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SUBSEQUENCES AND REARRANGEMENTS OF SEQUENCES IN *FK* SPACES

ROBERT DEVOS

The purpose of this paper is to study *FK* spaces which contain all subsequences or all rearrangements of a given sequence. Using a result of Bennett and Kalton we are able to show that if a separable *FK* space contains all subsequences or all rearrangements of a sequence with two or more finite cluster points, then it contains m. We are also able to show that if ℓ^p contains all rearrangements of some sequence not in ℓ^p, then it is a wedge space. This leads to proofs that if X is a solid symmetric *FK* space, $X \setminus \ell^p \neq \phi$, $X \neq s$, then $X \neq \ell^p_A$ for any matrix A and if in addition X is not wedge then X and ℓ^p are not linearly homeomorphic, via a matrix, hence extending a result of Banach.

1. Recently there has been a large number of papers **[8]**, **[9]**, **[11]**, **[13]**, **[14]** and **[15]** considering subsequences and rearrangements of sequences in c_A and ℓ_A. In this paper we consider these operations in an *FK* space setting and are able to generalize many of these results.

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Let s denote the space of all complex-valued sequences. An *FK* space is a vector subspace of s which is also a Fréchet space, (complete linear metric) with continuous coordinates. A *BK* space is a normed *FK* space. Some discussion of *FK* spaces is given in **[19]**. Well-known examples of *BK* spaces are the spaces m, c, c_0 of bounded, convergent, null sequences respectively, all with $\|x\|_\infty = \sup |x_k|$,

$$\ell^p = \left\{x \in s : \|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} < \infty\right\} \qquad (1 \leqq p < \infty)$$

(and we write $\ell = \ell^1$.)

Let m_0 be the linear span of all sequences of 0's and 1's and E^∞ the set of all finite sequences; that is, sequences all but finitely many of whose terms are zero. We shall assume that all *FK* spaces contain E^∞. Let A be a matrix, E an *FK* space, $E_A = \{x \in s : Ax \in E\}$ is well known to be an *FK* space.

Let $e = (1, 1, 1, \cdots)$, $e^j = (0, \cdots, 0, 1, 0, \cdots)$ (with 1 in rank j). We denote the nth section of an element $x \in E$ by $P_n x = \sum_{i=1}^{n} x_i e^i$ and say that x has AK provided that $P_n x \to x$ in E. The FK space E is called wedge when $e^n \to 0$ in E.

The α and β duals of a subset X of s are defined by

$$X^\alpha = \left\{ y \in s\colon \sum_{i=1}^{\infty} |x_i y_i| < \infty \quad \text{for each} \quad x \in X \right\}$$

$$X^\beta = \left\{ y \in s\colon \sum_{i=1}^{\infty} x_i y_i \quad \text{converges for each} \quad x \in X \right\}.$$

E is solid if $x \in E$ implies $(a_i x_i) \in E$ for each $a \in m$. Let Σ denote all permutations (rearrangements) of the positive integers. E is symmetric if $x \in E$ implies $x_\sigma = (x_{\sigma(i)}) \in E$ for each $\sigma \in \Sigma$.

In **[6]**, R. C. Buck proved the Tauberian theorem that if x is nonconvergent, then no regular summability matrix can sum every subsequence of x. I. J. Maddox in **[15]** improved Buck's theorem by showing that if A sums every subsequence of a divergent real sequence then $c_A \supset m$.

In **[11]**, J. A. Fridy proved a theorem analogous to Buck's, in which subsequence is replaced by rearrangement. T. A. Keagy in **[13]** extends Fridy's theorem as Maddox extended Buck's.

In the following two theorems, we consider subsequences and rearrangements of a sequence in an FK space. Theorem 2, along with the facts

(i) c_A is always separable;

(ii) if $x \notin m$ and every subsequence (rearrangement) of x is in c_A then $\exists\ N$ such that $a_{jn} = 0$ for $n \geqq N$, and this implies that $c_A = s$; gives us their results.

THEOREM 1. *Let E be an FK space $\supseteq E^\infty$. The following are equivalent.*

(a) *There exists an $x \in E$ with the properties*:
  (i) *for some p, q real numbers, $p \neq q$, pe and qe are subsequences of x.*
  (ii) *E contains all subsequences of x.*

(b) $E \supseteq m$

(c) $E \supseteq m_0$

(d) *$e \in E$ and there exists a $y \in E$ with the properties*:
  (i) *for some p, q real numbers, $p \neq q$, pe and qe are subsequences of y.*
  (ii) *E contains all rearrangements of y.*

Proof. Clearly (b) $\Rightarrow$ (a), (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d).

(c) $\Rightarrow$ (b) Bennett and Kalton's extension of Seevers results Theorem 1, p. 513 of **[5]**.

(a) $\Rightarrow$ (c) E contains all sequences of p's and q's hence E contains all sequences of 0's and 1's.

(d) $\Rightarrow$ (c) Let z be a sequence of 0's and 1's such that only finitely many $z_i = 1$ or $= 0$. Since $e \in E$ and $E^\infty \subseteq E$ then $z \in E$. Let z be a sequence of 0's and 1's with an infinite number of $z_i = 0$ and an infinite number of $z_i = 1$.

Let $r(k)$ and $s(k)$ be such that $z_{r(k)} = 1$, $z_{s(k)} = 0$ for all k and $\{r(k)\} \cup \{s(k)\} = \mathbf{Z}^+$.

Let y^1, y^2, y^3, y^4 be rearrangements of y such that

$$\begin{array}{lll} y^1_{r(2k)} = p, & y^1_{s(k)} = q & \\ y^2_{r(2k)} = q, & y^2_{s(k)} = p, & y^2_{r(2k-1)} = y^1_{r(2k-1)} \\ y^3_{r(2k-1)} = p, & y^3_{s(k)} = q & \\ y^4_{r(2k-1)} = q, & y^4_{s(k)} = p, & y^4_{r(2k)} = y^3_{r(2k)}. \end{array}$$

Hence

$$\frac{1}{3(p-q)}\left[(y^1 - y^2) + (p-q)e + (y^3 - y^4) + (p-q)e\right] = z$$

and so $z \in E$. Since z was arbitrary it follows that $E \supseteq m_0$.

Using a form of the closed graph theorem due to Kalton, Bennett and Kalton as Theorem 25 p. 577 of **[4]** prove

THEOREM (BENNETT-KALTON). *If E is a separable FK space $\supseteq E^\infty$ and $E + c_0 \supseteq m_0$ then $E \supseteq m$.*

Using this theorem and arguments similar to those of Theorem 1, we have

THEOREM 2. *Let E be a separable FK space $\supseteq E^\infty$. The following are equivalent.*

(a) *$\exists\ x \in E$ with at least two distinct finite cluster points and E contains all subsequences of x.*

(b) *$E \supseteq m$.*

(c) *$E \supseteq m_0$.*

(d) *$\exists\ y \in E$ with at least two distinct finite cluster points, E contains all rearrangements of y and $e \in E$.*

LEMMA 1. *Let Y be a linear sequence space, $x \in Y \backslash \ell^p$ such that every rearrangement of x belongs to Y. Then there exists a $z \in Y \backslash \ell^p$ such* *that every rearrangement of z belongs to Y and $|z_i| = 0$ for an infinite number of subscripts.*

Proof. Let y be a rearrangement of x such that the even coordinates form a sequence which is not in ℓ^p and the sequence $(y_{4n} - y_{4n-2}) \notin \ell^p$. Let y' be the rearrangement of x which permutes the $4n$th and the $4n-2$nd slots of y. Let $z = y - y'$. The odd coordinates of z are 0 and $z \in Y \backslash \ell^p$. Clearly any rearrangement of z belongs to Y.

THEOREM 3. *Let $A = (a_{ij})$ be a matrix, α^n the nth column of A and $1 \leqq p < \infty$. If there exists an $x \in \ell^p_A \backslash \ell^p$ such that every rearrangement of x belongs to ℓ^p_A then $\|\alpha^n\|_p \to 0$.*

Proof. By a Lemma in **[11]**, each row of A is in c_0. If $x \notin m$ then the rows of A are in E^∞, for if $\exists$ p such that $(a_{pn})_{n=1}^\infty \notin E^\infty$ then $\exists$ a rearrangement of x such that $\Sigma\, a_{p,k} x_{\sigma(i)}$ is not convergent. Let β^n be the nth row. If $\exists$ N such that $P_N \beta^n - \beta^n = 0$ for all n then $\ell^p_A = s$ and $\|\alpha^n\|_p = 0$ for $n \geqq N$. If N does not exist then $\exists$ a monotonic increasing sequence of positive integers $(p(k))$ and a rearrangement x_σ of x such that

$$\left| \sum_i a_{p(k),i} x_{\sigma(i)} \right| \geqq 1,$$

which implies $x_\sigma \notin \ell^p_A$, a contradiction; so N exists. If $x \in m$, we may assume $\|x\|_\infty \leqq \frac{1}{2}$. Suppose $\|\alpha^n\|_p \not\to 0$, then there exists $\epsilon > 0$ and an increasing sequence of integers r such that $\|\alpha^{r_i}\|_p \geqq \epsilon$, for all i. We now define a subsequence $(\ell(k))$ of r and $(m(k))$ of positive integers. Let $\ell(1) = r_1$, $m(0) = 0$ and $m(1)$ be such that $\|\alpha^{\ell(1)} - P_{m(1)}\alpha^{\ell(1)}\|_p < \frac{1}{2}\epsilon$. Since the rows are in c_0, pick $\ell(2) > \ell(1)$ such that $\|P_{m(1)}\alpha^{\ell(2)}\|_p < \frac{1}{4}\epsilon$. Pick $m(2) > m(1)$ such that $\|\alpha^{\ell(2)} - P_{m(2)}\alpha^{\ell(2)}\|_p < \frac{1}{4}\epsilon$.

Proceeding in this manner we inductively define increasing sequences $(\ell(k))$ (a subsequence of r) and $(m(k))$ such that

$$\|\alpha^{\ell(k)}\|_p \geqq \epsilon$$

$$\| P_{m(k)}\alpha^{\ell(k+1)}\|_p < \frac{1}{2^{k+1}}\,\epsilon$$

$$\| P_{m(k)}\alpha^{\ell(k)} - \alpha^{\ell(k)}\|_p < \frac{1}{2^k}\,\epsilon.$$

Hence

$$\| (P_{m(k)} - P_{m(k-1)})\alpha^{\ell(k)}\|_p \geqq \tfrac{1}{2}\epsilon. \qquad (k \geqq 2)$$

By Lemma 1, $\exists\, z \in \ell^p_A \backslash \ell^p$ such that $|z_i| = 0$ for $i \neq \ell(k)$ for some k and $\|z\|_\infty \leqq 1$ since $\|x\|_\infty \leqq \frac{1}{2}$. Hence

$$\left(\left| \sum_{k=1}^{\infty} a_{n,\ell(k)} z_{\ell(k)} \right| \right) \in \ell^p$$

call it γ^0. Let

$$\gamma^1 = |\alpha^{\ell(1)} - P_{m(1)}\alpha^{\ell(1)}|$$

(i.e. the absolute value of each term)

$$\gamma^n = |\alpha^{\ell(n)} - (P_{m(n)} - P_{m(n-1)})\alpha^{\ell(n)}| \quad \text{for} \quad n \geqq 1$$

$$\|\gamma^n\|_p \leqq \frac{1}{2^n}\epsilon + \frac{1}{2^n}\epsilon = \frac{1}{2^{n-1}}\epsilon.$$

Let $\delta = \Sigma_{i=0}^{\infty} \gamma^i$. Since $\Sigma_{i=0}^{\infty} \|\gamma^i\|_p < \infty$, it follows that $\delta \in \ell^p$. Let $m(s-1) < q \leqq m(s)$

$$\begin{aligned} |a_{q,\ell(s)} z_{\ell(s)}| &\leqq \left| \sum_{k=1}^{\infty} a_{q,\ell(k)} z_{\ell(k)} \right| + \sum_{\substack{k=1 \\ k \neq s}}^{\infty} |a_{q,\ell(k)} z_{\ell(k)}| \\ &\leqq \left| \sum_{k=1}^{\infty} a_{q,\ell(k)} z_{\ell(k)} \right| + \sum_{\substack{k=1 \\ k \neq s}}^{\infty} |a_{q,\ell(k)}| \\ &\leqq \delta_q. \end{aligned}$$

Hence the sequence

$$\delta' = z_{\ell(1)} P_{m(1)} \alpha^{\ell(1)} + \sum_{k=2}^{\infty} z_{\ell(k)} (P_{m(k)} - P_{m(k-1)}) \alpha^{\ell(k)} \in \ell^p.$$

But

$$\begin{aligned} \|\delta'\|_p^p &= \|z_{\ell(1)} P_{m(1)} \alpha^{\ell(1)}\|_p^p + \sum_{k=2}^{\infty} |z_{\ell(k)}|^p \|(P_{m(k)} - P_{m(k-1)}) \alpha^{\ell(k)}\|_p^p \\ &\geqq |z_{\ell(1)}|^p \left(\frac{\epsilon}{2}\right)^p + \sum_{k=2}^{\infty} |z_{\ell(k)}|^p \left(\frac{\epsilon}{2}\right)^p \end{aligned}$$

which implies $z \in \ell^p$, a contradiction. Hence $\|\alpha^n\|_p \to 0$.

This theorem was stated for $p = 1$ in the Notices by Keagy **[14]**. In **[2]** Bennett defined the concept of a wedge space. He then proves several equivalent conditions one of them being $E \supset z^\alpha$ for some $z \in c_0$. As Theorems 36 and 41, he shows ℓ^p_A is wedge iff $\|\alpha^n\|_p \to 0$ where α^n is the nth column of A.

COROLLARY 1. *Let X be a non-wedge FK space, $y \in X \backslash \ell^p$ such that $y_\sigma \in X$ for all $\sigma \in \Sigma$. Then $X \neq \ell^p_A$ for any matrix A.*

COROLLARY 2. *Let $X \neq s$ be a solid symmetric FK space $X \backslash \ell^p \neq \phi$. Then $X \neq \ell^p_A$ for any matrix A.*

Proof. In **[12]** Garling proves that $X \subseteq m$; but all wedge spaces contain unbounded sequences hence X is nonwedge.

Since ℓ^q is always solid symmetric we have

COROLLARY 3. *If $q > p$ then $\ell^q \neq \ell^p_A$ for any matrix A.*

This was proved using wedge spaces by Bennett in **[2]** and other techniques by DeVos in **[10]**.

THEOREM 4. *Let X be a non-wedge FK space with AK, $y \in X \backslash \ell^p$ such that $y_\sigma \in X$ for all $\sigma \in \Sigma$. Then X cannot equal ℓ^p_A nor can it be a closed subspace of ℓ^p_A for any matrix A.*

Proof. Let $z \in m_0$ be chosen such that $z_{n(k)} = 1$ and $z_i = 0$ for $i \neq n(k)$ where $(n(k))$ is an increasing sequence of positive integers such that $!e^{n(k)}! \geqq c > 0$ where $!!$ is the paranorm of X and $\|\alpha^{n(k)}\|_p < 1/2^k$ where $\alpha^{n(k)}$ is the $n(k)$ column of the matrix A. $z \notin X$ and $z \in \ell^p_A$ with AK hence z is the closure of X in ℓ^p_A. Hence X is not closed in ℓ^p_A.

Garling in **[11]** defines the spaces

$$\mu_z = \left\{ x \in s : \sup_{\sigma \in \Sigma} \sum_{i=1}^{\infty} |x_{\sigma(i)} z_i| < \infty \right\}$$

and shows that μ_z is a symmetric solid BK space. As Proposition 11 he shows for $z \in c_0$, $\mu_z \supsetneqq \ell'$. Combining these results we add another condition to Bennett's Theorem 36.

THEOREM 5. *The following conditions are equivalent for any matrix A.*

(i) ℓ_A *is a* (weak) *wedge space*

(ii) $\|\alpha^n\|_1 \to 0$

(iii) $\exists\ x \in \ell_A \backslash \ell$ *such that* $x_\sigma \in \ell_A$ *for all* $\sigma \in \Sigma$.

For $p > 1$, the converse of Theorem 3 is false. For the following example let all sequences be real. In **[16]** Ruckle defines the sequence h such that $h_n = n^{1/p} - (n-1)^{1/p}$ and shows that $\mu_h \subsetneqq \ell^p$. Let A be the matrix such that

$$a_{1n} = h_n \quad \text{and} \quad a_{pn} = 0 \quad \text{for} \quad p > 1;$$

Thus, $\ell^p_A = s_A = h^\beta \supset \mu_h$. Let $x \in h^\beta$ such that $x_\sigma \in h^\beta$ for all permutations σ. Then $x_\sigma \in h^\alpha$ for all permutations σ. Hence $x \in \mu_h$ which implies $x \in \ell^p$.

Banach in [1] shows that if $p \neq q$, $q \geqq 1$ then ℓ^p and ℓ^q are not linearly homeomorphic. He does this by showing that their linear dimensions are incomparable. If X and Y are linear topological spaces then $\dim_\ell X \leqq \dim_\ell Y$ iff X is isomorphic to a closed subspace of Y. The following theorems which follow easily from Theorem 3 are extensions of these results.

THEOREM 6. *Let X be a nonwedge FK space such that $\exists\ x \in X \backslash \ell^p$ with $x_\sigma \in X$ for all $\sigma \in \Sigma$. Then X and ℓ^p are not linearly homeomorphic via a matrix.*

THEOREM 7. *Let X be a nonwedge FK space with AK such that $\exists$ $x \in X \backslash \ell^p$ with $x_\sigma \in X$ for all $\sigma \in \Sigma$. Then* $\dim_\ell X \not\leqq \dim_\ell \ell^p$.

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VILLANOVA UNIVERSITY