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NONSINGULAR DEFORMATIONS OF A DETERMINANTAL SCHEME

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We will be considering an affine algebraic scheme X over a field k, which is determinantal, defined by the vanishing of the $l \times l$ minors of a matrix R.

We will show that deforming the constant and linear terms of the entries in the matrix R gives an almost everywhere flat deformation of X, and that under certain simple conditions, and in particular if the dimension of X is sufficiently low, this deformation has generically nonsingular fibers.

Essentially the same results were obtained simultaneously by D. Laksov [3] using more general theorems on transversality of mappings. He quotes a result of T. Svanes indicating that the codimension result, identical in both versions, is the best obtainable (see Example 3).

This article is a generalization of an earlier result about nonsingular deformations of Cohen-Macaulay schemes of codimension 2 (Schaps [4]). Moreover, since determinantal schemes were introduced by Macaulay as a generalization of complete intersection, the theorem proven in this paper can be regarded as a generalization of Bertini's theorem, that the generic deformation of a complete intersection is nonsingular.

The precise definition of a determinantal scheme is as follows:

DEFINITION. An affine scheme $X = \operatorname{Spec}(k[Z_1, \cdots, Z_q]/J)$ is determinantal if J is generated by all the $l \times l$ minors of an $m \times n$ matrix R of polynomials, and X is equidimensional of codimension $(m - l + 1)(n - l + 1)$.

On the course of the theorem, we will need to use the generic determinantal scheme, constructed as follows: Let $Y = (Y_{ij})$, $i = 1, \cdots, m$, $j = 1, \cdots n$, be a set of indeterminates, and let P_l^Y be the ideal generated in $k[Y]$ by the $l \times l$ minors of the matrix $[Y_{ij}]$. Then it is known that P_l^Y is a prime ideal of height $(m - l + 1)(n - l + 1)$. This number is thus the maximal codimension that can be obtained by a scheme generated by minors of this order in an $m \times n$ matrix. We will use a recent result by Hochster and Eagon [2], that every determinantal scheme is Cohen-Macaulay.

We will now proceed to the main theorem and its corollary. One example is included with the proof, a second reserved to the end.

THEOREM. *Let* $X = \mathrm{Spec}\,(B)$ *be a determinantal scheme,* B *being the quotient of* $k[Z] = k[Z_1, \cdots, Z_q]$ *by an ideal generated by the* $l \times l$ *minors of some* $m \times n$ *matrix* $R = [r_{ij}]$. *If* l *equals* 1, *or* $m = n = l$ *or* $q < (m - l + 2)(n - l + 2)$, *then* X *has a flat deformation whose generic fiber is nonsingular.*

Proof. If $m = n = l$, X is just a hypersurface, and we replace R by the 1×1 matrix $[\det R]$. Let $\tilde{X}$ be the algebraic family of deformations of X defined by the $l \times l$ minors of the deformed matrix $\tilde{R} = [\tilde{r}_{ij}]$, where

$$\tilde{r}_{ij} = r_{ij} + U_{ij} + \sum_{t=1}^{q} V^t_{ij} Z_t ,$$

the U and V being indeterminates. $\tilde{X}$ is itself determinantal, isomorphic to the product $\mathrm{Spec}\,(k[Y]/P^Y_l) \times \mathrm{Spec}\,(k[V, Z])$, of the generic determinantal scheme of type (m, n, l) with the affine space of dimension $q(mn + 1)$. The isomorphism is induced by $\phi\colon k[Y, V, Z] \to k[U, V, Z]$, with

$$\begin{aligned} \phi\colon Y_{ij} &\longrightarrow \tilde{r}_{ij}(U, V, Z) \\ \phi, \phi^{-1}\colon V^t_{ij} &\longrightarrow V^t_{ij} \\ \phi, \phi^{-1}\colon Z_t &\longrightarrow Z_t \\ \phi^{-1}\colon U_{ij} &\longrightarrow Y_{ij} - \tilde{r}_{ij}(0, V, Z) . \end{aligned}$$

This gives $\operatorname{codim} \tilde{X} = (m - l + 1)(n - l + 1)$, and also allows us to determine the singular locus of $\tilde{X}$, which will induce singularities on the fibers. (The remaining fiber singularities come from tangencies between $\tilde{X}$ and the fiber of the ambient space.) The singular locus of the generic determinantal scheme is also determinantal, generated by the $(l - 1) \times (l - 1)$ minors. Thus the singular locus of $\tilde{X}$ is the set on which rank $\tilde{R} < l - 1$. For $l = 1$, $\tilde{X}$ is a nonsingular complete intersection. For $l > 1$, the singular locus has codim $(m - l + 2)(n - l + 2)$. Since $q < (m - l + 2)(n - l + 2)$, in that case, by hypothesis, the projection of this locus has positive codimension, so in either case there is an open subset of $S = \mathrm{Spec}\,(k[U, V])$ over which rank $\tilde{R} \geqq l - 1$.

Before proceeding to the proof of the second part of the theorem, we will introduce some new notation. Let $\mu \subset \{1, \cdots, m\}$ and $\nu \subset \{1, \cdots, n\}$ designate sets of rows or columns, respectively, and let

$\#$ prefixed to a set designate its cardinality. Then if $\#\mu = \#\nu = l$, we let $f_{\mu\nu}$ be the subdeterminant of $\tilde{R}$ with rows μ and columns ν (without any adjustment of sign). Similarly if $\#\mu = \#\nu = l - 1$, we let $b_{\mu\nu}$ be the subdeterminant with rows μ and columns ν.

We now assume that rank $\tilde{R} \geqq l - 1$ over some subset N of S, open in the Zariski topology. By making a translation of coordinates if necessary, we may assume that the origin is in N. Let π be the projection onto S.

LEMMA 2. *Under these hypotheses, $\pi^{-1}(N)$ is locally a complete intersection, generated in each open affine $\tilde{X}_b$, $b = b_{\mu_0\nu_0}$, by $(1/b)f_{\alpha\beta}$ for all α, β such that $\mu_0 \subset \alpha$, $\nu_0 \subset \beta$.*

Proof. It is clear from our hypothesis on the rank of $\tilde{R}$ that the sets $\tilde{X}_b$, $b = b_{\mu_0\nu_0}$ with $\#\mu_0 = \#\nu_0 = l - 1$, do indeed cover $\pi^{-1}(N)$. We will fix μ_0 and ν_0. Let I be the ideal in $k[U, V, Z]_b$ generated by the $(m - l + 1)(n - l + 1)$ functions

$$h_{ij} = \pm b^{-1} f_{\alpha\beta}$$

where $\alpha = \mu_0 \cup \{i\}$, $\beta = \mu_0 \cup \{j\}$, and the sign is so adjusted that in the expansion of $f_{\alpha\beta}$ along the row i, $\tilde{r}_{ij}b$ will have positive coefficient. If $i \in \mu_0$ or $j \in \nu_0$, let us write $h_{ij} = 0$.

Let $f_{\mu\nu}$ be any other $l \times l$ subdeterminant. We will show that $f_{\mu\nu} \in I$. Decomposing $f_{\alpha\beta}$ along the ith row, and dividing by b, we have, after the adjustment of sign

$$h_{ij} = \tilde{r}_{ij} + b^{-1} \sum_{t \in \nu_0} \pm\, b_{\mu_0\sigma} \tilde{r}_{it}\ ,$$

where $\sigma = (\nu_0 \cup \{j\}) - \{t\}$, and the sign of $\tilde{r}_{it}$ is $(-1)^{\alpha(t) - \alpha(j)}$, where $\alpha(t)$ and $\alpha(j)$ are the respective positions of these numbers in α, regarded as an ordered set. Let us denote by $\tilde{r}_t$ the tth column of $\tilde{R}$ and add to the jth column, for any $j \notin \nu_0$, the partial sum

$$b^{-1} \sum_{t \in \nu_0 \cap \nu} \pm\, b_{\mu_0\sigma} \tilde{r}_t\ .$$

The entry in row i of column j will then be

$$(*) \qquad h_{ij} - b^{-1} \sum_{t \in \nu_0 - \nu} \pm\, b_{\mu_0\sigma} \tilde{r}_{it}\ .$$

Since we have added multiples of columns from the index set ν, and in fact from the set $\nu \cap \nu_0$ of unchanged columns, the value of the minor $f_{\mu\nu}$ will be unaffected by this operation which replaces r_{ij} by $(*)$, applied to the $d = \#(\nu - \nu_0)$ columns of $\nu - \nu_0$. We now use the multilinearity of the determinant to decompose the sums in these d columns. Since $^{\#}(\nu_0 - \nu)$ is $d - 1$, $f_{\mu\nu}$ is thus the sum of d^2 determinants, each of which either contains a column h_j, or contains d columns which are multiples of the $d - 1$ columns $\tilde{r}_t$, $t \in \nu_0 - \nu$. Since these latter determinants all vanish, we have $f_{\mu\nu} \in I$.

EXAMPLE 1. Consider the determinantal scheme generated by the 2×2 minors of the matrix

$$\begin{bmatrix} Z_1 & Z_2 & Z_3 \\ 1 & Z_4 & Z_5 \\ 1 & 1 & Z_6 \end{bmatrix}.$$

We require the codimension to be $(3 - 2 + 1)(3 - 2 + 1) = 4$, and in fact we have a complete intersection generated by $Z_4 = 1$, $Z_1 = Z_2$, $Z_5 = Z_6$, $Z_3 = Z_1 Z_6$. We construct the matrix $\tilde{R} = [\tilde{r}_{ij}]$, where, for example,

$$\tilde{r}_{11} = Z_1 + U_{11} + V_{11}^1 Z_1 + \cdots + V_{11}^6 Z_6 .$$

Let $\mu_0 = \{3\}$, and $\nu_0 = \{3\}$.

Thus

$$b = \tilde{r}_{33} = Z_6 + U_{33} + \cdots$$

and

$$h_{11} = \tilde{r}_{11} - \frac{r_{31}}{b}\tilde{r}_{13} \qquad h_{12} = \tilde{r}_{12} - \frac{\tilde{r}_{32}}{b}\tilde{r}_{13}$$

$$h_{21} = \tilde{r}_{21} - \frac{\tilde{r}_{31}}{b}\tilde{r}_{23} \qquad h_{22} = \tilde{r}_{22} - \frac{\tilde{r}_{32}}{b}\tilde{r}_{23} .$$

Consider $f_{\mu\nu}$, $\mu = \{1, 2\}$, $\nu = \{1, 2\}$

$$\begin{aligned} f_{\mu\nu} &= \begin{vmatrix} \tilde{r}_{11} & \tilde{r}_{12} \\ \tilde{r}_{21} & \tilde{r}_{22} \end{vmatrix} \\ &= \begin{vmatrix} h_{11} + \frac{\tilde{r}_{31}}{b}\tilde{r}_{13} & h_{12} + \frac{\tilde{r}_{32}}{b}\tilde{r}_{13} \\ h_{21} + \frac{\tilde{r}_{31}}{b}\tilde{r}_{23} & h_{22} + \frac{\tilde{r}_{32}}{b}\tilde{r}_{23} \end{vmatrix} \\ &= \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} + \frac{\tilde{r}_{32}}{b}\begin{vmatrix} h_{11} & \tilde{r}_{13} \\ h_{21} & \tilde{r}_{23} \end{vmatrix} \\ &\quad \frac{\tilde{r}_{31}}{b}\begin{vmatrix} \tilde{r}_{13} & h_{12} \\ \tilde{r}_{23} & h_{22} \end{vmatrix} + \frac{\tilde{r}_{31}\tilde{r}_{32}}{b \cdot b}\begin{vmatrix} \tilde{r}_{13} & \tilde{r}_{13} \\ \tilde{r}_{23} & \tilde{r}_{23} \end{vmatrix} \end{aligned}$$

Therefore $f_{\mu\nu} \in I$.

We now continue with the proof of the main theorem. We let $c = (m - l + 1)(n - l + 1)$ be the codimension of X. We have $c \leqq q$. Define a scheme $\tilde{V}$ to be the subscheme of $\tilde{X}_b$ defined by the vanishing of the $c \times c$ minors of the Jacobian matrix $[\partial h_{ij}/\partial Z_t]$ for $i \notin \mu_0$, $j \notin \nu_0$. By the Jacobian criterion, the singular scheme of any fiber of $\tilde{X}_b$ is supported on its intersection with $\tilde{V}$. (EGA IV, 0.20.5.14). If therefore we can show that codim $\tilde{V} \geqq q + 1$, we will know that the fibers are nonsingular except over a proper subscheme of S, for if the closure of the projection of $\tilde{V}$ did not have positive codimension, there would be an open subset of the parameter space over which the fiber of $\tilde{V}$ would be nonempty, and thus of codimension less than or equal to q. This is possible only if $\tilde{V}$ has codimension less than or equal to q over this set. Take indeterminants W^t_{ij}, $1 \leqq t \leqq q$, $i \notin \mu_0$, $j \notin \nu_0$, corresponding to the entries in the $c \times q$ Jacobian matrix, and indeterminants Y_{ij} corresponding to the c generators h_{ij} of J_b. Let

$$\mathscr{W} = \operatorname{Spec} k[W, Y] .$$

Now $c \leqq q$, and thus P^W_c is an ideal of height $q - c + 1$. Therefore $P^W_c + (Y)$ is an ideal of height $q + 1$. Thus

$$\operatorname{codim}_{\mathscr{W}} \operatorname{Spec} k[W, Y]/P^W_c + (Y)$$

is $q + 1$. Let $\hat{U}$, $\hat{V}$ be the subsets of U, V consisting of all U_{ij}, V^t_{ij} such that $i \in \mu_0$ or $j \in \nu_0$. We wish to construct an isomorphism

$$(k[W, Y]/(P^W_c + (Y)))[\hat{U}, \hat{V}, Z]_b \longrightarrow k[U, V, Z]_b/J_b .$$

Here $b = b_{\mu_0\nu_0}$ as always, and thus $b \in k[\hat{U}, \hat{V}, Z]$. We will map

$$\begin{aligned} Z &\longrightarrow Z \\ \hat{U} &\longrightarrow \hat{U} \\ \hat{V} &\longrightarrow \hat{V} . \end{aligned}$$

Hence, the invertibility of b will be preserved.

As for the remaining indeterminates, we send

$$\begin{aligned} W^t_{ij} &\longrightarrow \partial h_{ij}/\partial Z_t \\ Y_{ij} &\longrightarrow h_{ij} . \end{aligned}$$

To construct the inverse mapping ϕ we write

$$\begin{aligned} h_{ij} &= \tilde{r}_{ij} + g_{ij}(\hat{U}, \hat{V}, Z) \\ \partial h_{ij}/\partial Z_t &= \partial r_{ij}/\partial Z_t + V^t_{ij} + \partial g_{ij}/\partial Z_t . \end{aligned}$$

Therefore we set

$$\phi(V_{ij}^t) = W_{ij}^t - \partial r_{ij}/\partial Z_t - \partial g_{ij}/\partial Z_t$$
$$\phi(U_{ij}) = Y_{ij} - r_{ij} - \sum \phi(V_{ij}^t) Z_t - g_{ij} .$$

Since the mappings also establish an isomorphism between the ambient spaces

$$\mathscr{X}_b = \operatorname{Spec}(k[U, V, Z]_b)$$

and

$$\mathscr{Y} = \operatorname{Spec}[W, Y, \hat{U}, \hat{V}, Z]_b ,$$

it is clear that

$$\begin{aligned}\operatorname{codim}_{\mathscr{X}_b} V &= \operatorname{codim}_{\mathscr{Y}} \operatorname{Spec}(k[W, Y, \hat{U}, \hat{V}, Z]/P_c^W + (Y))_b \\ &= \operatorname{codim}_{\mathscr{W}} \operatorname{Spec}(k[W, Y]/P_c^W + (Y)) \\ &= q + 1 .\end{aligned}$$

It remains to show that we can restrict $\tilde{X}$ to a flat deformation of $\tilde{X}$. We have proven above that the generic fiber of $\tilde{X}$ is locally the intersection of hypersurfaces. Since these are generically in general position, the generic fiber is nonempty and thus of codimension equal to the codimension of X. (Shafarevich, Chap. 1. §6 [5]).

Thus if we let W be the constructible subset of S over which the fibers have this codimension c, W will contain the origin, 0, and also a Zariski open subset of S. If $0 \in \overline{S - W}$, let m be the maximum dimension of the components of $\overline{S - W}$ containing 0, and let H be a regular subspace of S through 0 of codimension m. By choosing H in general position, we can insure that any properties of the fibers over an open subset of S will also be true of the generic fiber of $\tilde{X}$ over H, in particular, smoothness. If $0 \notin \overline{S - W}$, we will take H to be S. The restriction of $\tilde{X}$ to $H \cap W$ has equidimensional fibers, $H \cap W$ is open since the intersection of H with $\overline{S - W}$ consists of isolated points, and the restriction of X to this regular scheme is equidimensional, hence determinantal, hence Cohen-Macaulay. Since we may assume the generic fiber over H to be smooth, the theorem now follows from the lemma quoted below, a proof of which is included in Schaps [4]. The local version is in EGA IV, 6.1.5.

LEMMA. *Given a morphism of algebraic schemes $f: X \to Y$ of finite type, Y regular, X Cohen-Macaulay, and the closed fibers of X over Y equidimensional, then the map f is flat.*

EXAMPLE 2. If k is an infinite field, there is a large and important class of reduced schemes which can be represented as determinantal schemes, the union of all linear coordinate schemes of dimension p in q space, for $p < q$. One simply chooses a $q \times (p+1)$ matrix $A = [a_{ij}]$ over k such that all its maximal minors are nonzero, and lets $R = [a_{ij}Z_i]$. Let $s = p + 1$. The $\binom{q}{s}$ maximal minors are scalar multiples of the monomials ΠZ_i of degree s, and the scheme is thus supported on the union of the spaces.

$$Z_{i_1} = \cdots = Z_{i_{q-p}} = 0 \ ,$$

with distinct i_j.

The theorem tells us that this scheme has non-singular deformations for $q = p + 1$, a hypersurface, and for $q < 2(q - p + 1)$, that is, $q > 2(p - 1)$. D. Mumford conjectures that these are the only smoothable cases.

EXAMPLE 3. A counter-example for the case $q = (m - l + 2)(n - l + 2)$, $l > 1$, is the scheme generated by the minors of the matrix

$$\begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}$$

where $R = Y_{ij}$, $i \leqq m - l + 2$, $j \leqq n - l + 2$, and I is the identity matrix of order $l - 2$. X is actually the generic determinantal scheme of type $(m - l + 2, n - l + 2, 2)$, and therefore has an isolated singularity. By a result of T. Svanes (thesis, M.I.T., 1971), the generic member of any flat family deforming X will also have an isolated singularity.

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REFERENCES

1. M. Hochster and J. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math., Oct., 1971.

2. A. Grothendieck, *Elements de Geometrie Algebrique* (EGA) IV, §24, Institute Haute Etude Scientifique, 1960, 1961.

3. D. Laksov, *Deformations of determinatal schemes*, preprint, 1974, University of Oslo.

4. M. Schaps, *Deformations of Cohen-Macaulay schemes of codimension* 2, *and non-singular deformations of space curves*, Tel-Aviv University. To appear.

5. I. R. Shaferevich, *Foundations of Algebraic Geometry*. Russian Mathematical Surveys, Vol. **24**, No. 6, 1969, London Mathematical Society.

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