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UNIQUENESS THEOREMS FOR TAUT SUBMANIFOLDS

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1. Introduction and statements of theorems. Given two closed smooth manifolds, how do you tell if they are diffeomorphic? If you start out with a homotopy equivalence, Browder–Novikoff Theory breaks the problem up into: (1) finding all self-equivalences, (2) finding a normal bordism, and (3) the surgery obstruction on the normal bordism. In applications, however, one may encounter manifolds suspected of being diffeomorphic, where no obvious homotopy equivalence is present.

We describe such a situation: Let $\overset{\zeta}{\underset{M}{|}}$ be a simply connected, finite, simplicial complex with linear bundle. Let $K_i^{2n} \xrightarrow{f_i} M$, $i = 0$ or 1, $n \geqq 3$, be normal maps from closed smooth manifolds, i.e. $f_i^*(\zeta) = \nu(K_i)$. Suppose that f_1 and f_2 are normally bordant, f_i is n-connected, and that $B_n(K_0) = B_n(K_1)$. B_n here denotes the n-th Betti number. It follows from Poincaré's Duality and the universal coefficient theorem that K_0 and K_1 have isomorphic integral homology groups, but a map inducing this isomorphism is lacking. However,

THEOREM 1. *If n is odd, K_0 and K_1 are diffeomorphic.*

THEOREM 2. *If n is even, but not 2, and the intersection pairings on*

$$(\operatorname{Ker} f_0\colon H_n(K_0; Z) \to H_n(M; Z))/\mathit{torsion} \qquad \mathit{and}$$
$$(\operatorname{Ker} f_1\colon H_n(K_1; Z) \to H_n(M; Z))/\mathit{torsion}$$

are isometric and nonsingular, then K_0 and K_1 are diffeomorphic.

COROLLARY 1. *If M^{2n+2} is a compact, simply connected, smooth $2n+2$-manifold, n odd, and $K_0^{2n} \xrightarrow{i_0} M^{2n+2}$ and $K_1^{2n} \xrightarrow{i_1} M^{2n+2}$ are n-connected inclusions of closed submanifolds with $i_{0*}[K_0] = i_{1*}[K_1] \in H_{2n}(M^{2n+2}; Z)$, then if $B_n(K_0) = B_n(K_1)$, K_0 is diffeomorphic to K_1.*

COROLLARY 2. *Assume M^{2n+2} is a simply connected smooth $2n+2$-manifold, n even ($n \neq 2$), with $H_n(M; Z) = 0$. If i_0 and i_1 are as above, then if the intersection pairings on $H_n(K_0; Z)/\mathit{torsion}$ and $H_n(K_1; Z)/\mathit{torsion}$ are isometric, K_1 is diffeomorphic to K_2.*

REMARK 1. The above corollaries are specialized by replacing the connectivity assumptions with the assumption that the submanifolds are taut (definition: π_i (M-neighborhood (K), ∂) $= 0$, $i \leqq n$).

REMARK 2. It follows from Corollary 1 and [2] that if n is odd and $\pi_1(M^{2n+2}) = 0$, every homology class $X \in H_{2n}(M^{2n+2}; Z)$ is represented by a simplest submanifold with an n-connected inclusion map, K^0, and every other submanifold with an n-connected inclusion map is diffeomorphic to $K^t = K^0 \underbrace{\# S^n \times S^n \times \# \cdots \# S^n \times S^n}_{t\text{-copies}}$, $t \geqq 0$.

THEOREM 3. *If* $\pi_1(M^{2n+2}) = 0$, *and if* K_0 *and* K_1 *are submanifolds representing* $X \in H_{2n}(M^{2n+2}; Z)$ *with* n*-connected inclusion maps, then there are integers* j, k *such that* $K_0 \underbrace{\# S^n \times S^n \# \cdots \# S^n \times S^n}_{j\text{-copies}}$ *is diffeomorphic to* $K_1 \underbrace{\# S^n \times S^n \# \cdots \# S^n \times S^n}_{k\text{-copies}}$.

2. Proofs. (All coefficients will be integral unless stated.) *Thm* 1 $\Rightarrow$ *Cor* 1 *and Thm* 2 $\Rightarrow$ *Cor* 2: Let N be a very large integer. i: $CP^{N-1} \rightarrow CP^N$ is an isomorphism on $H_2(\;\;; Z)$ so there is a unique 2-plane bundle $\begin{matrix}\zeta\\ |\\ CP^N\end{matrix}$ extending $\nu_{CP^{N-1} \rightarrow CP^N}$. Since $i_{1*}[K_1] = i_{2*}[K_2]$, there is a homotopy h: $M \times I \rightarrow CP^N$ with $h_0^{-1}(CP^{N-1}) = K_0$ and $h_1^{-1}(CP^{N-1}) = K_1$. $h^*(\zeta)$ extends $\nu_{K_0 \times 0 \rightarrow M \times 0}$ and $\nu_{K_1 \times 1 \rightarrow M \times 1}$. Furthermore, the inclusions $K_0 \times 0 \rightarrow \begin{matrix}\nu(M) \oplus h^*(\zeta)\\ |\\ M \times 0\end{matrix}$ and $K_1 \times 1 \rightarrow \begin{matrix}\nu(M) \oplus h^*(\zeta)\\ |\\ M \times 1\end{matrix}$ are normal maps and if h is transverse to CP^{N-1}, $h^{-1}(CP^{N-1})$ provides a normal bordism. This together with the hypothesis of Corollary 1 (or Corollary 2) is the hypothesis of Theorem 1 (or Theorem 2). Corollary 1 (or Corollary 2) follows.

Proof of Theorem 1. Let L, $\partial L = K_0 \cup -K_1$, be a normal bordism. By Theorem 1.2 [4] we may assume f: $L \rightarrow M$ is n-connected. It follows that $H_*(L, K_0) = 0$ for $* \neq n, n+1$. Our plan is to perform some normal n-surgery on interior (L) to produce L' with $H_n(L', K_0) = 0$ (this will leave $H_*(L, K_0)$ unchanged $* \leqq n-1$). By duality and U.C.T. $H_*(L', K_0) = 0$, and $L \overset{\text{diff}}{\cong} K_0 \times I \Rightarrow K_1 \overset{\text{diff}}{\cong} K_0$ by the h-cobordism theorem.

Let i_0: $K_0 \rightarrow L$ and i_1: $K_1 \rightarrow L$ be the inclusions. Consider the following commutative diagram:

$$\begin{tikzcd}
 & & 0 & & & \\
 & 0 \arrow[r] & \dfrac{H_{n+1}(M,L)}{i_{0*}(H_{n+1}(M,K_0))} \arrow[r, "\cong"] \arrow[u] & H_n(L,K_0) \arrow[r] & 0 & \\
H_{n+1}(L) \arrow[r, "f_{n+1}"] & H_{n+1}(M) \arrow[r] \arrow[u] & H_{n+1}(M,L) \arrow[r] \arrow[u] & H_n(L) \arrow[r] & H_n(M) \arrow[r, "f_n"] \arrow[u] & 0 \\
H_{n+1}(K_0) \arrow[r, "f_{0_{n+1}}"] \arrow[u, "i_{0*}"] & H_{n+1}(M) \arrow[r] \arrow[u, "id_*"] & H_{n+1}(M,K_0) \arrow[r] \arrow[u, "i_{0*}"] & H_n(K_0) \arrow[r, "f_{0_n}"] \arrow[u, "i_{0*}"] & H_n(M) \arrow[r] \arrow[u, "id_*"] & 0
\end{tikzcd}$$

Since f_0 factors through i_0, $\operatorname{coker}(f_{0_{n+1}}) \xrightarrow{\text{id}_*} \operatorname{coker}(f_{n+1})$ is epi. There is a short exact sequence

$$0 \to \underset{\substack{\cong \\ 0}}{\operatorname{coker}(f_{n+1})/\operatorname{coker}(f_{0_{n+1}})} \to \frac{H_{n+1}(M,L)}{i_{0*}H_{n+1}(M,K_0)} \to \operatorname{Ker} f_{n/_{i_{0*}(\operatorname{Ker} f_{0_n})}} \to 0$$

so there is a natural isomorphism

$$\operatorname{Ker} f_{n/_{i_{0*}(\operatorname{Ker} f_{0_n})}} \cong H_n(L,K_0).$$

We will consider the two modules identified.

Duality and the U.C.T. show $H_n(L,K_0)$ and $H_n(L,K_1)$ are isomorphic. So we have a noncommutative diagram:

$$\begin{tikzcd}
 & \ker f_{n/_{i_{0*}(\ker f_{0_n}}} = C_0 \\
A = \ker f_n \arrow[ur, "\text{epi}\ \pi_0"] \arrow[dr, "\text{epi}\ \pi_1"'] & \cong \\
 & \ker f_{n/_{i_{1*}(\ker f_{0_n}}} = C_1
\end{tikzcd}$$

We need an algebraic fact about such diagrams.

LEMMA 1. *If F is a field, there are elements $a_1, \cdots, a_k \in A$, such that $\{\pi_\epsilon(a_1) \otimes 1, \cdots, \pi_\epsilon(a_k) \otimes 1\}$ is a basis for $C_\epsilon \otimes_Z F$, $\epsilon = 0$ or 1.*

Proof. By induction. $\dim_F(C_0 \otimes_Z F) = \dim_F(C_1 \otimes_Z F)$. Suppose $a_1 \cdots a_j$ are already chosen so that $\pi_\epsilon(a_1) \otimes 1, \cdots, \pi_\epsilon(a_j) \otimes 1$ are independent. Let the spans of these vectors be $\text{span}_{j,0}$ and $\text{span}_{j,1}$. If these spans are proper subspaces of $C_0 \otimes_Z F$ and $C_1 \otimes_Z f$, then $\pi_\epsilon^{-1}(\text{span}_{j,\epsilon})$ are proper subspaces of $A \otimes_Z F$. Let a_{j+1} be any element of A such that $a_{j+1} \otimes 1 \in (A \otimes_Z F - (\bigcup_{\epsilon=0,1} \pi_\epsilon^{-1}(\text{span}_{j,\epsilon}))) \neq \varnothing$.

In order to compute the effect on $H_n(L,K_0)$ of normal n-surgeries along L, we use a diagram adapted from Lemma IV.3.2 **[1]**. (At this point it may be helpful for the reader to review the proof that the odd dimensional simply connected surgery obstruction vanishes, see Ch. IV, §3 of **[1]**. This argument is originally due to J. Milnor and M. Kervaire, see **[3]**.)

LEMMA 2. *If L' results from a normal surgery on $S^n \hookrightarrow L$, $[S^n] = x \in \ker f_n$, we have the following diagram*:

(*)

$$\begin{tikzcd}
 & & H_{n+1}(L', K_0) \arrow[d] & & \\
 & & Z \in \lambda' \arrow[d, "d'"] & & \\
H_{n+1}(L, K_0) \arrow[r, "\pi_1(x)\cdot", "\theta"'] & Z \arrow[r, "d"] & H_n(L_0, K_0) \arrow[r, "i_*"] \arrow[d, "i'_*"] & H_n(L, K_0) \arrow[r] & 0 \\
 & & H_n(L', K_0) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

with: (1) $\pi_0(x) = x_0 = i_* d(\lambda')$, *where* λ' *is the appropriate generator above.* (2) *The map* $\theta: H_{n+1}(L, K_0) \to Z$ *is given by intersection with* $\pi_1(x) \in H_n(L, K_1)$.

Proof. (1) See Lemma IV.3.2 **[1]**.

(2) $\theta: H_{n+1}(L, K_0) \to Z$ may be thought of as:

$$\begin{tikzcd}
H_{n+1}(L/K_0) \arrow[r, "j"] \arrow[rr, bend right, "\bar{j}"'] & H_{n+1}(L/K_0, L_0/K_0) \arrow[r, "\mathrm{exc}^{-1}"] & H_{n+1}(S^n \times D^{n+1}, S^n \times S^n) \arrow[r, "U(\ )"] & Z
\end{tikzcd}$$

where the last map is evaluation of the Thom class $U \in H^{n+1}(S^n \times D^{n+1}, S^n \times S^n)$. Let $a \in H_{n+1}(L/K_0)$. $j^*(\mathrm{exc}^{-1} U)(a) = U(\bar{j}_* a) = \theta(a)$.

$$\begin{tikzcd}
H^{n+1}(L, K_0) \arrow[r, "j^*"] \arrow[dd, "\cap [L, K_0 K_1]"', "\cap [L, K_0 \cup K_1]"] & H^{n+1}(L, L_0) \arrow[d] & & & H^{n+1}(S^n \times D^{n+1}, S^n \times \arrow[lll, "\mathrm{exc}^{-1}"'] \arrow[dd, "\cap [S^n \times D^{n+1}, S^n \times S^n]"'] \arrow[dll, bend right] \\
 & H^{n+1}(L, K_0 \cup K_1) \arrow[d] & \ni x & U \in \arrow[l, bend left] & \\
H_n(L, K_1) & H_n(L) \arrow[l, "p_1"'] & & & H_n(S^n \times \arrow[lll]
\end{tikzcd}$$

$j^*(\mathrm{exc}^{-1} U) \cap [L, K_0 \cup K_1] = \pi_1(x) \in H_n(L, K_1)$, so

$$\theta(a) = j^*(\mathrm{exc}^{-1} U)(a) = \pi_1(x) \cdot (a),$$

where · denotes algebraic intersection.

Lemma 3. *If we set* $y = i'_* d(\lambda)$, *then*:

(1) $H_n(L', K_0)/\langle y \rangle \cong H_n(L, K_0)/\langle x_0 \rangle$, $\langle\ \rangle$ *means "subgroup generated by".*

(2) *Suppose* (x_0) *has finite order s, then for some* $t \in Z$ $sd'(\lambda') + td(\lambda) = 0$. *Order* $(y) = \infty$ *iff* $t = 0$. *Order* $y = t$ *iff* $t \neq 0$.

(3) *If n is even (as in Theorem 2),* $t = 0$.

(4) *If n is odd (as in Theorem 1), L′ is not uniquely determined, and may be varied to* L'_m *by changing the framing of the surgery along x by m times the generator of* $\ker(\pi_n(SO(n+1)) \to \pi_n(SO)) \cong Z$. *So t is a function of* L'_m. *We have the relation*: $t(L'_m) = t(L') + 2ms$.

Proof. These facts correspond to Lemmas 3.2, 3.5, 3.8 and 3.11 respectively in Chapter IV of **[1]**.

Lemma 4. $L^{2n} \to \overset{\xi}{\underset{M}{\mid}}$ *is normally bordant rel* ∂ *to L′ satisfying* $H_*(L', K_0) = 0$, $* \leqq n-1$ *and* $H_n(L', K_0) =$ *torsion. (This lemma holds for n even or odd.)*

Proof. In Lemma 1 set $F = Q$, normal surgery on the classes $\{a_1, \cdots, a_k\}$ affords the desired L'. More precisely, assume $\mathrm{Rank}_Z(H_n(L, K_0)) > 0$. By setting $F = Q$ and $a_1 = x$ in Lemma 1, $\pi_0(x) = x_0$ and $\pi_1(x) = x_1$ are infinite order. Therefore, by diagram $(*)$ $\langle x_0 \rangle = Z$ and $\langle y \rangle =$ torsion. Let L' result from surgery on x. Now (1) of Lemma 3 implies $\mathrm{Rank}_Z(H_n(L', K_0)) = \mathrm{Rand}_Z(H_n(L, K_0)) - 1$. Proceed inductively.

Remark 3. It may not be possible to do the preceding construction without increasing the order of torsion $(H_n(L, K_0))$. The reason is that there are diagrams, for example:

$$\begin{tikzcd}
 & Z \\
Z + Z \arrow[ur, "4a + b", "\pi_0"'] \arrow[dr, "\pi_1", "a + 4b"'] & \\
 & Z
\end{tikzcd}$$

so that $\nexists\, x \in Z + Z$ with $\pi_\epsilon(x)$ generating a free summand. So unlike the classical case, killing the free part may increase the torsion!

Lemma 4′. *Given a prime, p,* $L^{2n} \to \overset{\xi}{\underset{M}{\mid}}$ *is normally bordant* rel ∂ *to L′ such that* $H_*(L', K_0) = 0$, $* \leqq n-1$, *and* $H_n(L', K_0) \otimes Z_p = 0$. *(This lemma holds for n even or odd.)*

Proof. Assume L satisfies the conclusion of Lemma 4. In Lemma 1 set $F = Z_p$ and $a_1 = x$. Now consider diagram $(*)$ with Z_p coefficient. The mod p reductions $(\pi_0(x))_p = (x_0)_p \in H_n(L, K_0; Z_p)$ and $(\pi_1(x))_p = (x_1)_p \in H_n(L, K_1; Z_p)$ are nonzero. d is the zero map so by (1) of Lemma 3:

$$\mathrm{Rank}_{Z_p}(H_n(L, K_0; Z_p)) = \mathrm{Rank}_{Z_p}(H_n(L', K_0); Z_p) + 1.$$

Using the U.C.T. and $H_{n+1}(L^{(')}, K_0; Z) = 0$ we have:

$$\mathrm{Rank}_{Z_p}(H_n(L, K_0; Z) \otimes Z_p) = \mathrm{Rank}_{Z_p}(H_n(L', K_0) \otimes Z_p) + 1.$$

Lemma 4′ now follows by induction on the above rank.

The argument is now restricted to n = odd. The prime 2 will play a special role.

Let L satisfy $H_*(L, K_0) = 0$, $* \leqq n-1$ and $H_n(L, K_0) \otimes Z_2 = 0$ (i.e., $H_n(L, K_0)$ is odd torsion).

LEMMA 5. *If n is odd and L is as above, and if a prime p/order $(H_n(L, K_0))$ there is a normal bordism* rel *∂L to L' (consisting of either one or two normal n-surgeries) with*: (1) $H_*(L', K_0) = 0$, $* \leqq n-1$, (2) *$H_n(L', K_0)$ odd torsion, and either*: (3) *order $H_n(L', K_0) \leqq$ order $H_n(L, K_0)$, and* (4) $\mathrm{Rank}_{Z_p}(H_n(L', K_0) \otimes Z_p) < \mathrm{Rank}_{Z_p}(H_n(L, K_0) \otimes Z_p))$ *or* (3′) *order $H_n(L', K_0) <$ order $H_n(L, K_0)$.*

Proof. In Lemma 1 set $F = Z_p$ and $a_1 = x$. As in the proof of Lemma 4′, any normal surgery based on x results in an L' with $\mathrm{Rank}_{Z_p}(H_n(L', K_0) \otimes Z_p) < \mathrm{Rank}_{Z_p}(H_n(L, K_0) \otimes Z_p)$. By (4) of Lemma 3 $\forall\, m$, $t(L'_m) = t(L') + 2ms$, where s = order $x_0 \in H_n(L, K_0)$. By (1) and (2) of Lemma 3, $t(L'_m) \neq 0$ iff $H_n(L'_m, K_0) =$ torsion, and order (torsion $H_n(L'_m, K_0)) \leqq$ order (torsion $H_n(L_m, K_0)$) iff $-s \leqq t(L'_m) \leqq s$. If (and only if) $t(L'_m) = 2s$, order $H_n(L', K_0) = 2$(order $H_n(L, K_0)$). We may choose m so that: *Case A.* $t(L'_m) \neq 0$ and $-s \leqq t(L'_m) \leqq s$ or *Case B.* $t(L'_m) = 2s$.

In each case diagram $(*)$ with Z_2 coefficients shows $\mathrm{Rank}_{Z_2}(H_n(L', K_0) \otimes Z_2) = 0$ or 1. In *Case A* we are finished if the above rank is zero; assume it is one. If $t(L'_m) = \pm s$, order $H_n(L', K_0) =$ order $H_n(L, K_0)$, so (2) and (3) above are satisfied and we are finished. So add the assumption: $-s < t(L'_m) < s$. Then order $H_n(L', K_0) <$ order $H_n(L, K_0)$. Now Lemma 1 with $F = Z_2$ provides $x' = a_1 \in \ker f'_n \colon H_n(L') \rightarrow H_n(M)$, so that if L'' is the result of a normal surgery on x' then $H_n(L'', K_0) \otimes Z_2 = 0$. $\exists\, r \in Z$ such that $-$order $x' \leqq t(L''_r) \leqq$ order x'. However, $t(L''_r) \neq 0$ as this would imply order $H(L''_r, K_0) = \infty$, contradicting $H_n(L''_r, K_0) \otimes Z_2 = 0$. So order $(H_n(L''_r, K_0)) \leqq$ order $(H_n(L'_m, K_0)) <$ order $(H_n(L, K_0))$ so (3′) above is satisfied. This completes Case A.

In Case B order $(H_n(L'_m, K_0)) = 2(\text{order } (H_n(L, K_0)))$.

LEMMA 6. *$\exists x' \in \ker f'_n$: $H_n(L'_m) \to H_n(M)$ such that $\pi_0(x')$ is the unique element of order 2, δ, in $H_n(L'_m, K_0)$ and $(\pi_1(x'))_2 \neq 0$.*

Proof. $H_n(L'_m, K_0) \cong H_n(L'_m, K_1) \cong Z_2 + \mathcal{O}$, where $\mathcal{O}$ is odd torsion. Consider compositions P_0, P_1

$$\begin{tikzcd}
 & & & 0 \\
 & & \mathcal{O} \arrow[ur] \arrow[dddd, dashed] & \\
 & H_n(L'_m, K_0) \arrow[ur] & & \\
\ker f'_n \arrow[ur, "\pi_0"] \arrow[dr, "\pi_1"'] \arrow[uurr, "P_0"] \arrow[ddrr, "P_1"] & & & \\
 & H_n(L'_m, K_1) \arrow[dr] & & \\
 & & Z_2 \arrow[dr] & \\
 & & & 0
\end{tikzcd}$$

Ker $P_0 \not\subset \ker P_1$, otherwise P_1 could be factored through P_0 by an epimorphism (dotted arrow). Let $x \in \ker P_0 - \ker P_1$. $\pi_0(x) = 0$ or δ, $(\pi_1(x))_2 \neq 0$. If $\pi_0(x) = \delta$, set $x' = x$ and we are finished. If $\pi_0(x) = 0$ let $y \in \ker f'_n$ satisfy $\pi_0(y) = \delta$. If $(\pi_1(y))_2 \neq 0$, set $x' = y$ and we are finished. If $(\pi_1(y))_2 = 0$, set $x' = x + y$. Now $\pi_0(x') = \pi_0(x + y) = 0 + \delta = \delta$ and $(\pi_1(x'))_2 = (\pi_1(x + y))_2 = (\pi_1(x))_2 + (\pi_1(y))_2 = 1 + 0 = 1$.

Let L''_r be the result of a normal surgery along x'. Again looking at Rank_{Z_2} tells us $t(L''_r) \neq 0, 2$. Therefore we can choose r so that $t(L''_r) = \pm 1$. It follows from diagram $(*)$ that $H_n(L''_r, K_0) \cong H_n(L'_m, K_0)/Z_2$. L''_r satisfies (3) above, but

$$\text{Rank}_{Z_p}(H_n(L''_r, K_0) \otimes Z_p) = \text{Rank}_{Z_p}(H_n(L'_m, K_0) < \text{Rank}_{Z_p}(H_n(L, K_0))$$

so L''_r also satisfies (4) above. This completes Case B.

$\text{Rank}_{Z_p}(H_n(L, K_0) \otimes Z_p)$ is finite so after a finite number of applications of Lemma 5, it is no longer possible to reduce the Z_p-rank without increasing order $(H_n(L, K_0))$. So we can find a normal cobordism rel ∂ of L to L' with (1), (2) and (3') satisfied. By inducting on order $(H_n(L, K_0))$, we have a normal bordism rel boundary to L' with $H_*(L', K_0) = 0$, all values of $*$. Theorem 1 now follows from the h-cobordism theorem.

Proof of Theorem 2. If n is even, surgery on a torsion class of $H_n(L, K_0)$ will always increase the $\text{rank}_Z(H_n(L, K_0))$ (see Lemma IV.3.8 of [1]). Remark 3 shows that it may be impossible to do surgery to lower $\text{Rank}_Z(H_n(L, K_0))$ without increasing order (torsion $H_n(L, K_0)$). This prevents the usual inductive argument on order torsion $H_n(L, K_0)$, and is the reason that Theorem 2 requires an additional assumption.

Let $\{\alpha_1^0, \cdots, \alpha_k^0\} \subset \ker f_0\colon\ H_n(K_0) \to H_n(M)$ be a basis for $(\ker f_0; Z)$/torsion and let $\{\alpha_1^1, \cdots, \alpha_k^1\} \subset (\ker f_1, Z)$ be the isometric image of $\{\alpha_1^0, \cdots, \alpha_k^0\}$. By the relative Hurewicz Theorem, the α's are spherical. The classes $\{(i_{0\cdot}(\alpha_1^0) - i_{1\cdot}(\alpha_1^1), \cdots, i_{0\cdot}(\alpha_k^0) - i_{1\cdot}(\alpha_k^1)\}$ are represented by framed, imbedded n-spheres, $s_1, \cdots, s_k$. Let L' be the result of a normal surgery on $\{s_1, \cdots, s_k\}$. Let h_0, h_1 denote the inclusions $K_0 \to L'$, $K_1 \to L'$.

LEMMA 7. $h_{0\cdot}(\ker f_0; Z) = h_{1\cdot}(\ker f_1; Z)$ *in the quotient* $H_n(L'; Z)$/*torsion.*

Proof. By induction on the number of surgeries. We will denote the inclusions at any stage by inc_0 and inc_1. Surgery on s_1 makes $\text{inc}_0(\alpha_1^0) = \text{inc}_1(\alpha_1^1)$. Assume that after surgery on $s_1, \cdots, s_j$, $\text{inc}_0(\alpha_i^0) = \text{inc}_1(\alpha_i^1)$, $1 \leqq i \leqq j$. Let $b_1, \cdots, b_j$ be simplicial chains with $\partial b_i = \alpha_i^0 \cup -\alpha_i^1$. Let $\pitchfork$ denote algebraic intersection. $\pitchfork_L(s_{j+1}, b_i) = \pitchfork_{K_0}(\alpha_{j+1}^0, \alpha_i^0) + \pitchfork_{K_1}(-\alpha_{j+1}^1, \alpha_i^1) = 0$ (using the isometry). As a result, $\text{inc}_0(\alpha_i^0)$ and $\text{inc}_1(\alpha_i^1)$ remain equal after surgery on s_{j+1} and $\text{inc}_0(\alpha_{j+1}^0) = \text{inc}_1(\alpha_{j+1}^1)$. The lemma follows.

Consider the following diagram with all kernels interpreted by applying the function $H_n(\ \ ; Z)$.

$$\begin{tikzcd}
 & & & 0 & \\
 & & & \ker f/h_{1\cdot}(\ker f_1) \arrow[u] \arrow[r, Leftrightarrow] & H_n(L, K_1) \arrow[dddd, phantom, "\cong"] \\
h_{1\cdot}\colon & \ker f_1 \arrow[dr] & & & \\
 & & \ker f \arrow[uur, "\pi_1"] \arrow[ddr, "\pi_0"] & & \\
h_{0\cdot}\colon & \ker f_0 \arrow[ur] & & & \\
 & & & \ker f/h_{0\cdot}(\ker f_0) \arrow[d] \arrow[r, Leftrightarrow] & H_n(L, K_0) \\
 & & & 0 &
\end{tikzcd}$$

Lemma 7 implies $h_1(\ker f_1) = h_0(\ker f_0)$ in the quotient $(\ker f)$/torsion.

Assuming the hypothesis of Theorem 2, we have:

LEMMA 8. *Suppose* $h_{1\cdot}(\ker f_1) = h_{0\cdot}(\ker f_0)$ *in the quotient* $(\ker f)$/*torsion. If* $\text{Rank}_Z H_n(L, K_0) > 0 \ \exists\ x \in \ker f$ *such that*

(1) $\pi_\epsilon(x) = x_\epsilon$ *generates a free summand of* $H_n(L, K_\epsilon)$, *for* $\epsilon = 0, 1$, *and*

(2) *If* L' *is the result of surgery on* x, $h_{1*}(\ker f_1) = h_{0*}(\ker f_0)$ *in the quotient* $(\ker f')$/*torsion.*

(3) $H_n(L, K_0) \cong H_n(L', K_0) \oplus Z$.

Proof. Let $y \in H_n(L, K_0)$ generate a free summand. Let $x' \in \ker f$ satisfy $\pi_0(x') = y$. $\pi_1(x')$ generates a free summand of $H_n(L, K_1)$ (Proof: Consider the above diagram modulo torsion.). Let b_i, $1 \leqq i \leqq k$, be as before. Let $a_i = \pitchfork(x', b_i)$. Since $(\ker f_{0*}, \pitchfork)$ is nonsingular, $\exists \beta \in \ker f_0$ such that $\pitchfork_{K_0}(\beta, \alpha_i) = -a_i \quad \forall \quad 1 \leqq i \leqq k$. Set $x = x' + i_0(\beta)$. $\pitchfork(x', b_i) = 0 \, \forall \, b_i$. (2) now follows. It is easy to check (3) using diagram $(*)$ of Lemma 2. (Compare with Remark 3.)

Let $(L'; K_0, K_1)$ be a normal bordism satisfying the conclusion of Lemma 7. By virtue of Lemma 8, we may do surgery to eliminate $H_n(L', K_0)$. The argument for this coincides with "the proof of Theorem IV.2.1 for $m = 2q + 1$, q even", page 104, **[1]**. As before, L' becomes an h-cobordism completing the proof of Theorem 2.

REMARK 4. If $n = 2$, this argument may be used to give another proof of: homotopy equivalence $\not\Rightarrow$ h-cobordism for simply connected 4-manifolds, see Wall, **[4]**.

The Proof of Theorem 3. We may again assume $H_*(L, K_0) = 0$, $* \neq n, n+1$. L may be described as $K_0 \times I \cup n$, $n+1$-handles. Let D_j^+ be the cores of the n-handles. Disks $D_j^- \to K_0 \times I$ may be chosen so that $f_\#[D_j^+ \cup D_j^-] = 0 \in \pi_n(M)$ as $f_\#\colon \pi_{n-1}(K_0) \to \pi_{n-1}(M)$ is an isomorphism and $f_\#\colon \pi_n(K_0) \to \pi_n(M)$ is surjective. Let H be the level set of L after the n-handles have been attached. The preceding statement implies that H is obtained from K_0 by a sequence of surgeries on $(n-1)$-spheres, ∂D_j^-, trivialized by some null-homotopy (D_j^-) in $K_0 \times I$. By general position $\{\partial D_j^-\}$ is isotopically trivial (we check $(n-1) + 1 < \frac{1}{2}(2n+1)$). Therefore, $H \overset{\text{diff}}{\cong} K_0 \#_j (S^n \times S^n)_j$. By turning L upside down, we obtain $H \overset{\text{diff}}{\cong} K \#_k (S^n \times S^n)_k$. This proves Theorem 3.

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