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DELOOPING THE CONTINUOUS K-THEORY OF A VALUATION RING

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In this note the continuous algebraic K-theory groups of a complete discrete valuation ring are described as the inverse limit of the ordinary algebraic K-theory of its finite quotient rings.

In [4] we defined continuous algebraic K-theory groups K_i^{top}, $i \geqq 2$, both for a complete discrete valuation ring $\mathcal{O}$ with finite residue field of positive characteristic p and for its fraction field and proved that K_2^{top} agrees with the fundamental group of the special linear group as defined in [2] by means of universal topological central extensions. The definition of K_i^{top} in [4] is in terms of BN-pairs and is similar to the theory K_i^{BN} of [5] which is known [6] to deloop to ordinary algebraic K-theory. The purpose of this note is to deloop $K_i^{\mathrm{top}}(\mathcal{O})$ in the sense of the following result: Let $\mathscr{P} \cap \mathcal{O}$ be the maximal ideal and let K_i be the algebraic K-theory groups of Quillen [3].

THEOREM. *For $i \geqq 2$ there is a natural isomorphism*

$$K_i^{\mathrm{top}}(\mathcal{O}) \cong \varprojlim_n K_i(\mathcal{O}/\mathscr{P}^n) .$$

In a forthcoming paper of the author and R. J. Milgram, this equaation allows us to use the continuous cohomology of $\mathrm{SL}(l, \mathcal{O})$ to compute the rank of the free part of $K_i^{\mathrm{top}}(\mathcal{O})$ as a module over the p-adic completion of the rational integers.

In § 2 a step in the proof of this theorem is used to describe the homotopy fiber of $BE(A)^+ \rightarrow BE(A/J)^+$ where J is an ideal in a commutative ring A such that $1 + J \subset A^*$. At least, we construct a space $B\{U_F(A, J)\}^+$ whose homotopy groups fit into the appropriate exact sequence.

Actually, in this paper we shall let

$$K_i^{\mathrm{top}}(\mathcal{O}) = \varprojlim_{m|n} [\varinjlim_{l} \pi_{i-1} \mathrm{SL}_n^{\mathrm{top}}(l, \mathcal{O})]$$

whereas in [4] the order of the inverse and direct limits is reversed. The above definition is perhaps better as it still gives the main results of [4]. To see the two are the same one would have to prove that

$$\longrightarrow \pi_{i-1} \mathrm{SL}_n^{\mathrm{top}} (l, \mathcal{O}) \longrightarrow \pi_{i-1} \mathrm{SL}_n^{\mathrm{top}} (l + 1, \mathcal{O}) \longrightarrow \cdots$$

eventually stabilizes to an isomorphism.

The theorem makes it clear that the natural map $K_i(\mathscr{O}) \to K_i^{\mathrm{top}}(\mathscr{O})$ comes from the ring maps $\mathscr{O} \to \mathscr{O}/\mathscr{P}^n$.

1. **Delooping.** Let n and l be fixed. The main step is to prove

PROPOSITION 1.1. *There is a natural homotopy equivalence*

$$\mathrm{SL}^{ab}(l, \mathscr{O}/\mathscr{P}^n) \cong \mathrm{SL}_n^{\mathrm{top}}(l, \mathscr{O})$$

such that if $m|n$ there is a homotopy commutative diagram

$$(*) \qquad \begin{array}{ccc} \mathrm{SL}^{ab}(l, \mathscr{O}/\mathscr{P}^n) & \cong & \mathrm{SL}_n^{\mathrm{top}}(l, \mathscr{O}) \\ \downarrow & & \downarrow \\ \mathrm{SL}^{ab}(l, \mathscr{O}/\mathscr{P}^m) & \cong & \mathrm{SL}_m^{\mathrm{top}}(l, \mathscr{O})\,. \end{array}$$

See [4] for notation. From this result and [6] we see that for $i \geqq 2$

$$\begin{aligned} \varinjlim_{l} \pi_{i-1}\,\mathrm{SL}_n^{\mathrm{top}}(l, \mathscr{O}) &= \varinjlim_{l} \pi_{i-1}\,\mathrm{SL}^{ab}\, l, \mathscr{O}/\mathscr{P}^n) \\ &= \pi_{i-1}\,\mathrm{SL}^{ab}(\mathscr{O}/\mathscr{P}^n) \\ &= K_i(\mathscr{O}/\mathscr{P}^n)\,. \end{aligned}$$

Here $\mathrm{SL}^{ab}(A)$ of [4] is the same as $E^{BN}(A)$ of [5]. The main theorem now follows from commutativity of (*).

For simplicity of notation let $S_l = \mathrm{SL}^{ab}(l, \mathscr{O}/\mathscr{P}^n)$ and $T_l = \mathrm{SL}_n^{\mathrm{top}}(l, \mathscr{O})$. Let P^l(resp. Q^l) be the complex whose k-simplices are $(k+1)$-tuples $(F_0 < F_1 < \cdots < F_k)$ where F_i is a linear (resp. affine) facette or R^l. $P^l \subset S_l$ by the imbedding $F \to U_F$ and $Q^l \subset T_l$ via $F \to U_F^n$. Let $\mathrm{st}_l(\Delta) < Q^l$ be the star of Δ consisting of all affine facettes F such that $\Delta < F$. Let $K_l < T_l$ be the subcomplex whose k-simplices $(\alpha_0 \cdot U_{F_0}^n < \cdots < \alpha_k \cdot U_{F_k}^n)$ have $F_i \in \mathrm{st}_l(\Delta)$.

Now for each affine facette $F \in \mathrm{st}_l(\Delta)$ there is a unique linear facette F' which contains F such that $F < G$ implies $F' < G'$. The map $\mathrm{st}_l(\Delta) \to P^l$ sending F to F' is an isomorphism of partially ordered sets. Let $\pi\colon \mathrm{SL}(l, \mathscr{O}) \to \mathrm{SL}(l, \mathscr{O}/\mathscr{P}^n)$ be reduction modulo $\mathscr{P}^n$. We claim that

$$(1.2) \qquad \pi(U_F^n) = U_{F'}$$

for $F \in \mathrm{st}_l(\Delta)$. This is clear for the fundamental chamber $C = \{x_l + 1 > x_1 > \cdots > x_l\}$ and also for any $F < C$. For an arbitrary $F \in \mathrm{st}_l(\Delta)$ choose an element w of the linear Weyl group W_0 so that $w \cdot F < C$. Thus by [4, Lemma 3]

$$\begin{aligned}\pi(U_F^n) &= \pi(w^{-1}(wU_F^n w^{-1})w)\\ &= w^{-1}\cdot\pi(U_{w\cdot F}^n)\cdot w\\ &= w^{-1}\cdot U_{w\cdot F'}\cdot w\\ &= U_{F'}\,.\end{aligned}$$

Moreover for each $F \in \mathrm{st}_l(\Delta)$ we have

$$\pi^{-1}(U_{F'}) = U_F^n \tag{1.2'}$$

These two equations imply the correspondence

$$\alpha\cdot U_F^n \longrightarrow \pi(\alpha)\cdot U_{F'}$$

preserves order and defines a simplicial isomorphism $K_l \to S_l$. Hence to prove (1.1) it suffices to show K_l is a deformation retract to T_l.

Let $f; g$: $Q^l \to Q^l$ be two simplicial maps arising from order preserving maps of vertices.

LEMMA 1.3. *There is a triangulation $(Q_l \times I)'$ of $Q_l \times I$ as a partially ordered set which refines the standard triangulation leaving $Q^l \times 0$ and $Q^l \times 1$ fixed and there is a simplicial map w: $(Q^l \times I)' \to Q^l$ such that*

(a) *$w|Q^l \times 0 = f$ and $w|Q^l \times 1 = g$*

(b) *if $\sigma = (v_0 < \cdots < v_n)$ is a simplex of the standard triangulation $Q \times I$, $v \in \sigma$ is a vertex in the new triangulation, and $e_{ij}(\lambda)$ is in $U_{w(v_s)}^n$ for $0 \leqq s \leqq k$, then*

$$e_{ij}(\lambda) \in U_{w(v)}^n\,.$$

This is the affine analogue of Lemma 3.3 of [6] and the proof is similar. For (b) compare (B) of Lemma 4 of [4].

Now let r: $Q^l \to \mathrm{st}_l(\Delta) \subset Q^l$ be defined by

$$r(F) = \begin{cases}\text{the unique affine facette of } \mathrm{st}_l(\Delta) \text{ which is}\\ \text{contained in the same linear facette as } F.\end{cases}$$

This is an order preserving map which is the identity on $\mathrm{st}_l(\Delta)$.

LEMMA 1.4. *For each affine facette F we have $U_F^n \subset U_{r(F)}^n$.*

Proof. If $w \in W_0$, then $w\cdot r(F) = r(w\cdot F)$ and $w\cdot F_F^n\cdot w^{-1} = U_{w\cdot F}^n$; so by choosing a w such that $w\cdot F$ is contained in the closure $\bar{C}_0$ of the fundamental linear chamber $C_0 = \{x_1 > \cdots > x_l\}$ we can assume $F \subset \bar{C}_0$. In this case $r(F) = C$. When $i > j$, $e_i - e_j \geqq 0$ on F; so for the generator $e_{ij}(\lambda)$ of U_F^n the element $\lambda \in \mathcal{O}$ can be arbitrary and $e_{ij}(\lambda) \in U_C^n$. When $i < j$, $e_i - e_j \leqq 0$ on F so $k(F, e_i - e_j)_n \geqq n = k(C, e_i - e_j)_n$; hence any generator $e_{ij}(\lambda)$ of U_F^n also belongs to U_C^n.

We can now complete the proof of (1.1). Apply Lemma 1.3 in the case $f = \mathrm{id}$ and $g = r$ to get $w: (Q^l \times I)' \to Q^l$ satisfying (a) and (b). The map $\rho: T_l \to Q^l$ taking $\alpha \cdot U^n_F$ to F is nondegenerate on simplices and so is $\rho \times 1: T_l \times I \to Q^l \times I$. Therefore the triangulation $(Q^l \times I)'$ induces a subdivision $(T_l \times I)'$ of $T_l \times I$. Let $\sigma = (\alpha_0 \cdot U^n_{F_0} < \cdots < \alpha_k \cdot U^n_{F_k})$ be a simplex of T_l and let v be a vertex of $\sigma \times I$. Let $u = (\rho \times 1)(v)$. By (1.4) we have $U^n_{F_0} \subset U^n_{w(v)}$. Hence by (b) of (1.3) we still have

$$U^n_{F_0} \subset U^n_{w(v)} \tag{1.5}$$

if v is any vertex of $(\sigma \times I)'$.

Let $R: T_l \to T_l$ be defined by $R(\alpha \cdot U^n_F) = \alpha \cdot U^n_{r(F)}$. This retracts T_l onto K_l. Define a homotopy $H: (T_l \times I)' \to T_l$ from the identity to R as follows: Let v be a vertex $(\sigma \times I)'$ and let $u = (\rho \times 1)(v)$. Let

$$H(v) = \alpha_0 \cdot U^n_{w(v)} .$$

Then (1.5) shows this is independent of the choice $\alpha_0 \in U^n_{F_0}$ so we get a well defined map.

2. A fibration in K-theory.

Let A be a commutative ring and $J \subset A$ be an ideal such that $1 + J \subset A^*$. Then $K_i(A) \to K_i(A/J)$ is surjective for $i = 1, 2$. In this section we build a space $B\{U_F(A, J)\}^+$ such that for $i \geqq 2$ there is a natural exact sequence

$$\begin{aligned} \cdots \longrightarrow K_{i+1}(A/J) &\longrightarrow \pi_i B\{U_F(A, J)\}^+ \\ &\longrightarrow K_i(A) \longrightarrow K_i(A/J) \longrightarrow \cdots . \end{aligned} \tag{2.1}$$

Let P^l denote the set of linear facettes in R^l and identify P^l as a subset of P^{l+1} by the map

$$(x_1, \cdots, x_l) \longrightarrow (x_1, \cdots, x_l, x_l) .$$

Let $P^\infty = \cup_l P^l$. If $F \in P^\infty$ define the subgroup $U_F(A, J)$ of the group $E(A)$ of elementary matrices to be the one generated by

(a) $e_{ij}(\lambda)$ where $\lambda \in A$ for $e_i - e_j > 0$ on F

(b) $e_{ij}(\lambda)$ where $\lambda \in J$ for $e_i - e_j < 0$ on F

(c) diagonal matrices diag $\{1 + \lambda_1, \cdots, 1 + \lambda_r\}$ of determinant one where $\lambda_i \in J$.

If $F < G$, then $U_F(A, J) < U_G(A, J)$. When $J = 0$, we just get the groups U_F of [4] and]5]. In this case we write $U_F(A, J) = U_F(A)$. Let $\pi: E(A) \to E(A/J)$ be reduction mod J. Then as in (1.2) and (1.2)' we have

$$\pi[U_F(A, J)] = U_F(A/J) \quad \text{and} \quad \pi^{-1}[U_F(A/J)] = U_F(A, J) . \tag{2.2}$$

Let $B\{U_F(A, J)\}$ be the realization of the simplicial space which in dimension $k \geqq 0$ is the disjoint union of the spaces

$$(F_0 < \cdots < F_k) \times BU_{F_0}(A, J)$$

where $F_i \in P^\infty$. Let $E\{\alpha \cdot U_F(A, J)\}$ be defined as the pullback

$$\begin{tikzcd} E\{\alpha \cdot U_F(A, J)\} \arrow[r] \arrow[d] & EG \arrow[d] \\ B\{U_F(A, J)\} \arrow[r] & BG \end{tikzcd}$$

where $G = E(A)$. When $J = 0$ we recover $E\{\alpha \cdot U_F\}$ as in [1]. Moreover just as in [1] the space $E\{\alpha \cdot U_F(A, J)\}$ has the homotopy type of the space $E^{BN}(A, J)$ whose k-simplices are $(k+1)$-tuples

$$\sigma_0 \cdot U_{F_0}(A, J) < \cdots < \alpha_k \cdot U_{F_k}(A, J)$$

where $\alpha \cdot U_F(A, J) < \beta \cdot U_G(A, J)$ iff $F < G$ and $\alpha \cdot U_F(A, J) \subset \beta \cdot U_G(A, J)$. As in [1] we have a homotopy fibration

$$E\{\alpha \cdot U_F(A, J)\} \longrightarrow B\{U_F(A, J)\} \longrightarrow BE(A) .$$

Suppose for the moment we have

LEMMA 2.3. *$\pi_1 B\{U_F(A, J)\}$ is perfect.*

Then essentially the same argument as in [1] shows that

$$(**) \qquad E\{\alpha \cdot U_F(A, J)\} \longrightarrow B\{U_F(A, J)\}^+ \longrightarrow BE(A)^+$$

is also a homotopy fibration. It follows from (2.2) that the map

$$E^{BN}(A, J) \longrightarrow E^{BN}(A/J)$$

given by $\alpha \cdot U_F(A, J) \rightarrow \pi(\alpha) \cdot U_F(A/J)$ is an isomorphism. By [6] we therefore have $\pi_{i-1} E^{BN}(A, J) = K_i(A/J)$ and the homotopy sequence of the fibration (**) gives (2.1).

To prove the lemma, it is enough to show the generators are products of commutators and the formula $w \cdot U_F \cdot w^{-1} = U_{u \cdot F}$ reduces the argument to the case where $F = C_0 = \{x_1 > x_2 > \cdots > x_l\}$ considered as lying in P^l. Here $l \geqq 3$. For generators $e_{ij}(\lambda)$ of $\pi_1(BU_{C_0})$ the third Steinberg relation $e_{ij}(\alpha\beta) = [e_{ij}(\alpha), e_{jk}(\beta)]$ shows $e_{ij}(\lambda)$ is a commutator: for example, if $\lambda \in J$ we have $e_{21}(\lambda) = [e_{23}(1), e_{31}(\lambda)]$. Now consider the generators $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \in 1 + J$, where λ is in the ith row and ith column and λ^{-1} is in the jth row and jth column. For simplicity take $i = 1$ and $j = 2$. Recall that if $M, N \in U_F$ are considered as generators of $\pi_1 BU_F$ their composition as loops is homotopic to

MN. Let $\lambda = 1 + \sigma$ and $\lambda^{-1} = 1 + \tau$ where $\tau, \sigma \in J$. We have the following matrix identity valid in $E(A)$:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\tau & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Thus modulo the commutator subgroup

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+\sigma & \sigma \\ -\sigma & 1-\sigma \end{pmatrix}.$$

Now let $D = \{x_1 = x_2 > \cdots > x_l\}$ and $C_0' = \{x_2 > x_1 > \cdots > x_l\}$. We have $U_{C_0} \supset U_D \subset U_{C_0'}$ and the matrix $\begin{pmatrix} 1+\sigma & \sigma \\ -\sigma & 1-\sigma \end{pmatrix}$ lies in U_D. Each of $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ belong to $U_{C_0'}$ and therefore by the above argument lie in the commutator subgroup. Therefore so does $\begin{pmatrix} 1+\sigma & \sigma \\ -\sigma & 1-\sigma \end{pmatrix}$, and we conclude that the loop $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ lies in the commutator subgroup.

It is probably true that

$$B\{U_F(A, J)\}^+ \longrightarrow BE(A)^+ \longrightarrow BE(A/J)^+$$

is a homotopy fibration.

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