PACIFIC JOURNAL OF MATHEMATICS
Vol. 65, No. 2, 1976

THE STRUCTURE OF SUBLATTICES OF THE PRODUCT OF n LATTICES

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The structure of sublattices of the product of n lattices is explored. Such a sublattice is decomposed and completely characterized in terms of $n(n-1)/2$ sublattices of the product of two lattices. A sublattice of the product of two lattices is represented in terms of several easily characterized sublattices. The sublattice characterizations provide analogous characterizations for those functions whose level sets are sublattices. A simple representation is also given for the sections of a sublattice of the product of two lattices.

Introduction. I will proceed to explore the structure of sublattices of the product of n lattices. It will be shown that such general sublattices can be represented in terms of some other sublattices which are quite simple to conceptualize and characterize.

The results on sublattice structure are given in § 1. In Theorem 1 a sublattice of the product of n lattices is decomposed so that it is completely characterized in terms of $n(n-1)/2$ sublattices of the product of two lattices. Lemma 2 and Corollary 1 give simple representations for sections of a sublattice of the product of two lattices. Theorem 2 represents sublattices of the product of two lattices by several easily characterized sublattices of this product. Theorem 3 combines previous results to provide a more refined characterization of sublattices of the product of n lattices.

Often sets are constructed as the intersection of level sets of a system of functions. For instance, this is frequently the case in defining the feasible region for optimization problems. To recognize when such sets are sublattices one must know what functions have sublattices as their level sets. Thus in § 2 the results of § 1 are translated into analogous characterizations for those functions whose level sets are sublattices.

These results are handy in dealing with structured optimization problems considered by the author [5, 6, 7, 9, 10, 11]. In [5, 9] a theory is developed for certain structured optimization problems in which each constraint set must be a sublattice. In order to recognize and generate domains which are sublattices (so the theory may be applied) as well as to envision the possible range of applicability of this theory it is useful to refer to the sublattice characterizations herein. The theory of [5, 9] is applied to diverse areas such as mathematical economics, optimal theory of production, shortest path problems, structured stochastic dynamic programming [5, 10], graphs and flows in networks [5, 7], and game theory [6]. For example, in [6] the results of [5, 9] are used together with Tarski's fixed point theorem [3, 4] to give conditions for the existence of an equilibrium point in an n-person nonzero-sum game, and several iterative procedures are given for constructing such an equilibrium point. Because the conditions on this game require that each player's decision be chosen from some sublattice of E^m, a characterization of such sublattices is again useful to recognize and generate games which fit this model and to perceive the model's possible range of application.

1. The structure of sublattices. If $S = \times_{i=1}^n S_i$ and $L = (x = (x_1, \cdots x_n): (x_j, x_k) \in T$ and $x \in S\}$ where T is a subset of $S_j \times S_k$ for some two distinct indices j and k, then L is a *bivariate* subset of S and T is the *jk-generator* of L. If $S_1, \cdots, S_n$ are lattices and T is the jk-generator of a bivariate subset L of $S = \times_{i=1}^n S_i$ then L is a sublattice of S if and only if T is a sublattice of $S_j \times S_k$.

THEOREM 1. *If $S_1, \cdots, S_n$ are lattices, $n > 1$, and $S = \times_{i=1}^n S_i$, then a set L is a sublattice of S if and only if it is the intersection of $n(n-1)/2$ bivariate sublattices of S.*

Proof. The sufficiency part is immediate because the intersection of sublattices is a sublattice.

Now suppose L is a nonempty sublattice of S. For $1 \leqq j \leqq n$, $1 \leqq k \leqq n$, and $j \neq k$, define

$T_{jk} = \{(x_j, x_k)$: there exists $y = (y_1, \cdots, y_n) \in L$ with $y_j = x_j$ and $y_k = x_k\}$ and

$$L_{jk} = \{x: (x_j, x_k) \in T_{jk}, x \in S\} .$$

Note that T_{jk} is a sublattice of $S_j \times S_k$ because L is a sublattice of S, and hence L_{jk} is a bivariate sublattice of S.

If $x = (x_1, \cdots, x_n) \in L$ then $(x_j, x_k) \in T_{jk}$ and thus $x \in L_{jk}$ for each $j \neq k$. Therefore,

$$L \subseteqq \bigcap_{j \neq k} L_{jk} . \tag{1}$$

Now pick $x \in \bigcap_{j\neq k} L_{jk}$. For each $j \neq k$ $x \in L_{jk}$ so $(x_j, x_k) \in T_{jk}$ and hence there exists $y^{jk} \in L$ with $y_j^{jk} = x_j$ and $y_k^{jk} = x_k$. For each j,

$1 \leqq j \leqq n$, let $y^j = \bigwedge_{k \neq j} y^{jk}$. Note that $y^j_j = x_j$ because $y^{jk}_j = x_j$ for all $k \neq j$, and $y^j \leqq x$ because $y^j_k \leqq y^{jk}_k = x_k$ for all $k \neq j$. Also, $y^j \in L$ since each $y^{jk} \in L$ and L is a sublattice of S. But $x = \bigvee_{j=1}^n y^j \in L$ because L is a sublattice. Thus,

$$L \supseteqq \bigcap_{j \neq k} L_{jk} . \tag{2}$$

By (1), (2), and $L_{jk} = L_{kj}$,

$$L = \bigcap_{1 \leqq j < k \leqq n} L_{jk} .$$

Theorem 1 (and almost all the subsequent material) was obtained by the author in 1971 and distributed as [8] in 1974. The referee has pointed out to me that Theorem 1 is a special case of a universal algebraic result of K. A. Baker and A. F. Pixley [1]. Baker and Pixley credited this lattice version of their result to unpublished work by G. M. Bergman. Bergman has included his result in a recent paper [2] in which he noted that he had discovered it in 1967.

Theorem 1 shows that a sublattice of the product of n lattices can be completely characterized in terms of sublattices of the product of two lattices. I now proceed to explore and characterize the structure of sublattices of the product of two lattices.

For a poset S and $x \in S$, define $[x, \infty) = \{y: x \leqq y, y \in S\}$ and $(-\infty, x] = \{y: y \leqq x, y \in S\}$.

If S_1 and S_2 are posets, $L \subseteqq S_1 \times S_2$, and either $[x_1, \infty) \times (-\infty, x_2] \subseteqq L$ for all $(x_1, x_2) \in L$ or $(-\infty, x_1] \times [x_2, \infty) \subseteqq L$ for all $(x_1, x_2) \in L$, then L is *bimonotone*. If S_1 and S_2 are chains then a bimonotone subset of $S_1 \times S_2$ is clearly a sublattice, but a bimonotone subset of the product of two lattices is not necessarily a sublattice. If S_1 and S_2 are posets and $L \subseteqq S_1 \times S_2$, then L generates two *bimonotone hulls*, $H_1(L) = \bigcup_{x \in L} [x_1, \infty) \times (-\infty, x_2]$ and $H_2(L) = \bigcup_{x \in L} (-\infty, x_1] \times [x_2, \infty)$, which are the smallest bimonotone sets containing L.

Since a bimonotone subset of the product of two chains is a sublattice, the bimonotone hulls of any subset of such a product must be sublattices. Lemma 1 shows that the bimonotone hulls of a sublattice of the product of two lattices are sublattices. However, as the following example shows, the bimonotone hulls of L are not necessarily sublattices if S_1 and S_2 are lattices but not chains and L is not a sublattice. Let $S_1 = S_2 = E^2$ with the usual relation $\leqq$ and $L = \{(0, 1, 0, 1), (1, 0, 1, 0)\}$. Then both bimonotone hulls contain $(0, 1, 0, 1)$ and $(1, 0, 1, 0)$ but the meet $(0, 0, 0, 0)$ and the join $(1, 1, 1, 1)$ are not in either bimonotone hull.

LEMMA 1. *If S_1 and S_2 are lattices and L is a sublattice of $S_1 \times S_2$, then the bimonotone hulls of L are sublattices.*

Proof. I will show that $H_1(L)$ is a sublattice. The proof for $H_2(L)$ follows symmetrically.

Pick $(x_1, x_2) \in H_1(L)$ and $(y_1, y_2) \in H_1(L)$. Then there exist $(\bar{x}_1, \bar{x}_2) \in L$ and $(\bar{y}_1, \bar{y}_2) \in L$ with $\bar{x}_1 \leqq x_1$, $x_2 \leqq \bar{x}_2$, $\bar{y}_1 \leqq y_1$, and $y_2 \leqq \bar{y}_2$. Because L is a sublattice of $S_1 \times S_2$, $(\bar{x}_1 \wedge \bar{y}_1, \bar{x}_2 \wedge \bar{y}_2) \in L$ and $(\bar{x}_1 \vee \bar{y}_1, \bar{x}_2 \vee \bar{y}_2) \in L$. Then $(x_1, x_2) \wedge (y_1, y_2) = (x_1 \wedge y_1, x_2 \wedge y_2) \in [\bar{x}_1 \wedge \bar{y}_1, \infty) \times (-\infty, \bar{x}_2 \wedge \bar{y}_2] \subseteqq H_1(L)$, and $(x_1, x_2) \vee (y_1, y_2) = (x_1 \vee y_1, x_2 \vee y_2) \in [\bar{x}_1 \vee \bar{y}_1, \infty) \times (-\infty, \bar{x}_2 \vee \bar{y}_2] \subseteqq H_1(L)$. Thus $H_1(L)$ is a sublattice.

If $L \subseteqq \times_{i=1}^{n} S_i$ then the *section* of L at $x_j \in S_j$ is $L^j(x_j) = \{(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n): (x_1, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_n) \in L\}$ and the *projection* of L on S_j is $\Pi_j L = \{x_j: L^j(x_j) \text{ is nonempty}\}$. If $S_1, \cdots, S_n$ are lattices and L is a sublattice of $\times_{i=1}^{n} S_i$, then each section $L^j(x_j)$ is a sublattice of $\times_{i \neq j} S_i$ and the projection $\Pi_j L$ is a sublattice of S_j for all j.

Theorem 2 will show that a sublattice of the product of two lattices can be represented as the intersection of its bimonotone hulls and the product of its two projections. Lemma 2 provides an intermediary result needed to establish Theorem 2 and shows a surprisingly simple characteristic of the sections of the product of two lattices. Corollary 1, a direct consequence of Lemma 2, shows that a section containing its infimum and supremum is simply the intersection of the appropriate projection and an interval.

LEMMA 2. *If S_1 and S_2 are lattices, L is a sublattice of $S_1 \times S_2$, $x_1 \in S_1$, $a_2 \in L^1(x_1)$, and $b_2 \in L^1(x_1)$, then $\Pi_2 L \cap [a_2, b_2] \subseteqq L^1(x_1)$.*

Proof. Pick $x_2 \in \Pi_2 L \cap [a_2, b_2]$. Because $x_2 \in \Pi_2 L$, there exists $y_1 \in S_1$ with $(y_1, x_2) \in L$. Because $a_2 \leqq x_2$ and L is a lattice, $(x_1 \vee y_1, x_2) = (x_1, a_2) \vee (y_1, x_2) \in L$. Because $x_2 \leqq b_2$ and L is a lattice, $(x_1, x_2) = (x_1 \vee y_1, x_2) \wedge (x_1, b_2) \in L$. Thus $x_2 \in L^1(x_1)$ and so $\Pi_2 L \cap [a_2, b_2] \subseteqq L^1(x_1)$.

COROLLARY 1. *If S_1 and S_2 are lattices, L is a sublattice of $S_1 \times S_2$, $x_1 \in S_1$, and $L^1(x_1)$ contains its infimum a_2 and its supremum b_2, then $L^1(x_1) = \Pi_2 L \cap [a_2, b_2]$.*

THEOREM 2. *If S_1 and S_2 are lattices and L is a sublattice of $S_1 \times S_2$, then L is the intersection of its two bimonotone hulls and the product of its two projections.*

Proof. Clearly $L \subseteqq H_1(L) \cap H_2(L) \cap \{\Pi_1 L \times \Pi_2 L\}$. Pick any

$\bar{x} \in H_1(L) \cap H_2(L) \cap \{\Pi_1 L \times \Pi_2 L\}$. Since $\bar{x} \in H_1(L)$ and $\bar{x} \in H_2(L)$, there exist $y \in L$ and $w \in L$ such that $y_1 \leqq \bar{x}_1$, $\bar{x}_2 \leqq y_2$, $\bar{x}_1 \leqq w_1$, and $w_2 \leqq \bar{x}_2$. Because L is a sublattice, $(y_1, w_2) = y \wedge w \in L$ and $(w_1, y_2) = y \vee w \in L$. Thus $w_2 \in L^1(y_1)$ and $y_2 \in L^1(y_1)$ so by Lemma 2 $\Pi_2 L \cap [w_2, y_2] \subseteqq L^1(y_1)$ and therefore $\bar{x}_2 \in L^1(y_1)$ and $(y_1, \bar{x}_2) \in L$. Also, $y_1 \in L^2(w_2)$ and $w_1 \in L^2(w_2)$ so by Lemma 2 $\Pi_1 L \cap [y_1, w_1] \subseteqq L^2(w_2)$ and therefore $\bar{x}_1 \in L^2(w_2)$ and $(\bar{x}_1, w_2) \in L$. Because L is a sublattice $\bar{x} = (y_1, \bar{x}_2) \vee (\bar{x}_1, w_2) \in L$, and so $L = H_1(L) \cap H_2(L) \cap \{\Pi_1 L \times \Pi_2 L\}$.

Note that under the hypotheses of Theorem 2 the bimonotone hulls are sublattices by Lemma 1. The converse of Theorem 2 is immediate when the bimonotone hulls are sublattices, but the example preceding Lemma 1 contradicts this converse generally.

If $S_1, \cdots, S_n$ are posets, $S = \times_{i=1}^n S_i$, L is a bivariate subset of S, T is the jk-generator of L, and T is bimonotone, then L is *bimonotone.*

The following is immediate from Theorem 2 and Lemma 1.

COROLLARY 2. *If $S_1, \cdots, S_n$ are lattices, $S = \times_{i=1}^n S_i$, and L is a bivariate sublattice of S, then L is the intersection of two bimonotone sublattices and $\times_{i=1}^n \Pi_i L$.*

Note that in Corollary 2 $\Pi_i L = S_i$ for at least $n - 2$ of the indices i.

The result of Theorem 3 is derived by applying Corollary 2 to Theorem 1.

THEOREM 3. *If $S_1, \cdots, S_n$ are lattices and $S = \times_{i=1}^n S_i$, then a set L is a sublattice of S if and only if it is the intersection of $n(n-1)$ bimonotone sublattices of S and $\times_{i=1}^n \Pi_i L$.*

2. The structure of sublattice-generating functions. Often sets are constructed as the intersection of level sets of a system of functions, and so it is useful to characterize those functions whose level sets are sublattices.

Suppose f is a function from a lattice S into a chain B. If each level set of f is a sublattice of S, then f is a *sublattice-generating function.* For $L \subseteqq S$, f is an *indicator function* for L if

$$f(x) = \begin{cases} b & \text{for} \quad x \in L \\ d & \text{for} \quad x \in S \quad \text{and} \quad x \notin L \end{cases}$$

where $b < d$ in B. If f is an indicator function for L, then f is a sublattice-generating function if and only if L is a sublattice of S. By this correspondence, properties of sublattices can be directly translated into properties of sublattice-generating indicator functions and properties of sublattice-generating functions can be translated into properties of sublattices. It is seen in Lemma 3 below that a sublattice-generating function that is bounded below is the pointwise supremum of a collection of sublattice-generating indicator functions, and thus properties of sublattice-generating indicator functions imply properties of sublattice-generating functions which are bounded below. The remarks following Lemma 3 give properties of sublattice-generating indicator functions and of sublattice-generating functions which are bounded below which correspond directly to properties of sublattices as given in §2.

LEMMA 3. *A function f from a lattice S into a chain B is a sublattice-generating function and bounded below if and only if it is the pointwise supremum of a collection of sublattice-generating indicator functions.*

Proof. The pointwise supremum, if it exists, of a collection of sublattice-generating functions is clearly also a sublattice-generating function because the intersection of sublattices is a sublattice. This, together with the fact that an indicator function is bounded below, establishes sufficiency.

Now suppose that f is a sublattice-generating function and bounded below. Then there exists $d \in B$ such that $d \leqq f(x)$ for all $x \in S$. For all $b \in B \cap [d, \infty)$, define

$$S^b = \{x \colon x \in S,\ f(x) < b\}$$

and

$$f^b\{x) = \begin{cases} d & \text{if} \quad x \in S^b \\ b & \text{if} \quad x \in S \quad \text{and} \quad x \notin S^b . \end{cases}$$

Since f is a sublattice-generating function and B a chain, each S^b is a sublattice of S and so f^b is a sublattice-generating indicator function for each $b \in B \cap [d, \infty)$. Pick any $\bar{x} \in S$. For any $b \in B \cap [d, \infty)$, if $f(\bar{x}) < b$ then $\bar{x} \in S^b$ and $f^b(\bar{x}) = d \leqq f(\bar{x})$, and otherwise $\bar{x} \notin S^b$ and $f^b(\bar{x}) = b \leqq f(\bar{x})$. Thus, $f^b(\bar{x}) \leqq f(\bar{x})$ for each $b \in B \cap [d, \infty)$. But $f(\bar{x}) \in B \cap [d, \infty)$ and $\bar{x} \notin S^{f(\bar{x})}$, and so $f^{f(\bar{x})}(\bar{x}) = f(\bar{x})$. Therefore

$$f(\bar{x}) = \sup_{b \in B \cap [d, \infty)} f^b(\bar{x}) .$$

In all subsequent remarks it will be assumed that the domain

S is the product of n lattices $S_1, \cdots, S_n$ and that the range of all functions is, for convenience, E^1.

A function is *univariate* if it varies in at most one coordinate. A function is *bivariate* if it varies in at most two coordinates. A function is *bimonotone* if it is isotone in one of its coordinates, antitone in one of its coordinates, and does not vary in the other $n - 2$ coordinates.

By Theorem 1 an indicator function of a sublattice of S can be represented as the pointwise supremum of $n(n-1)/2$ indicator functions of bivariate sublattices. Thus a sublattice-generating function which is bounded below is the pointwise supremum of a collection of indicator functions of bivariate sublattices, and so such a function is the pointwise supremum of $n(n-1)/2$ bivariate sublattice-generating functions.

An indicator function of a bimonotone set is bimonotone. Each level set of a bimonotone function is bimonotone. When each S_i is a chain it can be seen directly that univariate functions and bimonotone functions are sublattice-generating functions, as Veinott [personal communication] has previously observed.

By Corollary 2, an indicator function of a bivariate sublattice is the pointwise supremum of two bimonotone sublattice-generating indicator functions and two univariate sublattice-generating indicator functions.

By Theorem 3 a sublattice-generating indicator function is the pointwise supremum of $n(n-1)$ bimonotone sublattice-generating indicator functions and n univariate sublattice-generating indicator functions. Thus a sublattice-generating function which is bounded below is the pointwise supremum of a collection of bimonotone sublattice-generating indicator functions and univariate sublattice-generating indicator functions, and such a functior is therefore the pointwise supremum of $n(n-1)$ bimonotone sublattice-generating functions and n univariate sublattice-generating functions. Consequently, when each S_i is a chain, a sublattice-generating function which is bounded below is the pointwise supremum of $n(n-1)$ bimonotone functions and n univariate functions.

If $S_1, \cdots, S_n$ are chains, $f(x) = \sum_{i=1}^{n} f_i(x_i)$ where $x_i \in S_i$ for each i, and f is a sublattice-generating function on $\times_{i=1}^{n} S_i$, then f must be either univariate or bimonotone, as Veinott [personal communication] previously noted.

References

1. K. Baker, and A. Pixley, *Polynomial interpolation and the Chinese remainder theorem for algebraic systems*, Math. Zeitschr., **143** (1975), 165-174.

2. G. Bergman, *On the existence of subalgebras of direct products with prescribed d-fold projections*, technical report, University of California, Berkeley, (1975).

3. G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications, Vol. XXV, third edition, (1967).

4. A. Tarski, *A lattice-theoretical fixpoint theorem and its applications*, Pacific J. Math., **5** (1955) 285-309.

5. D. Topkis, *Ordered optimal solutions*, doctoral dissertation, Stanford University, (1968).

6. ———, *Equilibrium points in nonzero-sum n-person subadditive games*, technical report ORC 70-38, Operations Research Center, University of California, Berkeley, (1970).

7. ———, *Monotone minimum node-cuts in capacitated networks*, technical report ORC 70-39, Operations Research Center, University of California, Berkeley, (1970).

8. ———, *The structure of sublattices of the product of n lattices*, technical report, Hebrew University, Jerusalem, (1974).

9. ———, *Minimizing a subadditive function on a lattice*, technical report, Hebrew University, Jerusalem, (1975).

10. ———, *Applications of minimizing a subadditive function on a lattice* (forthcoming).

11. ———, *Topology and subcomplete sublattices* (forthcoming).

Received December 19, 1975

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