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CONSTRUCTION OF 2-BALANCED (n, k, λ) ARRAYS

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Let I_n denote the set of positive integers $\{1, 2, \cdots, n\}$, and I_n^λ the multiset consisting of λ copies of I_n. A submultiset S of I_n^λ is t-balanced if S can be partitioned into t parts such that the sums of all elements in each part are all equal. A t-balance (n, k, λ) array is a partition of I_n^λ into m multisets S_i, $i = 1, \cdots, m$, which are all of size k and t-balanced. In this paper, we give a necessary and sufficient condition for the existence of 2-balanced (n, k, λ) arrays. Furthermore, we show how 2-balanced (n, k, λ) arrays can be used to construct a class of neighbor designs used in serology, or to give coverings of complete multigraphs by k-cycles.

I. Introduction. Let I_n denote the set of positive integers $\{1, 2, \cdots, n\}$, and I_n^λ the multiset consisting of the multiset union ($^+$) of λ copies of I_n. Here we follow the notation of Knuth [**1**, Volume 2, p. 420 ex. 19] where multisets are defined as mathematical entities like sets, but may contain identical elements repeated a finite number of times. If A and B are multisets, we define their multiset union $A\ ^+ B$ as a multiset in the following way: An element x occurring a times in A and b times in B occurs $a + b$ times in $A\ ^+ B$. Submultisets of a multiset are similarly defined, with $A \subseteq B$ if x occurs a times in A implies x occurs $b \geqq a$ times in B. The cardinality of a multiset A, denoted by $|A|$, is the sum of the number of occurrences of all elements in A. We have clearly, $|A\ ^+ B| = |A| + |B|$. If $|A| = k$, we also call A a k-multiset.

If $S \subseteq I_n^\lambda$, let $\|S\|$ denote the sum of all elements in S. For example, if $S = \{1, 2, 2, 4\}$, then $|S| = 4$ and $\|S\| = 9$. *S is t-balanced* if S can be partitioned into t submultisets $S^{(1)}, S^{(2)}, \cdots, S^{(t)}$, such that $\|S^{(j)}\| = 1/t\ \|S\|$ for $j = 1, 2, \cdots, t$.

A *t-balanced (n, k, λ) array* is a partition of I_n^λ into m k-multisets S_i, $i = 1, \cdots, m$ which are all t-balanced. Thus a t-balanced (n, k, λ) array may be considered as an arrangement of the $n\lambda$ numbers in I_n^λ into an $m \times k$ matrix such that each row of the matrix is a t-balanced multiset.

As illustrations, we exhibit below a 3-balanced $(14, 7, 1)$ array and a 2-balanced $(8, 3, 3)$ array. The partitions of the S_i's into balanced submultisets are indicated by ";".

A 3-balanced $(14, 7, 1)$ array:

$$S_1 = (14; 1, 2, 11; 3, 5, 6)$$

$$S_2 = (9, 12; 8, 13; 4, 7, 10)$$

A 2-balanced $(8, 3, 3)$ array:

$$S_1 = (1, 2; 3)$$

$$S_2 = (1, 6; 7)$$

$$S_3 = (1, 6; 7)$$

$$S_4 = (2, 4; 6)$$

$$S_5 = (2, 5; 7)$$

$$S_6 = (3, 5; 8)$$

$$S_7 = (3, 5; 8)$$

$$S_8 = (4, 4; 8)$$

Note that the S_i's need not be distinct and the partition of each S_i into $S_i^{(j)}$'s needs not be uniform in sizes.

From the definition of a t-balanced (n, k, λ) array, it is clear that $k \geqq t$ and $\lambda n = mk$. Furthermore, since each S_i is t-balanced, $\|S_i\| \equiv 0 \pmod t$ and hence

$$\sum_{i=1}^{m} \|S_i\| = \|I_n^\lambda\| = \frac{\lambda n(n+1)}{2} \equiv 0 \pmod t.$$

For $t = k$, each S_i must consist of a single element occurring $t = k$ times and hence $\lambda \equiv 0 (\text{mod } t)$. Clearly t-balanced $(n, t, t\lambda')$ arrays exist for all positive integers n, t, and λ'. Also, 1-balanced (n, k, λ) arrays are just arrangements of λn integers into an $m \times k$ matrix and hence exist for all (n, k, λ) provided $\lambda n \equiv 0 \bmod k$. For $n = 1$, we must have $k \equiv 0 \pmod t$ and $\lambda \equiv 0 \pmod k$. Clearly, t-balanced $(1, k, \lambda)$ arrays exist trivially for those parameters.

In this paper, we deal mainly with the case $t = 2$, and establish necessary and sufficient conditions for the existence of 2-balanced (n, k, λ) arrays. The sufficiency proof will be constructive in nature. Furthermore, we show how 2-balanced (n, k, λ) arrays can be used to construct a class of neighbor designs used in serology, or to give coverings of complete multigraphs by k-cycles.

II. Necessary and sufficient conditions for the existence of 2-balanced (n, k, λ) arrays. In the rest of the paper we assume $t = 2$ and denote 2-balanced (n, k, λ) arrays by

$A(n, k, \lambda)$. From the remarks made in the previous section, we further assume $k \geqq 2$ and $n > 1$. We shall prove:

THEOREM 1. *Let $k \geqq 2$ and $n > 1$, then $A(n, k, \lambda)$ exists if and only if n, k, λ satisfy the following conditions*:

(a) $\lambda n \equiv 0 \pmod{k}$.

(b) $\lambda n(n+1) \equiv 0 \pmod{4}$.

(c) $\lambda \equiv 0 \pmod{2}$ *if* $k = 2$.

(d) $n > 2$ *if* $k = 3$.

The necessity of (a), (b) and (c) had already been shown. For $k = 3$, $n = 2$, the only possibility for the S_i's is $(1, 1; 2)$ and I_n^λ cannot be so partitioned, and hence (d) is also necessary. We prove below that conditions (a), (b), (c), (d) are also sufficient constructively. For clarity, the work is divided into a number of smaller steps.

First, in order to reduce the tedious effort of construction, we establish below a set of lemmas where we can deduce the construction of large classes of $A(n, k, \lambda)$'s from some "previously" constructed ones. For convenience, we introduce the following notations:

1. $N(k, \lambda) \equiv$ the set of all positive integers $n > 2$ such that n, k, λ satisfy the conditions (a), (b) and (c) in Theorem 1. (The construction of $A(2, k, \lambda)$ will be treated separately.)

2. While $A(n, k, \lambda)$ stands for a 2-balanced (n, k, λ) array, we shall also refer to the list of multisets S_i associated with $A(n, k, \lambda)$ as the rows of $A(n, k, \lambda)$.

3. $A(\{n\}, k, \lambda) \equiv a$ set of $A(n, k, \lambda)$'s for all $n \in N(k, \lambda)$.

4. We let $H_1 \Rightarrow H_2$ where H_1 and H_2 are various collections of $A(n, k, \lambda)$'s denote the statement: If we can construct the $A(n, k, \lambda)$'s in H_1, then we can construct the $A(n, k, \lambda)$'s in H_2.

Let $g = (k, \lambda)$ denote the g.c.d. of k and λ, $k^* = k/g$, $\lambda^* = \lambda/g$. The following lemma characterizes $N(k, \lambda)$:

LEMMA 1. *$N(k, \lambda)$ consists of all positive multiples of k^* which are > 2 if λ is even and all positive multiples of k^* which are congruent to* 0 *or* 3 mod 4 *if λ is odd, except for $k = 2$ where $N(2, \lambda) = \varnothing$ if λ is odd.*

The proof follows from (a), (b) in Theorem 1 and the definition of $N(k, \lambda)$.

From Lemma 1, we see that $N(k, \lambda)$ depends only on k^* and the parity of λ. Hence we have

LEMMA 2. *Let $k \geqq k' > 2$.*

1. *If $k/\lambda = k'/\lambda'$ and $\lambda \equiv \lambda'$* mod 2, *then $N(k, \lambda) \equiv N(k', \lambda')$.*

2. *If $\lambda \equiv g$* mod 2, *then $N(k, \lambda) \equiv N(k, g)$.*

3. *If* $\lambda \equiv 0,\ g \equiv 1 \bmod 2$, *then* $N(k, \lambda) \equiv N(k, 2g)$.

LEMMA 3. $\{A(n, k, \lambda_1),\ A(n, k, \lambda_2)\} \Rightarrow A(n, k, \lambda_1 + \lambda_2)$.

Proof. The rows of $A(n, k, \lambda_1)$ together with the rows of $A(n, k, \lambda_2)$ form $A(n, k, \lambda_1 + \lambda_2)$.

COROLLARY 1. $A(n, k, \lambda) \Rightarrow \{A(n, k, r\lambda),\ r = 1, 2, \cdots,\}$.

COROLLARY 2.
1. $A(\{n\}, k, g) \Rightarrow A(\{n\}, k, \lambda)$ *if* $\lambda \equiv g \bmod 2$, *and*
2. $A(\{n\}, k, 2g) \Rightarrow A(\{n\}, k, \lambda)$ *if* $\lambda \equiv 0,\ g \equiv 1 \bmod 2$.

Proof. From (2) and (3) of Lemma 2 and Corollary 1.

LEMMA 4. *If* $k_1/\lambda_1 = k_2/\lambda_2$, *then*

$$\{A(n, k_1, \lambda_1),\ A(n, k_2, \lambda_2)\} \Rightarrow A(n, k_1 + k_2, \lambda_1 + \lambda_2).$$

Proof. From $k_1/\lambda_1 = k_2/\lambda_2$, we see that $A(n, k_1, \lambda_1)$ and $A(n, k_2, \lambda_2)$ have the same number of rows. Let $\{S_i\}$ and $\{T_i\}$, $i = 1, 2, \cdots, m$ be the rows for $A(n, k_1, \lambda_1)$ and $A(n, k_2, \lambda_2)$ respectively. Then $\{R_i\} = \{S_i{}^+ T_i\}$, $i = 1, 2, \cdots, m$ form the rows for an $A(n, k_1 + k_2, \lambda_1 + \lambda_2)$ with $R_i = R_i^{(1)+} R_i^{(2)}$ where $R_i^{(j)} = S_i^{(j)+} T_i^{(j)}$, $j = 1, 2$.

COROLLARY 1. $A(n, k, \lambda) \Rightarrow \{A(n, rk, r\lambda),\ r = 1, 2, \cdots\}$.

COROLLARY 2. $A(n, k^*, \lambda^*) \Rightarrow A(n, k, \lambda)$.

COROLLARY 3. *Let* $k^* \geqq 3$, $g \equiv 1 \pmod 2$, *then*

$$A(\{n\}, k^*, \lambda^*) \Rightarrow A(\{n\}, k, \lambda).$$

Proof. Since $g \equiv 1 \pmod 2$, λ and λ^* have the same parity and hence $N(k^*, \lambda^*) \equiv N(k, \lambda)$ by (1) of Lemma 2. Hence Corollary 3 follows from Corollary 2.

Looking at $A(n, k, \lambda)$'s as $m \times k$ matrices of elements from I_n^λ, we may view the constructions by Lemmas 3 and 4 as vertical and horizontal compositions respectively, as illustrated by the following diagrams:

1. $$\boxed{\begin{array}{c} A(n, k, \lambda_1) \\ A(n, k, \lambda_2) \end{array}} \quad \Rightarrow \quad \boxed{A(n, k, \lambda_1 + \lambda_2)}$$

2. $$\boxed{A(n, k_1, \lambda_1)}\boxed{A(n, k_2, \lambda_2)} \quad \Rightarrow \quad \boxed{A(n, k_1 + k_2, \lambda_1 + \lambda_2)}$$

Corollary 3 states that we can reduce the construction of $A(\{n\}, k, \lambda)$ to the construction of $A(\{n\}, k^*, \lambda^*)$ provided $k^* \geqq 3$ and $g \equiv 1 \pmod 2$. The condition $g \equiv 1 \pmod 2$ may be removed by the following lemma:

LEMMA 5. (*even-even lemma*).

If $g \equiv 0 \pmod 2$, *then we can construct* $A(\{n\}, k, \lambda)$.

The construction is trivial since the rows of $A(n, k, \lambda)$ can consist of elements from I_n^λ occurring in pairs.

The following two lemmas take care of the situation when $k^* < 3$.

LEMMA 6. *Let* $k^* = 1$ *so that* $\lambda = k\lambda'$. *Then*

$$A(\{n\}, 3, 3\lambda') \Rightarrow A(\{n\}, k, \lambda)$$

Proof. We may assume k is odd, otherwise $A(\{n\}, k, \lambda)$ can be constructed by the even-even lemma. Lemma 6 is obvious if $k = 3$, hence we may assume $k \geqq 5$. Since $k - 3$ and $(k-3)\lambda'$ are both even, we can construct $A(\{n\}, k-3, (k-3)\lambda')$ by the even-even lemma. From Lemma 1, we have $N(k, \lambda) \equiv N(3, 3\lambda') \subseteq N(k-3, (k-3)\lambda')$ and hence $\{A(\{n\}, 3, 3\lambda'), A(\{n\}, k-3, (k-3)\lambda')\} \Rightarrow A(\{n\}, k, \lambda)$ by horizontal composition, and the lemma follows.

LEMMA 7. *Let* $k^* = 2$ *so that* $k = 2g$, $\lambda = g\lambda'$, λ' *odd. Then* $A(\{n\}, 6, 3\lambda') \Rightarrow A(\{n\}, k, \lambda)$.

Proof. We may assume g is odd and > 3 as in the proof of Lemma 6. The lemma then follows by the following horizontal composition:

$$\{A(\{n\}, 6, 3\lambda'), A(\{n\}, k-6, (g-3)\lambda')\} \Rightarrow A(\{n\}, k, \lambda).$$

Details are similar to the proof of Lemma 6.

Lemmas 8 and 9 below reduce the construction of $A(\{n\}, k^*, \lambda^*)$ to the construction of $A(\{n\}, \bar{k}, \bar{\lambda})$ for $3 \leqq \bar{k} \leqq 6$.

LEMMA 8. *Let* $(k, \lambda) = 1$. *Determine the integer* l *so that* $3 \leqq k - 4l < 7$. *Then*

$$A(\{n\}, k-4l, \lambda) \Rightarrow A(\{n\}, k, \lambda).$$

Proof. Let $n \in N(k, \lambda)$. Then $\lambda n = km$. Since $(k, \lambda) = 1$, we must have $n = ka$ and $m = \lambda a$. For $l \geqq 1$, let $n' = n - 4al = (k-4l)a$, then $n' \in N(k-4l, \lambda)$ since $\lambda n' = (k-4l)a\lambda = (k-4l)m$ and $n' \equiv n \pmod 4$. (The case $n'=2$ cannot occur since $n'=(k-4l)a \geqq 3$.) From an $A(n-4al, k-4l, \lambda)$, which by assumption we can construct, we construct $A(n,k,\lambda)$ as follows:

The elements in the multiset I_n^λ which are not in I_{n-4al}^λ, namely, λ copies of each of the $4al$ integers from $n-4al+1$ to n, are placed into an $m \times 4l$ matrix B such that every row of B is 2-balanced. One way this can be done is to fill the rows of B sequentially by λ strings of integers from $n-4al+1$ to n. Then every row of B consists of l sets of four consecutive integers. Since for every four consecutive integers x, $x+1$, $x+2$, $x+3$, $x+(x+3)=(x+1)+(x+2)$, the rows of B are clearly 2-balanced.

From B and $A(n-4al, k-4l, \lambda)$, we construct $A(n,k,\lambda)$ by horizontal composition where row i of $A(n,k,\lambda)=$ row i of $A(n-4al, k-4l, \lambda)^+$ row i of B.

When $\lambda \equiv 0 \pmod 2$, Lemma 8 can be strengthened to:

LEMMA 9. *Let* $(k,\lambda)=1$, $\lambda \equiv 0 \pmod 2$. *Determine the integer* l *so that* $3 \leqq k-2l < 5$. *Then*

$$A(\{n\}, k-2l, \lambda) \Rightarrow A(\{n\}, k, \lambda).$$

The proof of Lemma 9 is similar to that of Lemma 8 except that $n' = n-2al$ and the rows of the $m \times 2l$ matrix B can merely consist of numbers from $I_n^\lambda - I_{n-2al}^\lambda$ occurring in pairs.

LEMMA 10. $\{A(\{n\},k,\lambda), k=3,4,5,6\} \Rightarrow A(n,k,\lambda)$ *for all* $n \geqq 3$, k, λ *satisfying the conditions of Theorem* 1.

Proof. The proof of Lemma 10 sums up the applications of the reduction lemmas given above.

1. If $g=(k,\lambda)=1$, use Lemma 8.
2. If $g \equiv 0 \bmod 2$, use the even-even lemma.
3. If $g>1$ and odd, use
   (a) Lemma 6 if $k^* = k/g = 1$,
   (b) Lemma 7 if $k^*=2$, and
   (c) Corollary 3 of Lemma 4 if $k^* \geqq 3$ to reduce the construction of $A(n,k,\lambda)$ to $A(n,k^*,\lambda^*)$, and use Lemma 8 to construct $A(n,k^*,\lambda^*)$.

In the following, we shall construct $A(2,k,\lambda)$, and $A(\{n\},k,\lambda)$ for $k=3,4,5,6$.

III. Construction of $A(2,k,\lambda)$. Let $n=2$, then λ must be even, say $\lambda = 2\lambda'$. From $\lambda n = km$, we have $4\lambda' = km$. If $k \equiv$

0 (mod 2), we can construct $A(2, k, \lambda)$ by the even-even lemma. Hence, we may assume $k \geqq 5$, is odd, and hence $m \equiv 0 \pmod 4$. The rows of $A(2, k, \lambda)$ can then consist of 1, 1, 2, and $(k-3)/2$ pairs of 1's or 2's, which are available since both $\lambda - 2m = (k-4)(m/2)$ and $\lambda - m = (k-2)(m/2)$ are even. (These are the number of 1's and 2's left for the third to kth columns of $A(2, k, \lambda)$.)

IV. Construction of $A(n, 3, \lambda)$.

Case 1. $(3, \lambda) = 1$, $\lambda \equiv 1 \pmod 2$.

From Corollary 2 of Lemma 3, we need only construct $A(n, 3, 1)$ for $n \in N(3, 1)$. From Lemma 1, $n \in N(3, 1)$ if $n \equiv 0 \pmod 3$ and $n \equiv 0,3 \pmod 4$. We have two subcases:

1. $n = 12w \quad w = 1, 2, \cdots$

The S_i's $(i = 1, \cdots, 4w)$ are:

$$(1+2j,\ 11w - j;\ 11w + 1 + j) \qquad j = 0, 1, \cdots, w-1.$$

$$(2+2j,\ 8w - j;\ 8w + 2 + j) \qquad j = 0, 1, \cdots, w-2.^1$$

$$(2w,\ 6w+1;\ 8w+1)$$

$$(3w+1+2j,\ 6w - j;\ 9w + 1 + j) \qquad j = 0, 1, \cdots, w-1.$$

$$(3w+2+2j,\ 3w - j;\ 6w + 2 + j) \qquad j = 0, 1, \cdots, w-1.$$

It is easy to see that all the S_i's are 2-"balanced", and in the following, we verify that the S_i's are indeed a parition of $I_n^\lambda \equiv (1, 2, \cdots, 12w)$. We shall leave similar verifications for subsequent constructions to the reader.

The S_i's may be expanded into the following schematic diagram:

$$\begin{array}{lll}
(1, & 11w; & 11w+1) \\
\quad \Downarrow (1-2) & \ \uparrow (11) & \quad \downarrow (12) \\
(2w-1, & 10w+1; & 12w) \\
 & & \\
(2, & 8w; & 8w+2) \\
\quad \Downarrow (1-2) & \ \uparrow (8) & \quad \downarrow (9) \\
(2w-2, & 7w+2; & 9w) \\
 & & \\
\quad (2^*) & \quad (6^*) & \quad (8^*)
\end{array}$$

[1] Here and in all subsequent lists of S_i's vacuous if the range of j is empty.

$$(2w, \quad 6w+1; \quad 8w+1)$$

$$\begin{array}{lll} (3w+1, & 6w; & 9w+1) \\ \quad \Downarrow (4,5) & \uparrow (6) & \downarrow (10) \\ (5w-1, & 5w+1; & 10w) \end{array}$$

$$\begin{array}{lll} (3w+2, & 3w; & 6w+2) \\ \quad \Downarrow (4,5) & \uparrow (3) & \downarrow (7) \\ (5w, & 2w+1; & 7w+1) \end{array}$$

Where $\underset{b}{\overset{a}{\Downarrow}}$ means increasing from a to b by steps of two and $\underset{b}{\overset{a}{\downarrow}}$ increasing from a to b by steps of one. Similarly for $\underset{a}{\overset{b}{\uparrow}}$. Following the indicated order in parenthesis, we have $\underset{2w-1}{\overset{1}{\Downarrow}}(1-2)$, $\underset{2w-2}{\overset{2}{\Downarrow}}(1-2)$ account for numbers from 1 to $2w-1$, followed by $2w$ in (2^*), $2w+1$ to $3w$ in $\underset{2w+1}{\overset{3w}{\uparrow}}(3)$, etc. until $\underset{12w}{\overset{11w+1}{\downarrow}}(12)$ where all numbers in $I_n^\lambda = (1, 2, \cdots, 12w)$ are accounted.

2. $n = 12w + 3 \quad w = 0, 1, \cdots$
The S_i's $(i = 1, \cdots, 4w+1)$ are:

$$\begin{array}{ll} (1+2j,\ 11w+3-j;\ 11w+4+j) & j = 0, 1, \cdots, w-1. \\ (2+2j,\ 8w+2-j;\ 8w+4+j) & j = 0, 1, \cdots, w-1. \\ (3w+2+2j,\ 3w+1-j;\ 6w+3+j) & j = 0, 1, \cdots, w-1. \\ (3w+3+2j,\ 6w+1-j;\ 9w+4+j) & j = 0, 1, \cdots, w-1. \\ (2w+1,\ 6w+2;\ 8w+3). & \end{array}$$

Case 2. $(3, \lambda) = 1,\ \lambda \equiv 0 \pmod 2$.

Again from Corollary 2 of Lemma 3, we need only construct $A(n, 3, 2)$ for $n \in N(3, 2)$. From Lemma 1, $n \in N(3, 2)$ if $n \equiv 0 \pmod 3$. Although when $n \equiv 0, 3 \pmod 4$ we can construct $A(n, 3, 2)$ from $A(n, 3, 1)$, we give another construction below which is simple and uniform for all $n = 3w$, $w = 1, 2, \cdots$.

The S_i's $(i = 1, \cdots, 2w)$ are:

$$\begin{array}{ll} (1+j,\ 3w-1-2j;\ 3w-j) & j = 0, 1, \cdots, w-1. \\ (1+j,\ w+1+j;\ w+2+2j) & j = 0, 1, \cdots, w-1. \end{array}$$

From the schematic diagram below, it is easy to see that each of the numbers from 1 to $3w$ is used exactly twice.

$$\begin{array}{ccc} 1 & 3w-1 & 3w \\ \downarrow & \Uparrow & \uparrow \\ w & w+1 & 2w+1 \end{array}$$

$$\begin{array}{ccc} 1 & w+1 & w+2 \\ \downarrow & \downarrow & \Downarrow \\ w & 2w & 3w \end{array}$$

Case 3. $(3, \lambda) = 3$, $\lambda \equiv 1 \pmod 2$.

We need only construct $A(n, 3, 3)$ for all $n \in N(3, 3)$. $N(3, 3)$ consists of all numbers $\equiv 0, 3 \pmod 4$. Again, we have two subcases.

1. $n = 4w$, $w \geqq 1$. The S_i's $(i = 1, \cdots, 4w)$ are:

$$\begin{array}{ll} (1+2j,\ 3w-j;\ 3w+1+j) & j = 0, 1, \cdots, w-1, \\ (1+2j,\ 3w-j;\ 3w+1+j) & j = 0, 1, \cdots, w-1, \\ (2+2j,\ 3w-1-j;\ 3w+1+j) & j = 0, 1, \cdots, w-1, \\ (2+2j,\ w-1-j;\ w+1+j) & j = 0, 1, \cdots, w-2, \\ (w,\ 2w,\ 3w). & \end{array}$$

2. $n = 4w + 3$, $w \geqq 0$. The S_i's $(i = 1, \cdots, 4w+3)$ are:

$$\begin{array}{ll} (1+2j,\ 3w+2-j;\ 3w+3+j) & j = 0, 1, \cdots, w, \\ (1+2j,\ 3w+2-j;\ 3w+3+j) & j = 0, 1, \cdots, w-1, \\ (2+2j,\ 3w+1-j;\ 3w+3+j) & j = 0, 1, \cdots, w, \\ (2+2j,\ w-j;\ w+2+j) & j = 0, 1, \cdots, w-1, \\ (w+1,\ 3w+2;\ 4w+3). & \end{array}$$

Case 4. $(3, \lambda) = 3$, $\lambda \equiv 0 \pmod 2$.

We need only construct $A(n, 3, 6)$ for all $n \in N(3, 6)$, which consists of all positive integers > 2. Again, we have two subcases:

1. $n = 2w$, $w \geqq 2$. The S_i's $(i = 1, \cdots, 4w)$ are:

$$\begin{array}{ll} (1+2j,\ w-j;\ w+1+j) & j = 0, 1, \cdots, w-1, \\ (1+2j,\ w+1-j;\ w+2+j) & j = 0, 1, \cdots, w-2, \end{array}$$

$(2+2j,\ w-2-j;\ w+j)$ $\quad j=0,1,\cdots,w-3,$

$(2+2j,\ w-j;\ w+2+j)$ $\quad j=0,1,\cdots,w-2,$

$(1,\ 2w-2,\ 2w-1)$

$(1,\ 2w-1,\ 2w)$

$(2,\ 2w-2,\ 2w)$

$(w-1,\ w+1,\ 2w).$

2. $n=2w+1,\ w\geqq 1$. The S_i's $(i=1,\cdots,4w+2)$ are:

$(1+2j,\ w+1-j;\ w+2+j)$ $\quad j=0,1,\cdots,w-1,$

$(1+2j,\ w+1-j;\ w+2+j)$ $\quad j=0,1,\cdots,w-1,$

$(2+2j,\ w-j;\ w+2+j)$ $\quad j=0,1,\cdots,w-1,$

$(2+2j,\ w-1-j;\ w+1+j)$ $\quad j=0,1,\cdots,w-2,$

$(1,\ 2w;\ 2w+1)$

$(1,\ 2w;\ 2w+1)$

$(w,\ w+1;\ 2w+1).$

Note that we could have constructed most of the $A(n,3,6)$'s from $A(n,3,1)$, $A(n,3,2)$, and $A(n,3,3)$'s, leaving us with those values of $n=12w+1, 12w+2, 12w+5, 12w+10$ which do not belong in $N(3,1)\cup N(3,2)\cup N(3,3)$. However, the unified construction here is actually simpler.

This completes the construction of all $A(n,3,\lambda)$'s for $n\in N(3,\lambda)$.

V. Construction of $A(n,4,\lambda)$. We may assume λ is odd and $(4,\lambda)=1$ otherwise $A(n,4,\lambda)$ can be constructed by the even-even lemma. Hence we need only construct $A(n,4,1)$ for $n=4w,\ w\geqq 1$.

The S_i's are $(i=1,2,\cdots,w)$

$$(1+4j,\ 4+4j;\ 2+4j,\ 3+4j) \qquad j=0,1,\cdots,w-1.$$

Note the similarity of this construction and the construction of matrix B in Lemma 8. Another equally simple construction is

$$(w-j,\ 4w-j;\ 2w-j,\ 3w-j) \qquad j=0,1,\cdots,w-1.$$

VI. Construction of $A(n, 5, \lambda)$.

Case 1. $(5, \lambda) = 5$, $\lambda = 5\lambda'$.
We can construct $A(n, 5, \lambda)$ from $A(n, 3, 3\lambda')$ by Lemma 6.

Case 2. $(5, \lambda) = 1$, $\lambda \equiv 0 \pmod 2$.
We can construct $A(n, 5, \lambda)$ from $A(n', 3, \lambda)$ by Lemma 9.

Case 3. $(5, \lambda) = 1$, $\lambda \equiv 1 \pmod 2$.
From Corollary 2 of Lemma 3, we need only construct $A(n, 5, 1)$ for $n \in N(5, 1)$. From Lemma 1, $n \in N(5, 1)$ if $n \equiv 0 \pmod 5$ and $n \equiv 0, 3 \pmod 4$. We have two subcases:

1. $n = 20w$, $w \geqq 1$.
   The S_i's $(i = 1, \cdots, 4w)$ are:

$$(1 + j,\ 10w + 1 + j,\ 14w + 2 + j;\ 6w + 1 + j,\ 18w + 3 + 2j) \qquad j = 0, 1, \cdots, w - 2.$$

$$(w + 1 + j,\ 11w + 1 + j,\ 15w + 1 + j;\ 9w + 1 + j,\ 18w + 2 + 2j) \qquad j = 0, 1, \cdots, w - 1.$$

$$(2w + 1 + j,\ 4w + 1 + j,\ 13w + 2 + j;\ 3w + 1 + j,\ 16w + 3 + 2j) \qquad j = 0, 1, \cdots, w - 1.$$

$$(5w + 1 + j,\ 7w + j,\ 12w + 2 + j;\ 8w + 1 + j,\ 16w + 2 + 2j) \qquad j = 0, 1, \cdots, w - 1.$$

$$(w,\ 11w,\ 12w + 1;\ 8w,\ 16w + 1).$$

For example, let $w = 2$, the 8 S_i's are:

(1, 21, 30; 13, 39)

(3, 23, 31; 19, 38)

(4, 24, 32; 20, 40)

(5, 9, 28; 7, 35)

(6, 10, 29; 8, 37)

(11, 14, 26; 17, 34)

(12, 15, 27; 18, 36)

(2, 22, 25; 16, 33).

2. $n = 20w + 15,\ w \geqq 0.$
The S_i's $(i = 1, \cdots, 4w + 3)$ are:

$$(2+j,\ 10w+7+j,\ 14w+11+j;\ 6w+6+j,\ 18w+14+2j)\qquad j = 0, 1, \cdots, w-1.$$

$$(w+2+j,\ 11w+7+j,\ 15w+11+j;\ 9w+7+j,\ 18w+13+2j)\qquad j = 0, 1, \cdots, w-1.$$

$$(2w+3+j,\ 4w+3+j,\ 13w+10+j;\ 3w+3+j,\ 16w+13+2j)\qquad j = 0, 1, \cdots, w-1.$$

$$(5w+5+j,\ 7w+6+j,\ 12w+10+j;\ 8w+7+j,\ 16w+14+2j)\qquad j = 0, 1, \cdots, w-1.$$

$$(1,\ 5w+3,\ 5w+4;\ 2w+2,\ 8w+6)$$

$$(6w+5,\ 12w+7,\ 16w+12;\ 14w+10,\ 20w+14)$$

$$(12w+8,\ 12w+9,\ 16w+11;\ 20w+13,\ 20w+15).$$

For example, $w = 1$ gives the following $A(35, 5, 1)$:

$$(2, 17, 25; 12, 32)$$
$$(3, 18, 26; 16, 31)$$
$$(5, 7, 23; 6, 29)$$
$$(10, 13, 22; 15, 30)$$
$$(1, 8, 9; 4, 14)$$
$$(11, 19, 28; 24, 34)$$
$$(20, 21, 27; 33, 35).$$

VII. Construction of $A(n, 6, \lambda)$. We may assume $\lambda \equiv 1 \pmod 2$, otherwise $A(n, 6, \lambda)$ can be constructed by the even-even lemma. We have two cases.

Case 1. $(6, \lambda) = 1$.

Again from Corollary 2 of Lemma 3, we need only construct $A(n, 6, 1)$ for $n \in N(6, 1)$. From Lemma 1, $n \in N(6, 1)$ if $n = 12w$, $w \geqq 1$. It is easy to construct $A(12w, 6, 1)$ and one simple way is as follows:

Let $(a, b, c; d, e, f)$ and $(a', b', c'; d', e', f')$ be the rows for any $A(12, 6, 1)$, such as

$$(1, 3, 7; 2, 4, 5) \quad \text{and} \quad (6, 10, 12; 8, 9, 11).$$

Then the rows for $A(12w, 6, 1)$ may be

$$(a + 12j,\ b + 12j,\ c + 12j;\ d + 12j,\ e + 12j,\ f + 12j),$$
$$(a' + 12j,\ b' + 12j,\ c' + 12j;\ d' + 12j,\ e' + 12j,\ f' + 12j)$$
$$j = 0, 1, \cdots, w - 1.$$

Case 2. $(6, \lambda) = 3$.

We need only construct $A(n, 6, 3)$ for $n \in N(6, 3)$ which consist of all numbers $n = 4w$, $w = 1, 2, \cdots$. The construction again is easy and analogous to case 1. Let $A(4, 6, 3)$ be $(a, b, c; d, e, f)$ and $(a', b', c'; d', e', f')$, say, $(1, 1, 4; 1, 2, 3)$ and $(2, 3, 4; 2, 3, 4)$. Then the rows for $A(4w, 6, 3)$ may be:

$$(a + 4j,\ b + 4j,\ c + 4j;\ d + 4j,\ e + 4j,\ f + 4j),$$
$$(a' + 4j,\ b' + 4j,\ c' + 4j';\ d' + 4j,\ e' + 4j,\ f' + 4j)$$
$$j = 0, 1, \cdots, w - 1.$$

This completes the proof of Theorem 1.

VIII. Applications.

A. *Construction of neighbor designs.* Rees **[2]** introduced the concept and name of neighbor designs for use in serology. He wrote, "A technique used in virus research requires the arrangement in circles of samples from a number of virus preparations in such a way that over the whole set a sample from each virus preparation appears next to a sample from every other virus preparation." Figure 1 shows such an arrangement of a set of antigens (virus preparations) around an antiserum on a circular plate. On the plate, every antigen has as neighbors two other antigens.

More generally, a neighbor design is an arrangement of v kinds of objects on b such plates, each containing k objects, such that, each object is a neighbor of every other object exactly λ times. The same object may appear more than once in a plate but adjacent (neighboring) objects must be distinct.

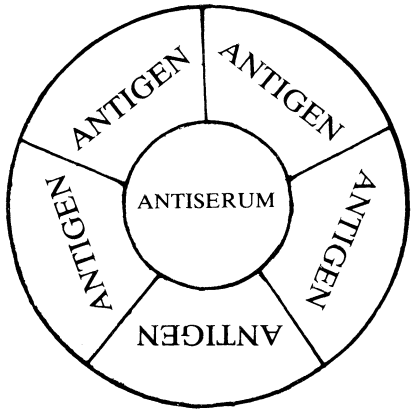

Figure 1

It can be easily shown that each object must appear exactly $r = \lambda(v-1)/2$ times on the b plates in a neighbor design. By simple counting, we also have $vr = bk$, hence $b = \lambda v(v-1)/2k$. Thus a necessary condition for a neighbor design with parameters v, k, λ to exist is that $k > 1$, and both r and b be positive integers. Denoting such a neighbor design by $ND(v, k, \lambda)$, it is clear that $ND(v, k, \lambda) \Rightarrow ND(v, k, t\lambda)$ since we can duplicate each plate of $ND(v, k, \lambda)$ t times.

Rees constructed $ND(v, k, \lambda)$ for every odd v with $k = v$, and for every $v \leqq 41$, $k \leqq 10$, $\lambda = 1$; some by using Galois field theory and others just by trial and error. Hwang **[2]** constructed some infinite classes of neighbor designs with parameters as follows:

1. $k > 2$, $v = 2k+1$, $\lambda = 1$.
2. $k \equiv 0 \pmod 2$, $v = 2^i k + 1$, $i = 1, 2, \cdots$, $\lambda = 1$.
3. $k \equiv 0 \pmod 4$, $v = 2jk + 1$, $j = 1, 2, \cdots$, $\lambda = 1$.

We show below how 2-balanced (n, k, λ) arrays may be used to construct a new class of $ND(v, k, \lambda)$'s.

Without loss of generality, we may let the set of v kinds of objects be designated by $V \equiv \{1, 2, \cdots, v\}$ and a plate B containing k objects $b_1, b_2, \cdots, b_k$, (not necessarily all distinct) from V, and arranged circularly in that order, by the sequence $(b_1, b_2, \cdots, b_k)$. For convenience, let $b_{k+1} = b_1$ so that the k pairs of neighbors in B are (b_{j+1}, b_j), $j = 1, 2, \cdots, k$. Let $B^{(l)} = (b_1^{(l)}, b_2^{(l)}, \cdots, b_k^{(l)})$, $l = 0, 1, \cdots, v-1$, be a set of v plates cyclically generated from B by the rule $b_j^{(l)} = b_j + l$ where $b_j^{(l)}$ is reduced mod v if necessary to an element of V. We call $B = B^{(0)}$ the *base plate* and the set of plates $B^{(l)}$, $l = 0, 1, \cdots, v-1$, the *full cyclic set of plates generated by* B and denote it by $[B]$. We may also view $[B]$ as a $v \times k$ matrix where the lth row is $B^{(l)}$. The columns of $[B]$ are then some cyclic permutations of $(1, 2, 3, \cdots, v)$.

For example, if $v = 7$ and $B = (1, 2, 4)$, then

$$[B] = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \\ 4 & 5 & 7 \\ 5 & 6 & 1 \\ 6 & 7 & 2 \\ 7 & 1 & 3 \end{pmatrix}.$$

For any pair of distinct objects a and b in V, let $d(a, b)$, the *distance between a and b*, be the *smallest positive residue mod v* which is congruent to either $(a - b)$ or $(b - a)$. For example, if $v = 7$, then $d(2, 6) = 3$, $d(1, 7) = 1$. We may also visualize $d(a, b)$ as the distance between a and b on the Hamiltonian cycle $(1, 2, \cdots, v, 1)$.

It is clear that $1 \leqq d(a, b) \leqq [v/2]$ and that in $[B]$, $d(b_{j+1}^{(l)}, b_j^{(l)}) = d(b_{j+1}, b_j)$ for all $l = 0, 1, \cdots, v - 1$. Furthermore, every pair $v_1, v_2 \in V$ with $d(v_1, v_2) = d(b_{j+1}, b_j)$ appears exactly once as neighbors in the $j + 1$th and jth column of $[B]$, except when $d(v_1, v_2) = d(b_{j+1}, b_j) = v/2$, ($v$ must then be even), where every such pair appears exactly twice. We state this result as

LEMMA 11. *Given any plate* $B = (b_1, b_2, \cdots, b_k)$, *let* $D_B \equiv \{d(b_{j+1}, b_j),\ j = 1, 2, \cdots, k\}$. *If* v_1, v_2 *are any two distinct objects in* V, *then the number of times* v_1, v_2 *appear as neighbors in* $[B]$ = *the number of occurrences of* $d(v_1, v_2)$ *in the multiset* D_B, *except for* $d(v_1, v_2) = v/2$, *where it is doubled.*

With Lemma 11, we prove:

THEOREM 2. $A(n, k, \lambda) \Rightarrow ND(2n + 1, k, \lambda)$.

Proof. For each row $S = (a_1, a_2, \cdots, a_k)$ of $A(n, k, \lambda)$ let $S = P^+N$ be a balanced partition of S. (The order in which the a_i's are written is immaterial.) Define $S^* = (d_1, d_2, \cdots, d_k)$ where $d_i = a_i$ if $a_i \in P$, and $d_i = -a_i$ if $a_i \in N$. Let $b_1, b_2, \cdots, b_k, b_{k+1}$ be constructed from S^* as follows:

$$b_1 = 1,$$

$$b_{j+1} = b_j + d_j = b_1 + \sum_1^j d_i \qquad j = 1, 2, \cdots, k,$$

reduced mod v if necessary so that $b_{j+1} \in V$. We have clearly, $b_{j+1} \neq b_j$, $b_{k+1} = b_1$ (since $\|S^*\| = 0$), and thus $B = (b_1, b_2, \cdots, b_k)$ form a plate with $D_B \equiv S$ since $d(b_{j+1}, b_j) = |d_j| = a_j$.

We assert that the vm plates $[B_1], \cdots, [B_m]$ constructed from rows $S_1, \cdots, S_m$ of $A(n, k, \lambda)$ in this manner do indeed form a $ND(2n + 1, k, \lambda)$. Since v is odd, by Lemma 11, every two distinct objects v_1, v_2 in V appear as neighbors in $\{[B_i], i = 1, 2, \cdots, m\}$ exactly the total number of times $1 \leqq d(v_1, v_2) \leqq n$ appears in $\{S_i, i = 1, 2, \cdots, m\}$, which is exactly λ.

B. *Coverings of Complete Multigraphs by k-Cycles.* By a complete multigraph K_v^λ we mean a multigraph on v vertices without self-loops and having exactly λ edges joining every pair of distinct vertices v_1, v_2. When $\lambda = 1$, this reduces to just the complete graph K_v. A plate B of a neighbor design $ND(v, k, \lambda)$ may be interpreted as a k-cycle $(b_1, b_2, \cdots, b_k, b_1)$ on K_v^λ and thus the b plates of $ND(v, k, \lambda)$ induces an edge cover of K_v^λ by b k-cycles. If the objects in B are distinct, then the cycle $(b_1, b_2, \cdots, b_k, b_1)$ is elementary. If the B's are constructed from the S's of an $A(n, k, \lambda)$ as described in Theorem 2, then the objects in B and all v plates in $[B]$ are distinct if and only if no proper substring T of $S^* = (d_1, d_2, \cdots, d_k)$ has $\|T\| \equiv 0 \bmod v$. Thus we have:

THEOREM 3. *Let* (n, k, λ) *satisfy the necessary conditions of Theorem* 1. *Then* K_{2n+1}^λ *can be covered by* $b = \lambda n(2n+1)/k$ k-*cycles. In particular, for* $\lambda = 1$, $3 \leqq k \leqq 6$, *the* b k-*cycles can be chosen to be elementary.*

Proof. The first part of the theorem follows the previous discussions and we need only verify the last statement. An inspection of the constructions for $A(n, k, 1)$ for $3 \leqq k \leqq 6$ shows that the S_i^*'s have no proper substring $\|T\| \equiv 0 \bmod v$, and hence the k-cycles so constructed are elementary.

REMARK 1. A more detailed analysis of our construction for $A(n, k, \lambda)$ shows that we can choose the S_i's such that there exist corresponding S_i^*'s free of proper substrings T with $\|T\| \equiv 0 \pmod v$ more generally than the cases stated in Theorem 3. For example, this is true for $k = 3, 4$, any λ, and $k = 5$, $(5, \lambda) = 1$, λ odd, or $\lambda = 2$. For $k = 4$, we have to (and can) avoid $S = (a, a; a, a)$ while $S^* = (a, b, -a, -b)$ is acceptable. However, the interesting problem of constructing $A(n, k, \lambda)$'s for all $n \in N(k, \lambda)$ with this property so that we can cover K_{2n+1}^λ with elementary k-cycles in this manner is still open.

REMARK 2. $ND(v, 3, \lambda)$'s are also triple systems. In **[3]**, we constructed triple systems directly for all parameters satisfying the necessary conditions $\lambda(v-1) \equiv 0 \pmod 2$, $\lambda v(v-1) \equiv 0 \pmod 6$, which is more general than the values $v = 2n+1, 3, \lambda$ constructable from $A(n, 3, \lambda)$'s in this paper. The neighbor designs constructed here have parameters satisfying $v = 2n+1$, $\lambda n \equiv 0 \pmod k$, $\lambda n(n+1) \equiv 0 \pmod 4$, $\lambda \equiv 0 \pmod 2$ when $k = 2$, and $v \neq 5$ when $k = 3$. In a forthcoming paper **[4]**, we use the results given here together with other constructions to show that we can always construct a neighbor design for all values of the parameters satisfying the following obvious necessary conditions: $\lambda(v-1) \equiv 0 \pmod 2$, $\lambda v(v-1) \equiv 0 \pmod{2k}$, $\lambda \equiv 0 \pmod 2$ if $k = 2$, and $k \equiv 0 \pmod 2$ if $v = 2$.

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