bar{\alpha} & 0 \end{bmatrix}.$$

Thus $V_{T_1}U_0 = V_{T_2}$ so that $V_{T_1}H^2_{\mathfrak{K}} = V_{T_2}H^2_{\mathfrak{K}}$ but $T_1 \neq T_2$. T_1, of course, is unitarily equivalent to T_2. Note that in this example $r(T_i) < 1$, $\|T_i^2\| < 1$ also.

Example 2 also shows that the hypothesis of Theorem 4 cannot be weakened to $r(T_i) < 1$.

Since $VH^2_{\mathfrak{K}} \subseteq V_TH^2_{\mathfrak{K}}$, we have $\|\tilde{T}\| = 1$ otherwise Theorem 4 would give us that $VH^2_{\mathfrak{K}} = V_TH^2_{\mathfrak{K}}$ which is a contradiction. We shall now show that there is an f such that $\|\tilde{T}f\| = \|f\|$. The proof will depend on the fact that if $\|X\| \leqq 1$, $r(X) < 1$, then

(9) $$H^2_{\mathscr{K}} \ominus V_X H^2_{\mathscr{K}} = \{(I - X^*X)^{1/2}(I - wX)^{-1}\phi \colon \phi \in \mathscr{K}\} \qquad \textbf{[8]}.$$

PROPOSITION 12. *Suppose that* $T \in \Theta$, $\|T\| < 1$. *Let* $V = V_T V_{T^*}$ *and* $\tilde{T}$ *be* S^* *acting in* $H^2_{\mathscr{K}} \ominus VH^2_{\mathscr{K}}$. *Then* $\|\tilde{T}f\| = \|f\|$ *for some* $f \in H^2_{\mathscr{K}} \ominus VH^2_{\mathscr{K}}$.

Proof. Since $VH^2_{\mathscr{K}} \subseteq V_T H^2_{\mathscr{K}}$ we have

(10) $$H^2_{\mathscr{K}} \ominus VH^2_{\mathscr{K}} = (H^2_{\mathscr{K}} \ominus V_T H^2_{\mathscr{K}}) \oplus (V_T H^2_{\mathscr{K}} \ominus V_T V_{T^*} H^2_{\mathscr{K}}).$$

Suppose that $f \in H^2_{\mathscr{K}} \ominus VH^2_{\mathscr{K}}$. Then from (10) we have $f = f_1 \oplus V_T f_2$ where $f_1 \in H^2_{\mathscr{K}} \ominus V_T H^2_{\mathscr{K}}$ and $f_2 \in H^2_{\mathscr{K}} \ominus V_{T^*} H^2_{\mathscr{K}}$. Thus there are $\phi, \psi \in \mathscr{K}$ such that $f_1 = (I - T^*T)^{1/2}(I - wT)^{-1}\phi$ and

$$f_2 = (I - TT^*)^{1/2}(I - wT^*)^{-1}\psi.$$

$\|\tilde{T}f\| = \|f\|$ if and only if $f(0) = 0$. But

$$f(0) = (I - T^*T)^{1/2}\phi - T^*(I - TT^*)^{1/2}\psi = (I - T^*T)^{1/2}(\phi - T^*\psi).$$

Hence $\|\tilde{T}f\| = \|f\|$ whenever $\phi = T^*\psi$.

Thus V is the scalar type of operator discussed in the earlier sections of this paper.

We shall conclude by calculating explicitly the relationship between $\tilde{T}$ and T. This will enable us to write down a nonnormal T' such that $U_{T'}$ is scalar for any $T \in \Theta$, $\|T\| < 1$. If $T \in \Theta$ is normal, T' will be a nonnormal 2-normal operator.

By Theorem 3 we may assume $\dim \mathscr{K}$ is infinite. Let $\mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_2$ where $\dim \mathscr{K} = \dim \mathscr{K}_1 = \dim \mathscr{K}_2$. Let E_i be an isometry from $\mathscr{K}$ onto $\mathscr{K}_i$. Thus if $\phi \in \mathscr{K}$, we have

(11) $$\phi = E_1\phi_1 \oplus E_2\phi_2 \quad \text{for unique} \quad \phi_1, \phi_2 \in \mathscr{K}.$$

Define B sending $\mathscr{K}$ onto $H^2_{\mathscr{K}} \ominus VH^2_{\mathscr{K}}$ by

(12) $$B\phi = (I - T^*T)^{1/2}(I - wT)^{-1}\phi_1 \oplus V_T(w)(I - TT^*)^{1/2}(I - wT^*)^{-1}\phi_2$$

where ϕ_1, ϕ_2 are defined by (11). The sum in (12) is the same orthogonal sum as in (10). It is clear from (11) and (12) that B is a one-to-one continuous linear transformation. From (10) we get that B is an isometry. Following **[8]** we define $\hat{T}$ as follows. If $\phi \in \mathscr{K}$, then $B\phi \in H^2_{\mathscr{K}} \ominus VH^2_{\mathscr{K}}$. Thus $S^*B\phi = \tilde{T}B\phi \in H^2_{\mathscr{K}} \ominus VH^2_{\mathscr{K}}$. Hence there is a $\psi \in \mathscr{K}$ such that $\tilde{T}B\phi = B\psi$. Define $\hat{T}\phi = \psi$. We have then that $\tilde{T}B = B\hat{T}$ and $\hat{T}$ is unitarily equivalent to $\tilde{T}$. Recall that $D =$

$(I - T^*T)^{1/2}$. Let $D_* = (I - TT^*)^{1/2}$. Take $\phi = E_1\phi_1$ so that $B\phi = D(I - wT)^{-1}\phi_1$. Then $\tilde{T}B\phi = D(I - wT)^{-1}T\phi_1 = B\psi$ where $\psi = E_1 T\phi_1$. Thus $\hat{T}E_1\phi = E_1T\phi_1$. That is, $\mathscr{k}_1$ is an invariant subspace for $\hat{T}$ and $\tilde{T}$ restricted to $\mathscr{k}_1$ is unitarily equivalent to T on $\mathscr{k}$.

Relative to the decomposition $\mathscr{k} = \mathscr{k}_1 \oplus \mathscr{k}_2$ we have

$$\hat{T} = \begin{bmatrix} E_1TE_1^* & X \\ 0 & Y \end{bmatrix}.$$

We now determine the two terms X, Y. Take $\phi = E_2\phi_2$ so that

$$B\phi = V_T(w)D_*(I - wT^*)^{-1}\phi_2.$$

Then

$$\begin{aligned}\tilde{T}B\phi &= S^*\{V_T(w)D_*(wT^*(I - wT^*)^{-1} + I)\}\phi_2 \\ &= S^*\{V_T(w)D_*\phi_2\} + V_T(w)D_*(I - wT^*)^{-1}T^*\phi_2.\end{aligned}$$

Notice that $V_T(w)D_*(I - wT^*)^{-1}T^*\phi_2 \in V_TH^2_{\mathscr{k}} \ominus VH^2_{\mathscr{k}}$. We shall show that $S^*\{V_T(w)D_*\phi_2\} \in H^2_{\mathscr{k}} \ominus V_TH^2_{\mathscr{k}}$. Since $\|T\| < 1$, we have

$$\begin{aligned}S^*\{V_T(w)D_*\phi_2\} &= S^*\{D^{-1}(wI - T^*)(I - wT)^{-1}D_*D_*\phi_2\} \\ &= S^*\{D^{-1}[w(I - wT)^{-1} - T^*(I - wT)^{-1}]D_*^2\phi_2\} \\ &= D^{-1}[(I - wT)^{-1} - T^*T(I - wT)^{-1}]D_*^2\phi_2 \\ &= D^{-1}(I - T^*T)(I - wT)^{-1}D_*^2\phi_2 \\ &= D(I - wT)^{-1}D_*^2\phi_2\end{aligned}$$

which is in $H^2_{\mathscr{k}} \ominus V_TH^2_{\mathscr{k}}$. Thus

$$\hat{T}E_2\phi_2 = E_1(I - TT^*)\phi_2 + E_2T^*\phi_2,$$

or

$$\hat{T} = \begin{bmatrix} E_1TE_1^* & E_1(I - TT^*)E_2^* \\ 0 & E_2T^*E_2^* \end{bmatrix}.$$

We have the following theorem.

THEOREM 5. *Suppose that* $[T^*T, T + T^*] = 0$, $\|T\| < 1$, *and let* $V = V_TV_{T^*}$. *Then* V *is scalar. Furthermore* S^* *restricted to* $H^2_{\mathscr{k}} \ominus VH^2_{\mathscr{k}}$ *is unitarily equivalent to*

$$T' = \begin{bmatrix} T & (I - TT^*) \\ 0 & T^* \end{bmatrix}$$

acting in $\mathscr{k} \oplus \mathscr{k}$. *The operator* T' *is such that* $\|T'\| = 1$, $1 \in \sigma_p(T'^*T')$, $r(T') < 1$, *and* $U_{T'}$ *is scalar.*

Proof. The only part of the theorem that needs to be verified is that $U_{T'}$ is scalar. But from the proof of Theorem 2 of **[1]** it is clear that if we write $V = cV_{X^T}U_1$ as in (5), then X is unitarily equivalent to T'. Thus $U_{T'}$ is scalar by Corollary 1.

Unfortunately T' does not help answer the question of whether $T \in \Theta$ implies T is subnormal. Neither T' nor T'^* are even hyponormal since $\|T'\| > r(T')$.

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