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ON STARSHAPED SETS AND HELLY-TYPE THEOREMS

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Suppose an ordered pair of sets (S, K) in a linear topological space is of Helly type $(n+1, n)$, i.e., for every $n+1$ distinct points in S there is a point in K which sees at least n of them via S. Then if S is closed, K compact, and $n \geqq 3$, the nontrivial visibility sets in K are pairwise nondisjoint. Sufficient conditions are obtained for S to be starshaped.

Let S be a subset of a linear topological space L. For points x, y in S, we say x *sees* y *via* S if and only if the segment $[x, y]$ lies in S. Further, the set S is said to be *starshaped* if and only if there is some point p in S such that, for every x in S, p sees x via S.

If S and K are subsets of L, with every point x in S is associated its *visibility set* $K(x)$, the set of all points of K which x sees via S.

We shall say (S, K) *has Helly-type* (s, r), where r and s are positive integers, $r \leqq s$, if for every s distinct points in S there is a point on K seeing at least r of them via S. Clearly, if (S, K) has Helly-type (s, r), and $0 \leqq i \leqq r-1$, then (S, K) has Helly-type $(s-i, r-i)$.

In this paper we obtain a solution to a problem posed by Valentine, concerning sets of Helly type which are unions of a finite number of starshaped sets [**3**, Prob. 6.7, p. 178], and also obtain some related results. Breen [**1**] has given conditions in the plane for a simply connected set to be a union of two starshaped sets. We replace simple connectedness by the following:

For S and K subsets of a linear topological space L, we shall say the ordered pair (S, K) has the *triangle property* if the interior of every triangle having an edge on K and the other edges in S is itself a subset of S.

If S is a closed subset of a linear topological space L, K is a compact convex subset of L of dimension k and (S, K) has the triangle property, then $K(x)$ is compact and convex for each $x \in S$. If (S, K) is of Helly type (r, r), for $r \geqq k+1$, then by Helly's theorem $\cap \{K(x)\colon x \in S\} \neq \emptyset$, and S is starshaped. However, it is possible under certain conditions to weaken the hypothesis considerably, and yet reach the same conclusion.

A collection of sets $\mathcal{K}$ is said to have "*piercing number*" j or a j-*partition* for a positive integer j, if $\mathcal{K}$ can be represented as a union of j collections, each with a nonvoid intersection.

The classical result on j-partitions is a theorem by H. Hadwiger and H. DeBrunner [**2**], which for convenience we state here as Theorem 1.

THEOREM 1. *For integers r, s and n, let $J(s, r, n)$ denote the smallest integer (if one exists) for which a j-partition is admitted by each family $\mathcal{K}$ of compact convex sets in R^n which has the (s, r) property, i.e., for every s members of $\mathcal{K}$, some r have a common point. Then $J(s, r, n) = s - r + 1$ whenever $r \leqq s$ and $nr \geqq (n-1)s + (n+1)$.*

REMARKS. When $j = 1$ and $r = n + 1$, Theorem 1 reduces to Helly's theorem.

If S is a closed subset of a linear topological space, K a compact convex subset of S of dimension n, such that (S, K) has the triangle property and is of Helly type (s, r), then for every $x \in S$, $K(x)$ is compact and convex, and the collection $\{K(x): x \in S\}$ has the (s, r) property.

Therefore, if $J(s, r, n) = j$, then the set S can be expressed as a union of j starshaped sets. However, for choices of s, r and n as small as $s = 4$, $r = 3$, $n = 2$, it is not known whether $J(4, 3, 2)$ exists.

If $n = 1$, then Theorem 1 implies that $J(s, r, 1) = s - r + 1$ if $r \geqq 2$, so that $J(s, 2, 1) = s - 1$. Consequently S will be the union of $s - 1$ starshaped sets if (S, K) has Helly-type $(s, 2)$ and K is a compact line segment. Also, since $J(r+1, r, 1) = 2$ for all $r \geqq 2$, $J(3, 2, 1) = J(4, 3, 1) = 2$. Consequently if (S, K) has Helly-type $(3, 2)$ or $(4, 3)$, where K is a compact line segment, then S is the union of two starshaped sets. Breen **[1]** proved this result for Helly-type $(3, 2)$ without the assumption that $K(x) \neq \emptyset$ for all x in S. We improve the $(4, 3)$ case by showing S will be starshaped. In fact, in Theorem 4, we obtain the more general result that if (S, K) is of Helly type $(2k+2, 2k+1)$ in a linear topological space, and K is of dimension k, then with a single exception S is starshaped. This result improves the prediction, from $J(2k+2, 2k+1, k) = 2$, that S would be a union of two starshaped sets. In Theorems 2 and 3, for (S, K) of Helly type $(n+1, n)$, without restrictions on dimension, sufficient conditions are obtained for the visibility sets to be pairwise nondisjoint (2), or for S to be starshaped (3).

We must first prove the following lemma.

LEMMA. *Let S and K be a closed and a compact subset, respectively, of a linear topological space L. If there exist x, w in S such that $K(x) \cap K(w) = \emptyset$ and $p \in K(x)$, $q \in K(w)$, then there exist t_0, τ_0 in $(0, 1)$ such that if $|t| < t_0$, $|\tau| < \tau_0$, then $K(y(t)) \cap K(z(\tau)) = \emptyset$, where $y(t) = tp + (1-t)x$, and $z(y) = \tau q + (1-\tau)w$.*

Proof. We first observe that for every x in S, $K(x)$ is compact: recall $K(x) = \{p \in K \cap S \mid [p, x] \subset S\}$. Let p be a limit point of $K(x)$. Select a sequence $\{p_n\}$ such that $p_n \in K(x)$ for every n and $p_n \to p$. For each n, the line segment $[p_n, x]$ is contained in S. By closure of S, $[p, x] \subset S$ and by closure of K, $p \in K$. Therefore $p \in$

$K(x)$. So $K(x)$ is a closed subset of a compact set and consequently compact.

Since $K(w)$ and $K(x)$ are compact and disjoint, there are disjoint open neighborhoods U, U' in L, such that $K(x) \subset U$ and $K(w) \subset U'$.

We wish to prove the existence of $t_0 > 0$ such that $0 < t < t_0$ implies $K(y(t)) \subset U$. Since t_0 exists trivially if $K(x) = \{x\}$, we may assume $K(x) \neq \{x\}$.

Assume no such t_0 exists. Then we can find a sequence of real numbers $\{t_n\}$, $t_n \to 0$ as $n \to \infty$, and a corresponding sequence of points $\{\alpha_n\}$ in $K \sim U$, such that for every n, $\alpha_n \in K(y(t_n))$.

By compactness of $K \sim U$, there is a point $\alpha_0 \in K \sim U$ and a subsequence of $\{\alpha_n\}$, called for convenience $\{a_i\}$, such that $\alpha_i \to \alpha_0$ as $i \to \infty$. Now for each i, $\alpha_i \in K(y(t_i))$, so the line segment from $y(t_i)$ to α_i is in S. By closure of S, the limiting line segment from x to α_0 is also in S. Therefore x sees α_0, contradicting the hypothesis, since α_0, not being in U, is clearly not in $K(x)$.

The same argument implies the existence of $\tau_0 > 0$ such that for $0 < \tau < \tau_0$, $K(z(\tau)) \subset U'$. We therefore conclude that for t, τ sufficiently small, $K(y(t)) \cap K(z(\tau)) = \emptyset$.

THEOREM 2. *Let S and K be, respectively, a closed and a compact subset of a linear topological space L, such that (S, K) is of Helly type $(n+1, n)$ for some $n \geqq 3$. Let $\mathcal{K} = \{K(x)\colon x \in S, K(x) \not\subset \{x\}\}$. Then $\mathcal{K}$ is pairwise nondisjoint.*

Proof. Suppose $\mathcal{K}$ fails to be pairwise nondisjoint and let $K(x)$ and $K(w)$ be members of $\mathcal{K}$ such that $K(x) \cap K(w) = \emptyset$. There exist neighborhoods U, U' such that $K(x) \subset U$, $K(w) \subset U'$, and $U \cap U' = \emptyset$. As in the proof of the lemma, select $p \in K(x)$, $p \neq x$, $q \in K(w)$, $q \neq w$, and then y on (x, p), z on (w, q) such that $K(y) \subset U$, $K(z) \subset U'$. There is no point in K seeing three of the four points x, y, w, z. Expanding the set $\{x, y, w, z\}$ if necessary, we have a contradiction of the hypothesis of Helly type $(n+1, n)$ for all $n \geqq 3$. Therefore $\mathcal{K}$ is pairwise nondisjoint.

A special case of Theorem 2 is of sufficient interest to be stated separately.

THEOREM 3. *Let S and K be a closed and a compact subset respectively, of a linear topological space L, such that (S, K) is of Helly type $(n+1, n)$ for some $n \geqq 3$. Let us further assume that for some $x_0 \in S$, $K(x_0) = \{p\}$, $p \neq x_0$. Then either S is starshaped relative to p or S is the union of an isolated point and a set starshaped relative to p.*

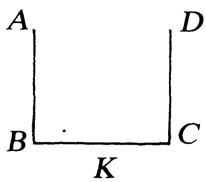
Proof. Suppose y_1 and y_2 are points in $S \sim \{p\}$, such that $K(y_i) \subset \{y_i\}$, $i = 1, 2$. The set $\{x_0, y_1, y_2\}$, suitably expanded, lacks the

$(n+1, n)$ property, since y_1 and y_2 do not see p, and x_0 sees neither y_1 nor y_2. Therefore there is at most one point y in $S \sim \{p\}$ such that $K(y) \subset \{y\}$.

We then have, by Theorem 2, that at most one point in S does not see p. Furthermore, any such point must be isolated, by the closure of S.

REMARK. It is possible for a point x_0 to be the only point with singleton visibility set. Consider the following example: Let $S = \{(x, y) \in R^2 \mid y \leqq x^2, 0 \leqq x \leqq 1, 0 \leqq y \leqq 1\}$, and $K = \{(1, y) \mid 0 \leqq y \leqq 1\}$. Let $x_0 = (0, 0)$. Then $K(x_0) = \{(1, 0)\}$. It is easily seen that (S, K) satisfies the hypothesis of Theorem 3, and that $(0, 0)$ is the only point with the required property.

REMARK. Theorems 2 and 3 do not hold when (S, K) is of Helly type (3.2). An example is shown below.



S is all of broken line segment $\overline{ABCD}$, in the plane, and $K = \overline{BC}$.

REMARK. Theorems 2 and 3 trivially fail if the hypothesis lacks the condition that $K(x) \neq \emptyset$, for every $x \in S$.

REMARK. Let S and K be subsets of a linear topological space L, such that (S, K) is of Helly type (3, 2). If there exist points $x, z \in S$ such that $K(x) = \{a\}$, $K(z) = \{b, c\}$, a, b, c, distinct, then S is a union of three starshaped sets, since an arbitrary w in S sees at least one of $\{a, b, c\}$ via S. As we see by Breen's example **[1]**, even with the restriction that S is a closed subset of the plane and K is a line segment we may need as many as three points to write S as a union of starshaped sets.

THEOREM 4. *Let S be a closed subset of a linear topological space L, and let K be a compact convex subset of S of finite dimension k. Suppose (S, K) has the triangle property and is of Helly type $(2k+2, 2k+1)$. Then S is the union of a starshaped set and at most one isolated point.*

Proof. Since the theorem is trivially true for $k = 0$, we assume $k > 0$. For arbitrary $x \in S$, $K(x)$ is compact, as was shown in the Lemma, and is also convex.

Suppose $K(x) \neq \emptyset$ for every $x \in S$. If, for arbitrary $\{x_i : x_i \in S, i = 1, 2, \cdots, k+1\}$, the set $\bigcap_{i=1}^{k+1} K(x_i) \neq \emptyset$, then Helly's theorem implies $\bigcap_{x \in S} K(x) \neq \emptyset$, so S is starshaped. Assume S is not starshaped. Then let j be a minimal integer such that $\bigcap_{i=1}^{j} K(x_i) = \emptyset$ for some collection

$\{x_i: x_i \in S, i = 1, 2, \cdots, j\}$. Then $j \geqq 2$ since $K(x) \neq \emptyset$ for all x, and $j \leqq k+1$ by assumption.

Consider $(S \sim \{x_1, \cdots, x_j\}, K)$. This pair is of Helly type $(2k+2-j, 2k+2-j)$: for given an arbitrary collection of $2k+2-j$ points from $S \sim \{x_1, \cdots, x_j\}$, augment the collection with $\{x_1, \cdots, x_j\}$, making a total of $2k+2$ points of S. By hypothesis at least $2k+1$ of these points must see a point of K in common. One point from the $2k+2$ points in S must fail to see the point in K, in fact, a point from the set $\{x_1, \cdots, x_j\}$ since otherwise the assumption that $\bigcap_{i=1}^{j} K(x_i) = \emptyset$ would be violated. Therefore all of the $2k+2-j$ points from $S \sim \{x_1, \cdots, x_j\}$ see the point in question.

Since $j \leqq k+1$, it follows that $2k+2-j \geqq k+1$, so the pair $(S \sim \{x_1, \cdots, x_j\}, K)$ is of Helly type $(k+1, k+1)$ as well, and consequently, by Helly's theorem $S \sim \{x_1, \cdots, x_j\}$ is starshaped. Then the closure of $S \sim \{x_1, \cdots, x_j\}$ is also starshaped. Our assumption that S is not starshaped implies that there is an integer i, $1 \leqq i \leqq j$, such that x_i is not in the closure of $S \sim \{x_1, \cdots, x_j\}$. Therefore x_i has a neighborhood containing no points of $S \sim \{x_1, \cdots, x_j\}$, and sees no points of K via S, which contradicts that $K(x_i) \neq \emptyset$. Therefore S is starshaped.

On the other hand, suppose for some $x_0 \in S$, $K(x_0) = \emptyset$. Then x_0 is the only point of S with empty visibility set, and $(S \sim \{x_0\}, K)$ is of Helly type $(2k+1, 2k+1)$. By Helly's theorem, the collection $\{K(y): y \in S \sim \{x_0\}\}$ has a nonvoid intersection, so $S \sim \{x_0\}$ is starshaped. S consists of the starshaped set $S \sim \{x_0\}$ and the point $\{x_0\}$. Closure of S implies that x_0 is isolated.

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