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ON THE GEOMETRY OF NUMERICAL RANGES

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A bounded convex set G in the plane is the numerical range of an operator on a separable Hilbert space if $G\backslash G^0$ is a countable union of arcs of conic sections and singletons. This result answers, in particular, a question raised by Joel Anderson.

0. Introduction. Throughout this paper T denotes a (bounded linear) operator on a separable complex Hilber space H. The numerical range of T, denoted by $W(T)$, is defined as the set of all complex numbers $(Tx \mid x)$ with $\|x\| = 1$. The problem of determining all (bounded and convex) subsets of the plane which are attainable as numerical ranges seems to be hard. In fact, the question of attainability is still unsettled for some very simple sets.

Joel Anderson has made several (still unpublished) contributions to the subject; they include the following elegant result in the case where H has finite dimension n: If $W(T)$ is contained in the closed unit disk and if it intersects the unit circle in more than n points, then it coincides with the unit disk. Anderson asked whether the assertion would be true if n were replaced by $\aleph_0$. He asked, in particular, whether the closed half-disk is attainable as the numerical range of an operator T; he proved that it is not if T is compact. See [4]. We shall show, as a corollary of a more general result, that the closed half-disk is attainable by a rank-one perturbation of a hermitian operator.

It should be noted, parhaps, that if the space has dimension $\geqq 2^{\aleph_0}$ then every bounded convex subset of the plane is the numerical range of a normal operator [5]. The fact that this is not true in the separable case is well known and easily demonstrated by a cardinality argument. In fact, if G is any convex open set whose closure contains uncountably many extreme points, then there exist convex sets with interior G which are not attainable. No concrete examples of non-attainable bounded convex sets are known to us. It is proved in this paper that a convex set G is the numerical range of an operator T if $G\backslash G^0 = E_0 \cup E_1$, where E_0 is countable and E_1 is a union of quadratic arcs.

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1. Preliminaries. Let G be a convex set in the plane and let λ be in the boundary of G. A line L is called a *support* of G at λ

if $\lambda \in L$ and G lies in one of the closed half-planes determined by L. A point $\lambda \in G$ is called an *extreme point* of G if λ does not belong to any open line segment lying in $W(T)$. An extreme point λ of G is called a *sharp point* of G if G has more than one support at λ. (We use the notations G^- and G^0 for the closure and the interior of a set G, respectively.)

We will make use of the following Theorems $A - E$. Most of them are known; we have included the references wherever possible and have provided proofs otherwise.

THEOREM A. *The numerical range $W(T)$ of T is convex and $W(aT + b) = aW(T) + b$ for all complex numbers a and b. Moreover if P is an arbitrary nonzero projection, then $W(PT \mid PH) \subseteqq W(T)$.* (See [3, Problem 166] and the references cited there.)

THEOREM B. *Let λ be a sharp point of $W(T)$. Then $T = S \oplus \lambda I$, where $\lambda \notin W(S)$s.* (See [1, Theorem 1].)

THEOREM C. *Let T be the direct sum of any family $\{T_\alpha\}_{\alpha \in J}$ of operators. Then $W(T) = \mathrm{Co}(\bigcup_{\alpha \in J} W(T_\alpha))$.* (Here $\mathrm{Co}(G)$ denotes the convex hull of a set G.)

The proof follows from the fact that if G is a convex set and if $\{\lambda_i\}$ is a sequence in G, then $\sum \alpha_i \lambda_i \in G$ for all sequences $\{\alpha_i\}$ of nonnegative numbers satisfying $\sum \alpha_i = 1$.

The following theorem is a combination of the results of [2] and [6]. We leave the proof to the reader.

THEOREM D. *Let T be hyponormal and let L be a support of $W(T)$. Then the set*

$$M = \{x \in H\colon (Tx \mid x) = \lambda \,\|\, x \,\|^2 \quad \textit{for some} \quad \lambda \in L\}$$

is a reducing (trivial or nontrivial) *invariant subspace of T and $W(T \mid M) \subseteqq L$. In particular $W(T)\backslash W(T)^0 = E_0 \cup E_1$, where E_0 is a countable set and E_1 is a conntable union of* (not necessarily closed) *line segments. Moreover if $W(T)\backslash W(T)^0 \neq \varnothing$, then T has a nontrivial normal part.*

The following theorem can be regarded as a converse to Theorem D.

THEOREM E. *Let G be a convex set with countably many extreme points. Then there exists a normal operator T such that $W(T) = G$.*

Proof. It follows from the assumption on G that

$$G \backslash G^0 = \bigcup_{\alpha \in J} E_\alpha ,$$

where J is a (possibly empty) countable index set and where E_α is either a singleton or a line segment $(\alpha \in J)$. Let T_α be a normal operator with numerical range E_α and let M_z be the multiplication by z in $L^2(G, dxdy)$. In view of Theorems C and D, $W(M_z \oplus \sum_{\alpha \in J} \oplus T_\alpha) = G$. (Note that the numerical range of M_z is G^0.)

2. **Main results.** In what follows a *quadratic arc* means a (not necessarily closed) smooth subarc of a conic section.

THEOREM 1. *Let G be any closed region whose boundary consists of a closed line segment L and a closed quadratic arc C. Then there exists an operator T with the following properties*:

(i) *T is irreducible,*

(ii) *$T = A + B$ where $W(A) \subsetneqq L$ and B is a normal operator of rank* 1,

(iii) $W(T) = G \backslash (L \cap C)$.

Proof. In view of Theorem A we can assume, without loss of generality, that L is the closed real interval $[-2, 2]$ and C lies in the upper half plane. Let (α, β) be the point on C at which the tangent to C is parallel to the x-axis. It is easy to verify that the equation of an arbitrary conic section having the above properties is given by the formula

$$\beta x^2 + b\beta y^2 - 2\alpha xy + (4 + \alpha^2 - b\beta^2)y - 4\beta = 0 , \tag{1}$$

where

$$b > (\alpha^2 - 4)/\beta^2 .$$

(The latter inequality follows from the fact that β is the maximum ordinate on the arc C.)

Let $\gamma = (1/2)(b\beta^2 - \alpha^2 + 4)^{1/2}$. Let H be a Hilbert space with an orthonormal basis $\{e_n\}_{1 \leqq n < \infty}$. Define an operator T as follows:

$$Te_1 = (\alpha + \beta i)e_1 + \gamma e_2 , \quad Te_2 = \gamma e_1 + e_3 ,$$

and

$$Te_n = e_{n-1} + e_{n+1} \quad \text{for} \quad n = 3, 4, \cdots .$$

We claim that T has the required properties (i)-(iii). The first two properties are easy; we prove only the last one. For $t \in [0, 1]$, let

M_t be the set of all unit vectors u in H such that $|(u \mid e_1)|^2 = t$. Let S be the weighted shift defined by $Se_1 = \gamma e_2$ and $Se_n = e_{n+1}$ for $n = 2, 3, \cdots$. Let $\Delta_t = \{2Re(Su \mid u): u \in M_t\}$. Computation shows that the points of $W(T)^-$ are determined by the following relations.

$$\begin{cases} \alpha t + \inf \Delta_t \leqq x \leqq \alpha t + \sup \Delta_t \,, \\ y = \beta t \,, \quad 0 \leqq t \leqq 1 \,. \end{cases} \tag{2}$$

Since $|(Su \mid u)|^2 \leqq (1-t)(\gamma^2 t - t + 1)$, it follows that $W(T)^-$ is a subset of all points (x, y) such that

$$\begin{cases} \beta x^2 + b\beta y^2 - 2\alpha xy + (4 + \alpha^2 - b\beta^2)y - 4\beta \leqq 0 \,, \\ \text{and} \qquad 0 \leqq y \leqq \beta \,. \end{cases}$$

For $t \neq 0$, let $\lambda = \pm[(1-t)/(1-t+\gamma^2 t)]^{1/2}$ and $u = t^{1/2}(e_1 + \lambda\gamma e_2 + \cdots + \lambda^{n-1}\gamma e_n + \cdots)$. It is easy to see that $\|u\| = 1$ and $(Su \mid u) = \pm[(1-t)(\gamma^2 t - t + 1)]^{1/2}$. Thus the arc

$$\begin{cases} \beta x^2 + b\beta y^2 - 2\alpha xy + (4 + \alpha^2 - b\beta^2)y - 4\beta = 0 \,, \\ \text{and} \qquad 0 < y \leqq \beta \end{cases}$$

lies in $W(T)$.

Let P be the projection onto the closed span of $\{e_3, e_4, \cdots\}$. Since $(1/2)PT \mid PH$ is the real part of a simple shift, $W(T) \supseteqq W(PT \mid PH) = (-2, 2)$. Therefore $W(T) = G\backslash\{-2, 2\}$ because $W(T)$ cannot have any sharp point (see Theorem B).

COROLLARY 1. *Let G be a convex set such that $G\backslash G^0 = E_0 \cup E_1$ where E_0 is a countable set and E_1 is a union of quadratic arcs. Then G is the numerical range of an operator.*

The proof is an easy consequence of Theorems C and 1 and the proof of Theorem E.

3. Remarks on the topology of $W(T)$. (a) Although $W(T)$ may not be closed, there always exists an operator S similar to T with $W(S)$ closed. The proof follows from the existence [7, Theorem 3] of an operator F of rank 2 such that $1 + F$ is invertible and

$$W((1+F)^{-1}T(1+F)) \supseteqq W_e(T) \,,$$

where $W_e(T) = \bigcap \{W(T+K)^-: K \text{ compact}\}$. Now let $S = (1 + F)^{-1}T(1+F)$. Since $W(S) \supseteqq W_e(S)$, it follows from [4, Theorem 1] that $W(S)$ is closed.

(b) Let $U = \{x \in H: \|x\| = 1\}$ and let $f(x) = (Tx \mid x)$ for $x \in U$. Obviously f is continuous in norm and $W(T)\backslash W(T)^0$ is the image of the closed set $f^{-1}(\partial W(T))$ under f. This shows that $W(T)\backslash W(T)^0$ is an analytic set. The following question seems to be open: is $W(T)\backslash W(T)^0$ a Borel set? In view of [4, Theorem 1] the answer is in the affirmative if $W_e(T)$ has a countable number of extreme points.

(c) Corallary 1 suggests another (probably open) question: is every compact convex set the numerical range of some operator on H? The corresponding question on open convex sets is easily answered in Theorem E.

REFERENCES

1. W. F. Donoghue, *On the numerical range of a bounded operater*, Michigan Math. J., **4** (1957), 261-263.
2. M. R. Embry, *the numerical range of an operator*, Pacific J. Math., **32** (1970), 647-650.
3. P. R. Halmos, *A. Hilbert Space Problem Book*, Van Nostrand, Princeton (1967).
4. J. S. Lancaster, *The boundary of the numerical range*, Proc. Amer. Math. Soc., **49** (1975), 393-398.
5. F. M. Pollack, *Numerical range and convex sets*, Canad. Math. Bull., **17** (1974), 295-296.
6. J. G. Stampfli, *Extreme points of the numerical range of hyponormal operators*, Michigan Math. J., **13** (1966), 87-89.
7. J. P. Williams, *Similarity and the numerical range*, J. Math. Anal. Appl., **36** (1969), 307-314.

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