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NON-HAUSDORFF MULTIFUNCTION GENERALIZATION OF THE KELLEY-MORSE ASCOLI THEOREM

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The paper generalizes the Kelley-Morse theorem to continuous point-compact multifunction context. The generalization, which is non-Hausdorff, contains the Ascoli theorem for continuous functions on a k_3-space by the authors and the known multifunction Ascoli theorems of Mancuso and of Smithson.

1. Introduction. The Kelley-Morse theorem [**3**, p. 236] is central among the topological Ascoli theorems for continuous functions on a k-space. It generalizes to the k_3-space theorem of [**1**], which contains all known Ascoli theorems for k-spaces or k_3-spaces.

Obviously a multifunction generalization depends on a multifunction extension of "even continuity". One such extension is that of Lin and Rose [**5**], but this was not applied in Kelley-Morse context. Another which was so applied [**7**, p. 24] is two-fold and leads to a two-fold multifunction Kelley-Morse theorem which, however, does not contain the Mancuso theorem [**6**, p. 470], nor the Smithson theorem [**9**, p. 259]. This paper gives a natural multifunction extension of the definition and leads to a multifunction theorem containing all the above-mentioned theorems.

2. Tychonoff sets. Let X and Y be nonempty sets. A *multifunction* is a point to set correspondence $f: X \to Y$ such that, for all $x \in X$, fx is a nonempty subset of Y. For $A \subseteq X$, $B \subseteq Y$ it is customary to write $f(A) = \bigcup_{x \in A} fx$, $f^-(B) = \{x : x \in X \text{ and } fx \cap B \neq \emptyset\}$ and $f^+(B) = \{x : x \in X \text{ and } fx \subseteq B\}$. If Y is a topological space, a multifunction $f: X \to Y$ is *point-compact* if fx is compact for all $x \in X$.

Let $\{Y_x\}_{x \in X}$ be a family of nonempty sets. The *m-product* $P\{Y_x : x \in X\}$ of the Y_x is the set of all multifunctions $f: X \to \bigcup_{x \in X} Y_x$ such that $fx \subseteq Y_x$ for all $x \in X$. In the case $Y_x = Y$ for all $x \in X$, the m-product of the Y_x, denoted Y^{mX}, is the set of all multifunctions on X to Y. In particular, if Y is a topological space, the symbol $(Y^{mX})_0$ will denote the set of all point-compact members of Y^{mX}. For $x \in X$, the *x-projection* $\mathrm{pr}_x : P\{Y_x : x \in X\} \to Y_x$ is the multifunction defined by $\mathrm{pr}_x f = fx$. If the Y_x are topological spaces, the *pointwise topology* τ_p on $P\{Y_x : x \in X\}$ is defined to be the topology having as open subbase the sets of the forms $\mathrm{pr}_x^-(U_x)$, $\mathrm{pr}_x^+(U_x)$, where U_x is open in Y_x, $x \in X$.

For $F \subseteq Y^{mX}$, $x \in X$, we write $F[x] = \bigcup_{f \in F} fx$. Let Y be a topological space. A subset F of Y^{mX} is *pointwise bounded* if $F[x]$ has compact closure in Y for all $x \in X$. A subset T of Y^{mX} is *Tychonoff* if, for every pointwise bounded subset F of T, $T \cap P\{\overline{F[x]}: x \in X\}$ is τ_p-compact. The following sets are Tychonoff:

(1) Y^X, by the classical Tychonoff theorem.

(2) Y^{mX}, by the theorem of Lin [4, p. 400].

(3) The set of all point-closed members of Y^{mX}, by Corollary 7.5 of [7, p. 17].

(4) $(Y^{mX})_0$, by Corollary 7.6 of [7, p. 17].

LEMMA 2.1. *If F is a pointwise bounded subset of a Tychonoff set T, then the τ_p-closure of F in T is compact.*

Proof. Let $\bar{F}$ denote the τ_p-closure of F in T. Since $T \cap P\{\overline{F[x]}: x \in X\}$ is a τ_p-compact subset of T, it suffices to show that $\bar{F} \subseteq P\{\overline{F[x]}: x \in X\}$. But this follows from Lemma 7.7 of [7, p. 17].

3. Even continuity. Let X and Y be topological spaces. A multifunction $f: X \to Y$ is *lower semi-continuous* (*upper semi-continuous*) if $f^-(U)(f^+(U))$ is open in X whenever U is open in Y. If f is both lower semi-continuous and upper semi-continuous it is called *continuous*. Henceforth, the set of all continuous multifunctions on X to Y will be denoted $\mathscr{C}(X, Y)$. The multifunction $(f, x) \to fx$ on $Y^{mX} \times X$ to Y, or any restriction, will be denoted by the symbol ω. Let $F \subseteq Y^{mX}$. A topology τ on F is said to be *jointly continuous* if $\omega: (F, \tau) \times X \to Y$ is continuous.

A subset F of Y^{mX} is *evenly continuous* if, whenever $x \in X$, K is a compact subset of Y and V is a neighborhood of K, there exist neighborhoods U, W of x, K, respectively, such that

(a) $f \in F$ and $fx \cap W \neq \emptyset$ imply $U \subseteq f^-(V)$, and

(b) $f \in F$ and $fx \subseteq W$ imply $U \subseteq f^+(V)$.

This extends the original Kelley-Morse definition [3, p. 235] by the substitution of compact subsets of Y for points of Y. It is easily verified that every member of an evenly continuous subset of Y^{mX} is lower semi-continuous. Moreover, every member of an evenly continuous subset of $(Y^{mX})_0$ is also upper semi-continuous, hence continuous.

LEMMA 3.1. *Let Y be a regular space. If F is an evenly continuous subset of Y^{mX}, then the τ_p-closure of F in Y^{mX} is evenly continuous.*

Proof. Let $\bar{F}$ denote the τ_p-closure of F in Y^{mX}. Let $x \in X$, let K be a compact subset of Y and let V be a closed neighborhood of K. There exist open neighborhoods U, W of x, K, respectively, such that, for all $f \in F$, $fx \cap W \neq \emptyset$ implies $U \subseteq f^-(V)$ and $fx \subseteq W$ implies $U \subseteq f^+(V)$. Let $g \in \bar{F}$ be such that $gx \cap W \neq \emptyset$. Let $\{g_\alpha\}$ be a net in F

which is τ_p-convergent to g. Since $\{h: h \in Y^{mX}$ and $hx \cap W \neq \emptyset\}$ is a τ_p-neighborhood of g, $g_\alpha x \cap W \neq \emptyset$ eventually, so $U \subseteq g_\alpha^-(V)$ eventually. Suppose that $U \not\subseteq g^-(V)$. Then, for some $u \in U$, $gu \subseteq Y - V$, so $g_\alpha u \subseteq Y - V$ eventually, which is a contradiction.

Now let $g \in \bar{F}$ be such that $gx \subseteq W$. Let $\{g_\alpha\}$ be a net in F which is τ_p-convergent to g. Since $\{h: h \in Y^{mX}$ and $hx \subseteq W\}$ is a τ_p-neighborhood of g, $g_\alpha x \subseteq W$ eventually, so $U \subseteq g_\alpha^+(V)$ eventually. Suppose that $U \not\subseteq g^+(V)$. Then, for some $u \in U$, $gu \cap (Y - V) \neq \emptyset$, so $g_\alpha u \cap (Y - V) \neq \emptyset$ eventually, which is a contradiction.

LEMMA 3.2. *If F is an evenly continuous subset of $(Y^{mX})_0$, then τ_p on F is jointly continuous.*

Proof. Let $\omega: (F, \tau_p) \times X \to Y$. Suppose that $(f, x) \in \omega^-(G)$, where G is open in Y. Choose $y \in fx \cap G$. There are neighborhoods U, W of x, y, respectively, such that $g \in F$ and $gx \cap W \neq \emptyset$ imply $U \subseteq g^-(G)$. Then $\{h: h \in F$ and $hx \cap W \neq \emptyset\} \times U$ is a neighborhood of (f, x) which is contained in $\omega^-(G)$.

Now suppose that $(f, x) \in \omega^+(G)$, where G is open in Y. There are neighborhoods U, W of x, fx, respectively, such that $g \in F$ and $gx \subseteq W$ imply $U \subseteq g^+(G)$. Then $\{h: h \in F$ and $hx \subseteq W\} \times U$ is a neighborhood of (f, x) which is contained in $\omega^+(G)$.

The following lemma generalizes an implicit lemma of Noble [8], stated explicitely as Lemma 1.4 in [7, p. 7]:

LEMMA 3.3. *Let $f \in \mathscr{C}(X \times Y, Z)$. If X is compact and Z is regular, then the set $F = \{f(x, \cdot): x \in X\}$ is evenly continuous.*

Proof. Let $y \in Y$, let K be a compact subset of Z and let V be an open neighborhood of K. Let W be a closed neighborhood of K which is contained in V. We construct a neighborhood U of y as follows: Since $f(\cdot, y)$ is continuous, $K_1 = f(\cdot, y)^-(W)$ and $K_2 = f(\cdot, y)^+(W)$ are closed in X, therefore compact. Thus the second projections $\mathrm{pr}_2: K_1 \times Y \to Y$, $\mathrm{pr}_2: K_2 \times Y \to Y$ are closed, so that

$$U_1 = Y - \mathrm{pr}_2[(K_1 \times Y) - f^-(V)], \qquad U_2 = Y - \mathrm{pr}_2[(K_2 \times Y) - f^+(V)]$$

are open in Y. Because $K_1 \subseteq f(\cdot, y)^-(V)$, $K_2 \subseteq f(\cdot, y)^+(V)$, we have $K_1 \times \{y\} \subseteq f^-(V)$, $K_2 \times \{y\} \subseteq f^+(V)$. Hence $y \notin \mathrm{pr}_2[(K_1 \times Y) - f^-(V)]$, $y \notin \mathrm{pr}_2[(K_2 \times Y) - f^+(V)]$, that is, $y \in U_1 \cap U_2 = U$.

We show that the neighborhoods U, W of y, K, respectively, satisfy the required implications: Let $g \in F$ be such that $gy \cap W \neq \emptyset$, so that $g = f(x, \cdot)$ for some $x \in K_1$. Let $u \in U$, so that $u \notin \mathrm{pr}_2[(K_1 \times Y) - f^-(V)]$.

Then $(x, y) \in f^-(V)$, that is, $gu \cap V \neq \varnothing$. Now let $g \in F$ be such that $gy \subseteq W$, so that $g = f(x, \cdot)$ for some $x \in X_2$. Let $u \in U$, so that $u \notin \mathrm{pr}_2[(K_2 \times Y) - f^+(V)]$. Then $(x, u) \in f^+(V)$, that is, $gu \subseteq V$.

Let X and Y be topological spaces. The *compact open topology* τ_c on Y^{mX} is defined to be the topology having as open subbase the sets of the forms $\{f\colon f(K) \subseteq U\}$, $\{f\colon fx \cap U \neq \varnothing$ for all $x \in K\}$, where K is a compact subset of X and U is open in Y. Obviously, τ_c is larger than τ_p.

A subset F of Y^{mX} *satisfies the condition* (G) if, for every τ_c-closed subset F_0 of F, $\bigcap_{f \in F_0} f^-(U)$ and $\bigcap_{f \in F_0} f^+(U)$ are open in X whenever U is open in Y. The following two lemmas relate this condition to even continuity:

LEMMA 3.4. *If Y is regular, then every subset of Y^{mX} satisfying the condition (G) is evenly continuous.*

Proof. Let F be a subset of Y^{mX} which satisfies the condition (G). Let $x \in X$, let K be a compact subset of Y and let V be an open neighborhood of K. Let W be an open neighborhood of K such that $K \subseteq W \subseteq \bar{W} \subseteq V$. Since $F_1 = \{h\colon h \in F$ and $hx \cap \bar{W} \neq \varnothing\}$, $F_2 = \{h\colon h \in F$ and $hx \subseteq \bar{W}\}$ are τ_c-closed in F, $U_1 = \bigcap_{h \in F_1} h^-(V)$ and $U_2 = \bigcap_{h \in F_2} h^+(V)$ are open in X. Then $U = U_1 \cap U_2$ is an open neighborhood of x.

Let $f \in F$ be such that $fx \cap W \neq \varnothing$. Then $f \in F_1$, so that $U \subseteq U_1 \subseteq f^-(V)$. Now let $f \in F$ be such that $fx \subseteq W$. Then $f \in F_2$, so that $U \subseteq U_2 \subseteq f^+(V)$.

LEMMA 3.5. *Every τ_c-compact evenly continuous subset of $(Y^{mX})_0$ satisfies the condition (G).*

Proof. Let F be a τ_c-compact evenly continuous subset of $(Y^{mX})_0$. Since F is τ_p-compact, it suffices, by Corollary 10.6 of [**7**, p. 23], to show that τ_p on F is jointly continuous. For this we apply Lemma 3.2.

Let X be a topological space and let $Y = (Y, \mathcal{U})$ be a uniform space. A subset F of Y^{mX} is *equicontinuous* if, for $(x, U) \in X \times \mathcal{U}$, there exists a neighborhood V of x such that, for all $f \in F$, $f(V) \subseteq U[fx]$ and $fz \cap U[y] \neq \varnothing$ whenever $(z, y) \in V \times fx$. The following two lemmas relate equicontinuity to even continuity:

LEMMA 3.6. *If $Y = (Y, \mathcal{U})$ is a uniform space, then every equicontinuous subset of Y^{mX} is evenly continuous.*

Proof. Let F be an equicontinuous subset of Y^{mX}. Let $x \in X$, let K be a compact subset of Y and let U be a symmetric member of $\mathcal{U}$. There is a neighborhood V of x such that, for all $f \in F$, $f(V) \subseteq$

$U[fx]$ and $fx \subseteq U[fz]$ for all $z \in V$. Write $W = U[K]$. Let $f \in F$ be such that $fx \cap W \neq \emptyset$. If $z \in V$ then $fx \subseteq U[fz]$, so that $U[fz] \cap W \neq \emptyset$, therefore $V \subseteq f^{-}(U^2[K])$. Now let $f \in F$ be such that $fx \subseteq W$. Then $f(V) \subseteq U[fx] \subseteq U^2[K]$, that is, $V \subseteq f^{+}(U^2[K])$.

LEMMA 3.7. *If Y is a uniform space, then every evenly continuous pointwise bounded subset of $(Y^{mX})_0$ is equicontinuous.*

Proof. Let F be an evenly continuous pointwise bounded subset of $(Y^{mX})_0$. Let $\bar{F}$ denote the τ_p-closure of F in $(Y^{mX})_0$. By the Lemmas 3.1, 3.2, τ_p on $\bar{F}$ is jointly continuous. Since $(Y^{mX})_0$ is a Tychonoff set, by Lemma 2.1, $\bar{F}$ is τ_p-compact. Then, by the Lemma 8 of Smithson [**9**, p. 258], $\bar{F}$ is equicontinuous.

4. Ascoli theorem. Let $X = (X, \tau)$ be a topological space. The *k-extension* of τ is the family $k(\tau)$ of all subsets U of X such that $U \cap K$ is open in K for every compact subset K of X. It is clear that $k(\tau)$ is a topology on X which is larger than τ. The topological space $kX = (X, k(\tau))$ is called the *k-extension* of X. A topological space X is called a *k-space* if $kX = X$. For an arbitrary topological space X, $kkX = kX$, so kX is a k-space. Familiar examples of k-spaces are the locally compact spaces and the spaces satisfying the first countability axiom.

Let X and Y be topological spaces. A function f: $X \to Y$ is called *k-continuous* if its restriction to each compact subset of X is continuous. Henceforth, the set of all continuous (k-continuous) functions on X to Y will be denoted $C(X, Y)(C_k(X, Y))$. It can be shown that a topological space X is a k-space if and only if $C_k(X, Y) = C(X, Y)$ for every topological space Y [**7**, p. 9]. A topological space X is a *k_3-space* if $C_k(X, Y) = C(X, Y)$ for every regular space Y. Thus a k-space is a k_3-space but not conversely. In fact, the product of uncountably many copies of the real line, which is not a k-space, is a k_3-space. We write $\mathscr{C}_0(X, Y) = (Y^{mX})_0 \cap \mathscr{C}(X, Y)$.

We note that if Y is regular, then $(\mathscr{C}_0(X, Y), \tau_c)$ is a regular space for every topological space X.

In a regular space there was introduced in [**7**, p. 11] the following equivalence relation R: xRy if every open neighborhood of x contains y. For a subset F of such a space, F^* denotes its R-saturation, that is, the smallest R-saturated set containing F.

THEOREM 4.1. *Let X, Y be topological spaces, let T be a Tychonoff subset of $(Y^{mX})_0$ and let $F \subseteq (T \cap \mathscr{C}(X, Y), \tau_c)$. If Y is regular, the following conditions are sufficient for the compactness of F:*

(a) *F^* is closed in $T \cap \mathscr{C}(X, Y)$.*
(b) *F is pointwise bounded, and*
(c) *F is evenly continuous.*

If X is a k-space and Y is regular, then the conditions (a), (b) *and* (c) *are necessary for the compactness of F.*

Proof. Sufficiency. Let $\bar{F}$ denote the τ_p-closure of F in T. Since $T \subseteq (Y^{mX})_0$, (c) implies, by Lemmas 3.1 and 3.2, that $\omega: (\bar{F}, \tau_p) \times X \to Y$ is continuous, and, in particular, that $\bar{F} \subseteq \mathscr{C}(X, Y)$. By Lemma 8.1 of [**7**, p. 18], $\tilde{\omega}: (\bar{F}, \tau_p) \to (\mathscr{C}(X, Y), \tau_c)$ is continuous. Since T is a Tychonoff set, (b) implies, by Lemma 2.1, that $\bar{F}$ is τ_p-compact, so $\tilde{\omega}(\bar{F}) = \bar{F}$ is a τ_c-compact subset of $T \cap \mathscr{C}(X, Y)$. But (a) implies $F \subseteq \bar{F} \subseteq F^*$, so, by Theorem 4.1 (b) of [**7**, p. 11], F is τ_c-compact.

Necessity. By Theorem 4.1 (c) of [**7**, p. 11], F^* is closed in $(T \cap \mathscr{C}(X, Y), \tau_c)$. It is clear that F is pointwise bounded. Since X is a k-space, by Theorem 9.4 of [**7**, p. 21], $\omega: (F, \tau_c) \times X \to Y$ is continuous. So by Lemma 3.3, $F = \{\omega(f, \cdot): f \in F\}$ is evenly continuous.

COROLLARY 1. *Let $F \subseteq (\mathscr{C}_0(X, Y), \tau_c)$. If Y is regular, the following conditions are sufficient for the compactness of F:*
(a) *F^* is closed in $\mathscr{C}_0(X, Y)$,*
(b) *F is pointwise bounded, and*
(c) *F is evenly continuous.*

If X is a k-space and Y is regular, then the conditions (a), (b) *and* (c) *are necessary for the compactness of F.*

COROLLARY 2. *Let $F \subseteq (C(X, Y), \tau_c)$. If Y is regular, the following conditions are sufficient for the compactness of F:*
(a) *F^* is closed in $C(X, Y)$,*
(b) *F is pointwise bounded, and*
(c) *F is evenly continuous.*

If X is a k-space and Y is regular, then the conditions (a), (b) *and* (c) *are necessary for the compactness of F.*

COROLLARY 3. *If Y is regular, a subset F of $(C_k(X, Y), \tau_c)$ is compact if and only if*
(a) *F^* is closed in $C_k(X, Y)$,*
(b) *F is pointwise bounded, and*
(c) *F is evenly continuous on compacta.*

Proof. For the sufficiency, we note that $C_k(X, Y) = C(kX, Y)$ and apply the Lemma 3.4 of [**7**, p. 11]. For the necessity, we consider F as a subset of $(C(kX, Y), \tau_c)$ and deduce from Corollary 2 the conditions (a),

(b) and the even continuity of F. Then it is clear that F, considered as a subset of $(C_k(X, Y), \tau_c)$, is evenly continuous on compacta.

COROLLARY 4. ([**1**,p. 635]). *Let* $F \subseteq (C(X, Y), \tau_c)$. *If* Y *is regular, the following conditions are sufficient for the compactness of* F:

(a) F *is closed in* $C(X, Y)$,

(b) F *is pointwise bounded, and*

(c) F *is evenly continuous.*

If X *is a* k_3*-space and* Y *is regular, then the conditions* (a), (b) *and* (c) *are necessary for the compactness of* F.

Proof. For the necessity, we note that, since Y is regular, $C_k(X, Y) = C(X, Y)$; then we apply Corollary 3 and Lemma 3.4 of [**7**, p. 11].

REMARKS. (1) By Lemmas 3.4, 3.5, the Corollary 1 is equivalent to the Theorem 10.10 of [**7**, pp. 23–24], which contains the Ascoli theorem of Gale [**2**, p. 304] and the multifunction Ascoli theorem of Mancuso [**6**, p. 470].

(2) Let Y be a uniform space. By Lemmas 3.6, 3.7 and Theorem 12.2 of [**7**, p. 28], the Corollary 1 in this context is equivalent to the Theorem 12.8 of [**7**, p. 31], which contains the multifunction Ascoli theorem of Smithson [**9**, p. 259].

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