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WEAK TYPE MULTIPLIERS FOR HANKEL TRANSFORMS

WILLIAM C. CONNETT AND ALAN L. SCHWARTZ

The main result of this paper is that weak type multiplier theorems for Jacobi expansions yield weak type multiplier theorems for Hankel transforms.

In recent papers the authors studied multiplier theorems for ultraspherical and Jacobi expansions. An interesting paper of Igari suggested a new approach to multiplier theorems that used asymptotics instead of the elaborate machinery used earlier.

This paper extends the method of Igari to give the first weak type multiplier theorem for Hankel transforms. This extension is important in itself and because this method together with the authors' results for Jacobi multipliers will generalize to the "radial" functions associated with the other compact symmetric spaces.

Let $\{P_n^{(\alpha,\beta)}(x)\}$ be the Jacobi polynomials with indices (α, β). The functions $\{P_n^{(\alpha,\beta)}(\cos\theta)\}$ are orthogonal with respect to the measure $d\mu(\theta) = (\sin\theta/2)^{2\alpha+1}(\cos\theta/2)^{2\beta+1}d\theta$. For measurable f on $[0, \pi]$ define

$$||f||_p = \left\{\int_0^\pi |f(\theta)|^p d\mu(\theta)\right\}^{1/p}$$

and

$$f\hat{\ }(n) = \int_0^\pi f(\theta)P_n^{(\alpha,\beta)}(\cos\theta)d\mu(\theta)$$

so that if

$$h_n^{-1} = \int_0^\pi [P_n^{(\alpha,\beta)}(\cos\theta)]^2 d\mu(\theta)\ ,$$

then

$$f(\theta) \sim \sum_{n=0}^\infty f\hat{\ }(n)h_n P_n^{(\alpha,\beta)}(\cos\theta)$$

where equality holds at least for finite series.

The multiplier transformation defined by the function $\phi(x)$ is denoted by T_ϕ, where at least formally,

$$T_\phi f(\theta) \sim \sum_0^\infty \phi(n)\hat{f}(n)h_n P_n^{(\alpha,\beta)}(\cos\theta)\ .$$

The operator T_ϕ is said to be of strong type p, $1 < p < \infty$ if

$$\| T_\phi \|_p = \| \phi(n) \|_p = \sup \| T_\phi f \|_p \quad (\| f \|_p = 1) .$$

The operator is said to be of weak type p if

$$\langle\!\langle T_\phi \rangle\!\rangle_p = \langle\!\langle \phi(n) \rangle\!\rangle_p = \sup \lambda[\mu\{x: | T_\phi f(\theta) | > \lambda\}]^{1/p} \quad (\lambda > 0, \ \| f \|_p = 1) .$$

The same objects of harmonic analysis are also needed for the Hankel transforms. Let $d\eta(x) = x^{2\alpha+1}dx\mathscr{J}_\alpha(x) = x^{-\alpha}J_\alpha(x)$ and briefly define:

$$| g |_p = \left(\int_0^\infty | g(x) |^p d\eta(x)\right)^{1/p} ,$$

$$\hat{g}(x) = \int_0^\infty g(y)\mathscr{J}_\alpha(xy)d\eta(y)$$

(at least for g continuous with compact support) ,

$$g(x) \sim \int_0^\infty \hat{g}(y)\mathscr{J}_\alpha(xy)d\eta(y) ,$$

$$U_\varphi g \sim \int_0^\infty \phi(y)\hat{g}(y)\mathscr{J}_\alpha(xy)d\eta(y) ,$$

$| U_\phi |_p$ for the strong type p norm of the operator ,

$\langle U_\phi \rangle_p$ for the weak type p norm .

The idea of Igari [4] is to transform strong type multiplier theorems for Jacobi expansions into strong type multiplier theorems for Hankel transforms. His result is

THEOREM (Igari). *Let $\alpha, \beta > -1$ and assume ϕ is continuous on $(0, \infty)$ except on a null set. If $1 \leqq p < \infty$, then*

$$| \phi(x) |_p = \liminf_{\varepsilon \to 0} \| \phi(\varepsilon n) \|_p .$$

Igari's techniques are adapted here to prove a similar theorem for weak type multipliers.

THEOREM 1. *Under the same hypotheses as above*

$$\langle \phi(x) \rangle_p = \liminf_{\varepsilon \to 0} \langle\!\langle \phi(\varepsilon n) \rangle\!\rangle_p .$$

The theorem will be proved following a

LEMMA. *Let $E \subset R$ satisfy $\eta(E) < \infty$ and let $f_n(x) \to f(x)$* a.e. *in E as $n \to \infty$. Then if $s > 0$*

$$\eta\{x \in E: | f(x) | > s\} \leqq \liminf_{n \to \infty} \{x \in E: | f_n(x) | > s\} .$$

The lemma can be proved by applying Fatou's lemma to the characteristic function of $\{x \in E: | f_n(x) | > s\}$.

Proof of Theorem 1. It will be sufficient to prove the theorem for compactly supported infinitely differentiable functions, so let g be one such with support in $(0, M)$.

Let $g_\lambda(\theta) = g(\lambda\theta)$ for λ so large that $\lambda\pi > M$, and let

$$G(\tau, \lambda) = \sum_{n=0}^{\infty} \phi\left(\frac{n}{\lambda}\right) g_\lambda^\wedge(n) P_n^{(\alpha,\beta)}\left(\cos\frac{\tau}{\lambda}\right),$$

$$G^N(\tau, \lambda) = \sum_{n=0}^{N[\lambda]} \phi\left(\frac{n}{\lambda}\right) g_\lambda^\wedge(n) P_n^{(\alpha,\beta)}\left(\cos\frac{\tau}{\lambda}\right),$$

$$H^N(\tau, \lambda) = G(\tau, \lambda) - G^N(\tau, \lambda),$$

$$G(\tau) = \int_0^\infty \phi(y) g^\wedge(y) \mathscr{J}_\alpha(\tau y) d\eta(y) = U_\phi g(\tau).$$

A careful reading of Igari's argument shows that there is a constant B independent of λ, N and K such that once λ is sufficiently large

$$\int_0^K |H^N(\tau, \lambda)|^2 d\eta(\tau) \leqq B/N^2. \tag{1}$$

He shows that $G^N(\tau, \lambda)$ converges everywhere to a function $G^N(\tau)$ and that a subsequence of $G^N(\tau)$ converges a.e. in $(0, \infty)$ to $G(\tau)$.

Let $\varepsilon > 0$, $\delta > 0$, $K > 0$, and $s > 0$ and define

$$M_\lambda = \left\langle\left\langle \phi\left(\frac{n}{\lambda}\right)\right\rangle\right\rangle_p^p \|g_\lambda\|_p^p s^{-p}.$$

If $\phi(n/\lambda)$ defines a weak type p operator on Jacobi series then

$$\mu\{\varphi: |G(\lambda\varphi, \lambda)| > s\} \leqq M_\lambda.$$

Then if λ is sufficiently large

$$\eta\{\tau \leqq K: |G(\tau, \lambda)| > s\} \leqq 2^{2\alpha+1}\lambda^{2\alpha+2}M_\lambda(1 + \varepsilon).$$

Let $N^2 > B/\varepsilon\delta^2$ so that relation (1) implies

$$\eta\{\tau \leqq K: |H^N(\tau, \lambda)| > \delta\} \leqq \varepsilon,$$

and $G^N(\tau, \lambda) = G(\tau, \lambda) - H^N(\tau, \lambda)$ so that

$$\{\tau \leqq K: |G^N(\tau, \lambda)| > s + \delta\} \subset \{\tau \leqq K: |G(\tau, \lambda)| > s\} \cup \{\tau \leqq K: |G(\tau, \lambda)| \leqq s \text{ and } |H^N(\tau, \lambda)| > \delta\},$$

thus

$$\eta\{\tau \leqq K: |G^N(\tau, \lambda)| > s + \delta\} \leqq 2^{2\alpha+1}\lambda^{2\alpha+2}M_\lambda(1 + \varepsilon) + \varepsilon. \tag{2}$$

Now choose $\lambda_j \to \infty$ such that

$$\lim_{j\to\infty}\left\langle\left\langle\phi\left(\frac{n}{\lambda_j}\right)\right\rangle\right\rangle_p^p = \liminf_{\lambda\to\infty}\left\langle\left\langle\phi\left(\frac{n}{\lambda}\right)\right\rangle\right\rangle_p^p$$

and observe that $\lim_{\lambda\to\infty} 2^{\alpha+1}\lambda^{2\alpha+2}\|g_\lambda\|_p^p = |g|_p^p$ so

$$\lim 2^{2\alpha+1}\lambda_j^{2\alpha+2}M_{\lambda_j} = M = \liminf_{\lambda\to\infty}\left\langle\left\langle\left(\phi\frac{n}{\lambda}\right)\right\rangle\right\rangle_p^p |g|_p^p s^{-p} ,$$

thus if λ is replaced by λ_j in (2) and then j increased, the lemma implies

$$(3) \qquad \eta\{\tau \leqq K: |G^N(\tau)| > s + \delta\} \leqq M(1+\varepsilon) + \varepsilon .$$

If N_j is now chosen so that $G^{N_j}(\tau) \to G(\tau)$ a.e. in $(0, \infty)$, then relation (3) and the lemma yield

$$\eta\{\tau \leqq K: |G(\tau)| > s + \delta\} \leqq M(1+\varepsilon) + \varepsilon .$$

Now, first the ε, then the δ, and finally the K can be removed by standard arguments to yield

$$\eta\{\tau: |U_\phi g(\tau)| > s\} \leqq \liminf_{\lambda\to\infty}\left\langle\left\langle\phi\left(\frac{n}{\lambda}\right)\right\rangle\right\rangle_p^p |g|_p^p s^{-p}$$

and the Theorem is proved.

To transfer strong type multiplier theorems from Jacobi series to Hankel transforms Igari's theorem should be used, for weak type results Theorem 1 is the tool; for instance ($[\alpha]$ is the greatest integer not exceeding α).

THEOREM 2. *Let $\alpha \geqq \beta > -1/2$ and $m = [\alpha] + 2$; assume that ϕ is m times continuously differentiable in $(0, \infty)$ and let*

$$[A_m(\phi)]^2 = \sup_x |\phi(x)|^2 + \sup_M M^{-1}\int_M^{2M} |x^m\phi^{(m)}(x)|^2 x^{-1}dx ,$$

then U_ϕ is of weak type 1 *and*

$$\langle U_\phi\rangle \leqq CA_m(\phi)$$

where C does not depend on ϕ.

Proof. Define $\Delta^0\phi(n) = \phi(n)$ and $\Delta^k\phi(n) = \Delta^{k-1}\phi(n+1) - \Delta^k\phi(n)$ for $k = 1, 2, \cdots$ and let $\phi_\varepsilon(x) = \phi(\varepsilon x)$.

It can be shown (see Bonami and Clerc [1], Theorem 4.12) that

$$|\Delta^m\phi_\varepsilon(n)| \leqq \varepsilon^{m-1}\int_{\varepsilon n}^{\varepsilon(n+m)} |\phi^{(m)}(x)|dx .$$

Consequently by the Schwarz inequality and the Mean Value Theorem for integrals there is constant C independent of n and ϕ such that

$$\frac{1}{n}|n^m \Delta^m \phi_\varepsilon(n)|^2 \leqq m \int_{\varepsilon n}^{\varepsilon(n+m)} |x^m \phi^{(m)}(x)|^2 x^{-1} dx .$$

Thus if

$$[B_m(\phi)]^2 = \sup_n |\phi(n)|^2 + \sup_M M^{-1} \sum_{n=M}^{2M} |n^m \Delta^m \phi(n)|^2 ,$$

then there is a constant C_1 independent of ϕ such that

$$B_m(\phi_\varepsilon) \leqq C_1 A_m(\phi) . \tag{4}$$

The main result of [3] actually shows that if $\alpha \geqq \beta > -1/2$ and $m = [\alpha] + 2$ there is a constant C_2 such that

$$\langle\langle T_\phi \rangle\rangle_1 \leqq C_2 B_m(\phi) .$$

Thus the result follows from Theorem 1 and (4).

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UNIVERSITY OF MISSOURI—ST. LOUIS