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ANOTHER MARTINGALE CONVERGENCE THEOREM

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A classical martingale theorem is generalized to "martingale like" sequences. The method of proof is a generalization of Doob's proof by "downcrossings".

Introduction. Let (Ω, B, P) be a probability space, $\{B_n\}$ an increasing sequence of sub sigma fields of B. Let $\{f_n, B_n, n \geqq 1\}$ be an adapted sequence of P-integrable random variables.

The sequence is said to be a *martingale in the limit* if

$$\lim_{n\to\infty} \sup_{\bar{n}>n} |f_n - E(f_{\bar{n}} \mid B_n)| = 0 \qquad P \quad \text{a.e.}$$

If was proven in an earlier paper, Mucci [3] that every uniformly integrable martingale in the limit converges both L_1 and P a.e., generalizing the corresponding martingale theorem. The purpose of the present note is to prove that every L_1-bounded martingale in the limit converges pointwise to an integrable random variable, thereby generalizing another classical martingale theorem. We recall that a sequence $\{f_n\}$ is said to be L_1-bounded if $\sup_n \int |f_n| < \infty$.

THE THEOREM. *Let $\{f_n, B_n, n \geqq 1\}$ be an L_1-bounded martingale in the limit. Then there exists $f \in L_1$ with $f_n \to f$ P a.e.*

Proof. Fix $a < b$, two arbitrary real numbers. We define, following the classical proof:

$\varphi(a, b)$ is the number of "downcrossings" of $\{f_n\}$ from above b to below a. Our objective will be to show that $P(\varphi(a,b) = \infty) = 0$ so that $P(\underline{\lim}\, f_n \leqq a < b \leqq \overline{\lim}\, f_n) = 0$, thereby determining that $f = \lim_n f_n$ exists almost everywhere, and since

$$\int |f| < \underline{\lim} \int |f_n| < \infty; \quad \text{that} \quad f \in L_1.$$

Our procedure consists in defining a "modified" number of downcrossings $\bar{\varphi}(a, b)$ and showing that $P(\bar{\varphi}(a,b) = \infty) = 0$ and further that, almost everywhere,

$$\bar{\varphi}(a,b) < \infty \quad \text{implies} \quad \varphi(a,b) < \infty.$$

We begin by defining a sequence of stopping times:

$$\tau_0 = 0.$$

Now let $\{\alpha_n\}$ be a decreasing sequence of positive numbers with $\Sigma\, \alpha_n < \infty$, and let N be a fixed positive integer.

Define τ_{2n-1} as the first $m \leqq N$ such that:

(1) $m > \tau_{2n-2}$

(2) $f_m > b$

(3) $\sup_{\bar{m}>m} |f_m - E(f_{\bar{m}} \mid B_m)| < \alpha_n$.

If no such m exists, set $\tau_{2n-1} = N$.

Likewise, define τ_{2n} as the first $m \leqq N$ such that:

($\bar{1}$) $m > \tau_{2n-1}$

($\bar{2}$) $f_m < a$

($\bar{3}$) $\sup_{\bar{m}>m} |f_m - E(f_{\bar{m}} \mid B_m)| < \alpha_n$.

If no such m exists, set $\tau_{2n} = N$. We have

$$\int f_{\tau_{2n-1}} - \int f_{\tau_{2n}} = \sum_1^N \int_{(\tau_{2n-1}=k)} (f_k - E(f_N \mid B_k)) + \sum_1^N \int_{(\tau_{2n}=k)} (E(f_N \mid B_k) - f_k) < 2\alpha_n.$$

Thus

$$(*) \qquad \sum_1^\infty \int (f_{\tau_{2n-1}} - f_{\tau_{2n}}) < 2 \sum_1^\infty \alpha_n = 2\alpha.$$

We want an inequality in the other direction.

Define

$$\bar{\varphi}(N, a, b) = \sum_1^\infty (I_{(f_{\tau_{2n-1}} \geqq b)} \cdot I_{(f_{\tau_{2n}} \leqq a)} \cdot I_{(\sup_{m \geqq 0} |E(f_{\tau_{2n}+m} \mid B_{\tau_{2n}}) - f_{\tau_{2n}}| < \alpha_n)}$$

the number of times we make a "downcrossing" subject to conditions (3), ($\bar{3}$) on our stopping times.

We have

$$\sum_1^\infty (f_{\tau_{2n-1}} - f_{\tau_{2n}}) \geqq (b-a)\,\bar{\varphi}(N, a, b) - |b| - |f_N|.$$

Taking integrals, defining

$$\bar{\varphi}(a, b) = \varliminf_{N\to\infty} \bar{\varphi}(N, a, b),$$

and using Fatou's lemma and $(*)$, we have

$$(**)\qquad \int \bar{\varphi}(a,b) < \frac{1}{b-a}\left[|b| + 2\alpha + \sup_n \int |f_n|\right] < \infty.$$

Therefore $P(\bar{\varphi}(a,b) < \infty) = 1$.

Let us now define

$$\Omega_0 = (\bar{\varphi}(a,b) < \infty) \cap \left(\lim_{n\to\infty} \sup_{\bar{n}>n} |f_n - E(f_{\bar{n}} \mid B_n)| = 0\right).$$

Clearly $P(\Omega_0) = 1$ and we will be finished if we can show that $\varphi(a,b) < \infty$ on Ω_0. Now, for a particular $\omega \in \Omega_0$, let $\bar{\varphi}(a,b) = M$. Suppose $\varphi(a,b) = \infty$.

Then we can find a sequence $\{n_k\}$ where $f_{n_{2k-1}} \geqq b$, $f_{n_{2k}} \leqq a$ and where (3), $(\bar{3})$ hold. This contradicts $\bar{\varphi}(a,b) = M$.

Corollary 1. *Let $\{f_n, B_n, n \geqq 1\}$ be a martingale in the limit, and let $r \geqq 1$. Then there exists $f \in L_r$ such that $f_n \to f$ both P a.e. and in the L_r-norm $\Leftrightarrow \{|f_n|^r\}$ is uniformly integrable.*

Proof. If $\{|f_n|^r\}$ is uniformly integrable, then $\{f_n, B_n\}$ is L_1-bounded, hence $f_n \to f\, P$ a.e. The rest follows by the usual classical arguments. (See Neveu, p. 57.)

Corollary 2. *Let $s_n = \Sigma_1^n \xi_k$ where $\{\xi_k\}$ is an independent sequence. Then $s_n \to s \in L_1$ both P a.e. and L_1 provided $\{s_n\}$ is Cauchy in the L_1-norm.*

Proof. The Cauchy condition is equivalent to $\{s_n, B_n, n \geqq 1\}$ being a martingale in the limit (here $B_n = \sigma(\xi_1 \cdots \xi_n)$). Further,

$$\sup_n \int |s_n| \leqq \int |s_M| + \sup_{n \geqq M} \int |s_n - s_M| < \infty.$$

References

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