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RELATIONS BETWEEN PACKING AND COVERING NUMBERS OF A TREE

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Let P_k denote the size of the largest subset of nodes of a tree T with n nodes such that the distance between any two nodes in the subset is at least $k+1$; let C_k denote the size of the smallest subset of nodes of T such that every node of T is at distance at most k from some node in the subset. We determine various relations involving P_k and C_k; in particular, we show that $P_k + kC_k \leqq n$ if $n \geqq k+1$ and that $P_{2k} = C_k$.

1. Introduction. The *distance* between nodes x and y in a graph G is the number $d(x, y)$ of edges in any shortest path in G that joins x and y. (For definitions not given here see [1] or [5].) A subset $\mathscr{P}$ of nodes of G is a *k-packing* if $d(x, y) > k$ for all pairs of distinct nodes x and y of $\mathscr{P}$; the *k-packing number* of G is the number $P_k = P_k(G)$ of nodes in any largest k-packing in G. A subset $\mathscr{C}$ of nodes of G is a *k-covering* if for every node x in G there is at least one node y in $\mathscr{C}$ such that $d(x, y) \leqq k$; the *k-covering number* of G is the number $C_k = C_k(G)$ of nodes in any smallest k-covering of G.

Our object here is to establish various relations between $P_k(T)$ and $C_k(T)$ when T is a tree with n nodes. We consider the case $k = 1$ in §2 and determine those values of α and β for which there exists a tree T such that $P_1(T) = \alpha$ and $C_1(T) = \beta$. We derive upper bounds for $P_k(T)$ and $C_k(T)$ in §3. In §4 we show that $P_k(T) + kC_k(T) \leqq n$ for any tree T with n nodes when $n \geqq k + 1$ and we show that this inequality is, in a sense, best possible. Finally, in §5 we show that $P_{2k} = C_k$.

The quantities $P_1(G)$ and $C_1(G)$ have been considered before under different names. For example, $P_1(G)$ and $C_1(G)$ are called the independence number and the domination number of G in [5; Chap. 13]; and they are called the coefficients of internal and external stability in [1; Chap. 4]. Some inequalities for $P_1(G)$ and $C_1(G)$ are given in [2; Chaps. 13 and 14] but some of these are unnecessarily weak when G is a tree.

2. Relations between P_1 and C_1. In what follows T will always denote an arbitrary tree with n nodes. For convenience, we shall frequently write P and C for $P_1(T)$ and $C_1(T)$.

THEOREM 1. *If* $n \geqq 2$, *then* $P + C \leqq n$.

Proof. If $\mathscr{P}$ denotes a 1-packing of P nodes in T then each node of $\mathscr{P}$ must be joined to at least one node not in $\mathscr{P}$ if $n \geqq 2$. Thus the $n - P$ nodes not in $\mathscr{P}$ constitute a 1-covering of T. Hence, $C \leqq n - P$, as required.

COROLLARY 1. *If* $n \geqq 2$, *then* $1 \leqq C \leqq (1/2)n \leqq P \leqq n - 1$.

Proof. It is obvious that $C \geqq 1$ and $P \leqq n - 1$ when $n \geqq 2$. The remaining inequalities follow from Theorem 1 and the easily established fact that $C \leqq P$ (see [5; p. 211]); they may also be proved directly by observing that the sets of nodes of T whose distances from a given node x are odd or even, respectively, are both 1-packings and 1-coverings. We remark that the inequalities $C_1(G) \leqq (1/2)n \leqq P_1(G)$ hold for any nontrivial connected bipartite graph G with n nodes.

THEOREM 2. *If* $n \geqq 1$, *then* $P + 2C \geqq n + 1$.

Proof. Let $\mathscr{C}$ denote a 1-covering of C nodes of T and let R denote the subgraph determined by the $n - C$ nodes not in $\mathscr{C}$. If R has j components and e edges then $e = n - C - j$ (see [5; p. 68]) and it is easy to see that $P \geqq j$. Since each node of R is joined to at least one node of $\mathscr{C}$ and since T has $n - 1$ edges altogether it follows that

$$e \leqq (n - 1) - (n - C) = C - 1 .$$

Hence,

$$P \geqq j = n - C - e \geqq (n - C) - (C - 1) = n - 2C + 1 ,$$

as required. (It will follow from Theorem 3 that the inequalities $1/2(n + 1 - P) \leqq C \leqq n - P$, implied by Theorems 1 and 2 are, in a sense, best possible.)

The next result is obtained by combining the inequalities $P \geqq (1/2)n$ and $P + 2C \geqq n + 1$.

COROLLARY 2. *If* $n \geqq 1$ *and* $0 \leqq \lambda \leqq 2$, *then*

$$P + \lambda C \geqq \frac{1}{2}\Big(1 + \frac{1}{2}\lambda\Big)n + \frac{1}{2}\lambda ;$$

in particular,

$$P + C \geqq \left\{\frac{3}{4}n + \frac{1}{2}\right\} ,$$

where $\{x\}$ *denotes the least integer not less than* x.

It is not difficult to construct trees for which equality holds in the last inequality. We remark that it follows from results in [3] and [4] that the average value of $P + C$ over the n^{n-2} trees with n labelled nodes is approximately $.927n$ for large values of n.

THEOREM 3. *If* α *and* β *are positive integers such that*

$$\alpha \geqq \frac{1}{2}n\,, \tag{1}$$

$$\alpha + \beta \leqq n\,, \tag{2}$$

and

$$\alpha + 2\beta \geqq n + 1\,, \tag{3}$$

then there exists a tree T *with* n *nodes such that* $P(T) = \alpha$ *and* $C(T) = \beta$.

Proof. Let $\nu = n - \alpha - \beta$. It follows from (1) that $\beta + \nu \leqq (1/2)n$ and this implies that $n + 1 - 2\beta - 2\nu \geqq 1$; furthermore, it follows from (3) that $\nu \leqq \beta - 1$ or $\beta - 1 - \nu \geqq 0$. Let T denote the tree constructed as follows: $n - 1$ nodes are split into ν sets of four nodes, $\beta - 1 - \nu$ sets of two nodes, and $n + 1 - 2\nu - 2\beta$ sets consisting of a single node; a path is formed on the nodes in each set and the node at one end of each of these paths is joined to an nth node. (The tree arising when $n = 13$, $\alpha = 7$, and $\beta = 4$ is illustrated in Figure 1.) It is not difficult to verify that this construction

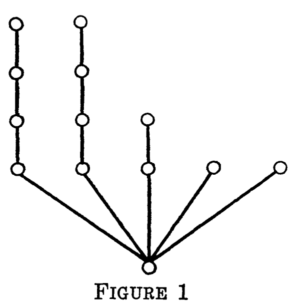

FIGURE 1

is indeed possible and that

$$P(T) = 2\nu + (\beta - 1 - \nu) + (n + 1 - 2\nu - 2\beta) = n - \beta - \nu = \alpha$$

and

$$C(T) = 1 + \nu + (\beta - 1 - \nu) = \beta\,,$$

as required.

3. Upper bounds for P_k and C_k. In what follows k and n will denote arbitrary positive integers.

THEOREM 4. *If* $n \geqq [1/2(k+3)]$ *then*

$$P_k \leqq [2n/(k+2)] \tag{4}$$

if k is even, and

$$P_k \leqq [(2n-2)/(k+1)] \tag{5}$$

if k is odd.

Proof. If x is any node in any k-packing $\mathscr{P}$ with P_k nodes of T, let $N(x) = \{u : u \in T \text{ and } d(x, u) \leqq j\}$ where $j = [(1/2)k]$. Since the tree T is connected and has at least $[1/2(k+3)] \geqq j+1$ nodes, it follows that $|N(x)| \geqq j+1$ for all $x \in \mathscr{P}$. Furthermore, the sets $\{N(x) : x \in \mathscr{P}\}$ are disjoint; for if $u \in N(x) \cap N(y)$ where $x \neq y$, then $d(x, y) = d(x, u) + d(u, y) \leqq 2j \leqq k$ and this would contradict the definition of $\mathscr{P}$. Hence,

$$n \geqq \sum_{x \in \mathscr{P}} |N(x)| \geqq P_k \cdot (j+1)$$

and this implies inequality (4).

If $k = 2j+1$ we may further assert that no edge joins a node u of any set $N(x)$ to a node v of any other set $N(y)$ where $x \neq y$; for if there were such an edge, then $d(x, y) = d(x, u) + d(u, v) + d(v, y) \leqq 2j+1 = k$ and this would again contradict the definition of $\mathscr{P}$. If $P_k = 1$ inequality (5) certainly holds. If $P_k \geqq 2$ there must exist at least one node of T that is not in any set $N(x)$, where $x \in \mathscr{P}$, for T would not be connected otherwise. Hence,

$$n \geqq 1 + \sum_{x \in \mathscr{P}} |N(x)| \geqq 1 + P_k \cdot (j+1)$$

when $k = 2j+1$, and this implies inequality (5).

If $\mathscr{P}$ is any maximal k-packing of P_k nodes in a tree T, then $\mathscr{P}$ is also a k-covering of T; for if there were a node x in T such that $d(x, y) > k$ for each node y in $\mathscr{P}$ then $\mathscr{P} \cup \{x\}$ would be a larger k-packing in T which is impossible. This implies that $C_k \leqq P_k$ for any tree T (this result is given in [5; p. 211] when $k = 1$, as was mentioned earlier). Hence, Theorem 4 provides an upper bound for C_k also; a better bound is given in the following result.

THEOREM 5. *If* $n \geqq k+1$, *then*

$$C_k \leqq [n/(k+1)] \; . \tag{6}$$

Proof. Suppose one of the longest paths in T joins nodes x and y. Let

$$D_i = \{u: u \in T \quad \text{and} \quad d(x, u) \equiv i \,(\text{mod}\,(k+1))\}$$

for $0 \leqq i \leqq k$. We may assume $D_i \neq \emptyset$ for each i, for otherwise the node x itself would constitute a k-covering and inequality (6) would certainly hold. We now show that each set D_i is a k-covering of T.

Let z denote any node of T and suppose $d(x, z) = l$. If $l \geqq i$ then $i + m(k+1) \leqq l < i + (m+1)(k+1)$ for some nonnegative integer m. Let u denote the unique node on the path joining x and z such that $d(x, u) = i + m(k+1)$; then $u \in D_i$ and $d(u, z) \leqq k$ as required. If $l < i$ let v denote the unique node on the path joining x and y such that $d(x, v) = i$; then $v \in D_i$ and

$$d(z, v) = d(z, y) - d(v, y) \leqq d(x, y) - d(v, y) = d(x, v) = i \leqq k \; ,$$

as required.

The k-coverings $\{D_i: 0 \leqq i \leqq k\}$ are disjoint and together they exhaust the nodes of T; hence, at least one of them has at most $[n/(k+1)]$ nodes. This suffices to complete the proof of the theorem.

It is not difficult to construct trees for which equality holds in (4), (5), and (6) for all admissible values of k and n.

4. A relation between P_k and C_k.

THEOREM 6. *If* $n \geqq k+1$, *then*

$$P_k + kC_k \leqq n \; . \tag{7}$$

Proof. If $k = 1$ this is the same as Theorem 1, so we shall assume henceforth that $k \geqq 2$.

Let $\mathscr{P}$ denote a k-packing of P_k nodes in T. If $x \in \mathscr{P}$ let $E(x) = \{u: u \in T \text{ and } d(u, x) = 1\}$; these sets are nonempty and disjoint when $k \geqq 2$ and no edge joins two nodes of the same set $E(x)$. Select one node u_x from each set $E(x)$ and let R denote the graph obtained from T as follows: remove each node x of $\mathscr{P}$ and all edges incident with x, and insert new edges joining each node u_x to each of the other nodes of $E(x)$. It is not difficult to see that R is a tree with $n - P_k$ nodes.

If r and s are nodes in $E(x)$ and $E(y)$, respectively, where $x \neq$

y, then $d(r, s) \geqq k - 1$; for, if $d(r, s) \leqq k - 2$ then $d(x, y) \leqq k$ and this would contradict the definition of $\mathscr{P}$. This implies the following observation:

(*) If a path in R of length at most $k - 1$ contains a new edge of the type ru_x where $r \in E(x)$, then the path does not contain any nodes of any other set $E(y)$ where $y \neq x$.

Let $\mathscr{C}$ denote any smallest $(k-1)$-covering of R. We shall show that the nodes of $\mathscr{C}$ constitute a k-covering of T. Let z denote any node of T. If $z \notin \mathscr{P}$ then $z \in R$ and there exists a node $v \in \mathscr{C}$ such that $d(v, z) \leqq k - 1$ in R. If there are no new edges in the path $p(v, z)$ from v to z in R then all the edges of $p(v, z)$ are in T and $d(v, z) \leqq k - 1$ in T also. If there is just one new edge in the path $p(v, z)$ of the type ru_x where $r \in E(x)$, then $d(v, z) \leqq k$ in T since the edge ru_x can be replaced by the two edges rx and xu_x in T. If there is more than one new edge in the path $p(v, z)$ then these new edges must all join pairs of nodes from the same set $E(x)$, in view of observation (*). But all new edges of this type are incident with the node u_x. Hence, there can be only two such edges in $p(v, z)$, they must occur consecutively, and they must be of the form ru_x and $u_x s$. But then $d(v, z) \leqq k - 1$ in T also since the edges ru_x and $u_x s$ in $p(v, z)$ can be replaced by the edges rx and xs in T.

If $z \in \mathscr{P}$ then then there exist nodes $r \in E(z)$ and $v \in \mathscr{C}$ such that $d(v, r) \leqq k - 1$ in R and the path $p(v, r)$ from v to r does not pass through any other nodes of $E(z)$. This path cannot contain any new edges by observation (*). Hence, $d(v, z) = d(v, r) + 1 \leqq k$ in T, as required.

If $n = k + 1$ then $C_k = P_k = 1$ and inequality (7) certainly holds. If $n \geqq k + 2$, it follows from Theorem 4 that $n - P_k \geqq k$. Hence, when $n \geqq k + 2$, we may apply Theorem 5 to the tree R and conclude that $|\mathscr{C}| \leqq (n - P_k)/k$. Since $C_k \leqq |\mathscr{C}|$, this implies that $P_k + kC_k \leqq n$, as required.

We now show that inequality (7) is best possible when $n = m(k + 1)$ for $m = 1, 2, \cdots$. Let H denote the tree with n nodes constructed as follows: the n nodes are split into m sets of $k + 1$ nodes each; a path of length k is formed on the nodes in each set; and, finally, the nodes at one end of these paths are joined so as to form a path of length $m - 1$. (The tree H arising when $n = 20$ and $k = 3$ is illustrated in Figure 2.) It is not difficult to verify that $P_k + kC_k = m + km = n$ for the tree H. We leave it as an exercise for the reader to show that there exists a tree with n nodes for which $C_k =$

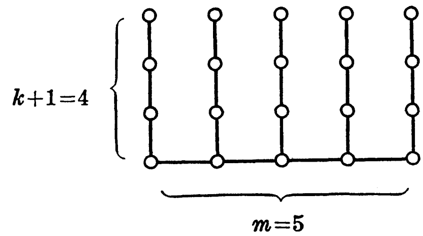

FIGURE 2

$[(n - P_k)/k]$ for arbitrary values of n and k such that $n \geqq k + 1$.

No inequality of the type

$$(1 + \varepsilon)P_k + (k - \varepsilon)C_k \leqq n \ ,$$

where ε is any positive constant, can be valid for all trees with sufficiently many nodes. To show this let J denote the tree with $n = m(k + 1) + 1$ nodes formed by joining a new node to one of the nodes of H in the way illustrated in Figure 3 when $n = 21$ and $k = 3$. It is easy to see that

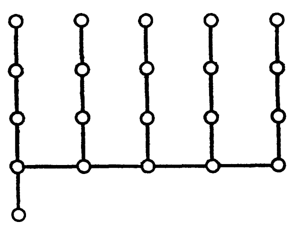

FIGURE 3

$$(1 + \varepsilon)P_k + (k - \varepsilon)C_k = (1 + \varepsilon)(m + 1) + (k - \varepsilon)m = n + \varepsilon$$

for the tree T. It might be of some interest to determine best possible upper bounds in terms of n for $lP_k + (k + 1 - l)C_k$ when $l > 1$.

It might also be of some interest to determine best possible upper bounds in term of n for $P_k + C_k$. It follows from Theorems 4 and 6 that $P_k + C_k \leqq 3n/(k + 2)$ when k is even, but this is probably not best possible in general.

There does not seem to be any natural nontrivial analogue of Theorem 2 when $k \geqq 2$, at least one that does not involve additional parameters or assumptions, since it is easy to construct trees for which $P_k = C_k = 1$ when $k \geqq 2$.

5. The equality of P_{2k} and C_k.

THEOREM 7. *If* $k \geqq 1$, *then* $P_{2k} = C_k$.

Proof. Let $\mathscr{P}$ denote a $2k$-packing consisting of P_{2k} nodes of the tree T and let $\mathscr{C}$ denote a k-covering consisting of C_k nodes of T. It is easy to see that for each node x in $\mathscr{C}$ the set $N(x) = \{u\colon u \in T \text{ and } d(x, u) \leqq k\}$ contains at most one node y in $\mathscr{P}$. Since every node y in $\mathscr{P}$ belongs to at least one set $N(x)$ it follows that $P_{2k} \leqq C_k$. It remains to show that $P_{2k} \geqq C_k$.

Let $\{x_0, x_1, \cdots, x_m\}$ denote the nodes of any longest path in the tree T. If $m \leqq 2k$, then $P_{2k} = C_k = 1$; so we may suppose that $m \geqq 2k + 1$. Let T' denote the smallest subtree of T containing all nodes z of T such that $d(x_k, z) > k$; that is, T' is the subtree determined by all nodes v of T such that either $d(x_k, v) > k$ or there exists some node, say z_v, such that $d(x_k, z_v) > k$ and the unique path joing z_v and x_m in T contains v. The subtree T' is nonempty since $d(x_k, x_m) > k$.

Let $\mathscr{P}'$ denote a largest $2k$-packing consisting of P'_{2k} nodes of T' and let $\mathscr{C}'$ denote a smallest k-covering consisting of C'_k nodes of T'. It is easy to see that $\mathscr{C} = \mathscr{C}' \cup \{x_k\}$ is a k-covering of T; consequently,

$$C_k \leqq C'_k + 1\,. \tag{8}$$

Suppose there exists a node y in $\mathscr{P}'$ such that $d(x_k, y) \leqq k$. Let B_y denote the subtree of T' determined by all nodes s of T' such that the unique path from s to x_m contains y; in particular, the node z_y, defined earlier, is in B_y. We assert that y is the only node of $\mathscr{P}'$ in B_y. For, if there were a second such node, say w, then $d(w, y) \geqq 2k + 1$; this would imply that

$$\begin{aligned} d(w, x_m) = d(w, y) + d(y, x_m) &\geqq 2k + 1 + d(x_k, x_m) - d(y, x_k) \\ &\geqq 2k + 1 + m - k - k = m + 1\,, \end{aligned}$$

contradicting the assumption that $\{x_0, x_1, \cdots, x_m\}$ was a longest path in T.

The foregoing observations imply that we may replace each node y in $\mathscr{P}'$ for which $d(x_k, y) \leqq k$ by a node z_y in T' for which $d(x_k, z_y) > k$ and still have a $2k$-packing. We may thus suppose, without loss of generality, that $d(x_k, y) > k$ for every node y in $\mathscr{P}'$; this implies that $d(x_0, y) > 2k$ for every node y in $\mathscr{P}'$. Thus the set $\mathscr{P}' \cup \{x_0\}$ is a $2k$-packing of T and, consequently,

$$P_{2k} \geqq P'_{2k} + 1\,. \tag{9}$$

The tree T' has fewer nodes than T so we may assume, as our induction hypothesis, that

$$P'_{2k} \geqq C'_k . \tag{10}$$

It now follows, by inequalities (8), (9), and (10) that $P_{2k} \geqq C_k$, as required, and this completes the proof of the theorem.

Theorems 6 and 7 imply the following result.

COROLLARY 3. *If* $n \geqq k + 1$, *then*

$$P_k + kP_{2k} \leqq n ;$$

if $n > 2k + 1$, *then*

$$C_k + 2kC_{2k} \leqq n .$$

We remark that in general these packing and covering sets are not identical; in particular, for arbitrary k it is easy to construct a tree none of whose largest $2k$-packings are smallest k-coverings. Furthermore, trees are not the only graphs G with the property that $P_{2k}(G) = C_k(G)$ for all k. For example, any graph with a node joined to all the remaining nodes has this property. It seems difficult to characterize such graphs in general.

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