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A GEOMETRIC CHARACTERIZATION OF INDETERMINATE MOMENT SEQUENCES

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Hamburger and Stieltjes moment sequences are studied from the standpoint of the geometry of their moment spaces. Necessary and sufficient conditions are obtained that each of these sequences be indeterminate. The elements in the associated Jacobi and Stieltjes type continued fractions are characterized in terms of ratios of distances in the moment spaces.

1. Introduction. A sequence of real numbers $\{c_n\}_{n=0}^{\infty}$ is an H (Hamburger moment) sequence if there exists a bounded nondecreasing function γ on $(-\infty, \infty)$ such that

$$c_n = \int_{-\infty}^{\infty} t^n d\gamma(t) \qquad (n = 0, 1, 2, \cdots) . \tag{1}$$

The function γ, called a mass function for the sequence $\{c_n\}$, is normalized to be left continuous and such that $\gamma(0) = 0$. The sequence $\{c_n\}_{n=0}^{\infty}$ is an S (Stieltjes moment) sequence if it is an H sequence and there is a mass function γ for the sequence that is constant on $(-\infty, 0)$. An H sequence or an S sequence is determinate if the mass function γ for the sequence is unique. Otherwise the moment sequence is indeterminate.

The geometric approach of Carathéodory [2] for the classical moment problems has been extended and generalized by a number of authors (see [5]). In particular, Krein [6] initiated a geometric study of general Tchebycheff systems and Karlin and Shapley [4] rekindled interest in the geometry of moment sequences by their definitive memoir on the finite (Hausdorff) moment problem. The primary purpose of this paper is to provide, in the spirit of the works of Krein and of Karlin and Shapley, geometric characterizations for indeterminate H sequences and for indeterminate S sequences.

More specifically, let $\mathfrak{M}_{2m+1}$ denote the set of vectors $c = (c_0, c_1, \cdots, c_{2m})$ in Euclidean E^{2m+1} space such that there is a mass function γ on $(-\infty, \infty)$ for which (1) holds when $n = 0, 1, 2, \cdots, 2m$. For real $\lambda > 0$ and for c, c^* in $\mathfrak{M}_{2m+1}$, the vectors λc and $c + c^*$ are also in $\mathfrak{M}_{2m+1}$. Thus $\mathfrak{M}_{2m+1}$ is a convex cone in E^{2m+1}. For a given $c = (c_0, c_1, \cdots, c_{2m}) \in \mathfrak{M}_{2m+1}$, we consider the two dimensional section of this cone defined by

$$D_m = \{(x, y) \colon (x, y, c_2, c_3, \cdots, c_{2m}) \in \mathfrak{M}_{2m+1}\} .$$

In §3 we prove that D_m is a closed region bounded by a parabola (possibly degenerate) and that $D_m \subset D_{m-1}$. The intersection $\bigcap_{m=1}^{\infty} D_m$, called the *limit parabolic region*, is either a closed region containing (c_0, c_1) bounded by a proper parabola or a ray $\{(x, y) \colon x \geqq \underline{c}_0, y = c_1\}$ where $\underline{c}_0 \leqq c_0$. An H sequence is proved to be indeterminate if and only if the point (c_0, c_1) is an interior point of the limit parabolic region. For an S sequence there is a convex cone $\mathfrak{N}_{2m+1}$ corresponding to $\mathfrak{M}_{2m+1}$ and two dimensional sections E'_m of this cone are introduced in the same fashion as the sections D_m were defined from $\mathfrak{M}_{2m+1}$. An S sequence is proved to be indeterminate if and only if (c_0, c_1) is an interior point of $\bigcap_{m=1}^{\infty} E'_m$ in § 4. The final section of this paper provides a geometric interpretation of the coefficients of the J-fraction or S-fraction corresponding to $\sum_{m=0}^{\infty} c_m/z^{m+1}$ when $\{c_m\}_{m=0}^{\infty}$ is an H or an S sequence respectively.

2. Preliminaries. For a real sequence $\{c_n\}_{n=0}^{\infty}$ let

$$\varDelta_{n,p} = \begin{vmatrix} c_p & c_{p+1} & \cdots & c_{p+n} \\ c_{p+1} & c_{p+2} & \cdots & c_{p+n+1} \\ \vdots & \vdots & & \vdots \\ c_{p+n} & c_{p+n+1} & \cdots & c_{p+2n} \end{vmatrix} \quad (n, p = 0, 1, 2, \cdots) .$$

For brevity set $\varDelta_n = \varDelta_{n,0}$. A classical necessary and sufficient condition for $\{c_n\}_{n=0}^{\infty}$ to be an H sequence is that either (a) $\varDelta_n > 0$ for $n = 0, 1, 2, \cdots$ or (b) $\varDelta_n > 0$ for $n = 0, 1, 2, \cdots, m-1$ and $\varDelta_n = 0$ for $n = m, m+1, \cdots$ ([3], [7, p. 5]). An H sequence is called *positive definite* or positive *semidefinite* according as (a) or (b) holds. A positive semidefinite H sequence is always determinate. A positive definite H sequence is determinate if and only if at least one of the sequences

$$(2) \qquad \{\varDelta_n/\varDelta_{n-1,2}\} , \quad \{\varDelta_{n,2}/\varDelta_{n-1,4}\}$$

has limit zero as $n \to \infty$ ([3] [7, p. 72]). Let

$$(3) \qquad \frac{a_0^2}{z+b_1} - \frac{a_1^2}{z+b_2} - \cdots - \frac{a_n^2}{z+b_{n+1}} - \cdots .$$

be the J-fraction expansion of the formal power series $\sum_{n=0}^{\infty} c_n/z^{n+1}$. If $a_j \neq 0$ $(j = 0, 1, 2, \cdots, n)$, let

$$(4) \qquad P_n^*(z) = \frac{B_n(z)}{a_0 a_1 \cdots a_n} , \quad Q_n^*(z) = \frac{A_n(z)}{a_0 a_1 \cdots a_n} ,$$

where $B_n(z)$ and $A_n(z)$ respectively denote the nth denominator and nth numerator of the continued fraction (3). Since $\Delta_n = a_0^2 a_1^2 \cdots a_n^2 \Delta_{n-1}$ [8, p. 197], the polynomials P_n^*, Q_n^* are defined for $n \leqq m$ if $\Delta_n \neq 0$, $n \leqq m$. When $\Delta_n \neq 0$, $n \leqq m$, we have furthermore that

$$(5) \qquad F_m^* = \sum_{j=0}^{m} [P_j^*(0)]^2 = -\frac{1}{\Delta_m} \begin{vmatrix} 0 & 1 & 0 & \cdots 0 \\ 1 & c_0 & c_1 & \cdots c_m \\ 0 & c_1 & c_2 & \cdots c_{m+1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & c_m & c_{m+1} & \cdots c_{2m} \end{vmatrix},$$

$$(6) \qquad G_m^* = \sum_{j=0}^{m} P_j^*(0) Q_j^*(0) = -\frac{1}{\Delta_m} \begin{vmatrix} 0 & 0 & c_0 & \cdots c_{m-1} \\ 1 & c_0 & c_1 & \cdots c_m \\ 0 & c_1 & c_2 & \cdots c_{m+1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & c_m & c_{m+1} & \cdots c_{2m} \end{vmatrix},$$

and

$$(7) \qquad H_m^* = \sum_{j=0}^{m} [Q_j^*(0)]^2 = -\frac{1}{\Delta_m} \begin{vmatrix} 0 & 0 & c_0 & \cdots c_{m-1} \\ 0 & c_0 & c_1 & \cdots c_m \\ c_0 & c_1 & c_2 & \cdots c_{m+1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m-1} & c_m & c_{m+1} & \cdots c_{2m} \end{vmatrix}.$$

(See, for example, [9].)

Set $\Delta_{-1,1} = 1$ and assume $c_0 > 0$. Then a necessary and sufficient condition for $\{c_n\}_{n=0}^{\infty}$ to be an S sequence is that either (c) $\Delta_n > 0$ and $\Delta_{n,1} > 0$ for $n = 0, 1, 2, \cdots$ or (d) $\Delta_n > 0$ and $\Delta_{n-1,1} > 0$ for $n = 0, 1, 2, \cdots, m$; $\Delta_{m,1} \geqq 0$; and $\Delta_n = \Delta_{n,1} = 0$ for $n = m+1, m+2, \cdots$ [7, p. 6]. The polynomials P_n^*, Q_n^* can be defined for an S sequence by (4) where the continued fraction (3) is the even part of the S-fraction

$$(8) \qquad \frac{d_0}{z} - \frac{d_1}{1} - \frac{d_2}{z} - \frac{d_3}{1} - \cdots - \frac{d_{2n}}{z} - \frac{d_{2n+1}}{1} - \cdots$$

corresponding to the formal power series $\sum_{j=0}^{\infty} c_j/z^{j+1}$ [8, p. 73].

3. *H* sequences. Let $(c_0, c_1, c_2, \cdots, c_{2m}) \in \mathfrak{M}_{2m+1}$. For real x and y, the vector $(x, y, c_2, c_3, \cdots, c_{2m}) \in \mathfrak{M}_{2m+1}$ if and only if

$$(9)\qquad D_n(x, y) = \begin{vmatrix} x & y & c_2 & \cdots & c_n \\ y & c_2 & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n} \end{vmatrix} \geqq 0 \quad (n = 1, 2, \cdots, m) .$$

Suppose $\Delta_n > 0$ for $n < m$. Then by (5), (6), and (7) with each c_j replaced by c_{j+2} we have

$$(10)\qquad D_n(x, y) = \Delta_{n-1,2}\left\{x - \sum_{j=0}^{n-1}[Q_j(0) + yP_j(0)]^2\right\} ,$$

where $P_j(z)$, $Q_j(z)$ are the polynomials (4) for the sequence $c_2, c_3, \cdots, c_{2m}$. Since $\Delta_n > 0$ implies $\Delta_{n-1,2} > 0$ for $0 < n < m$, the set D_n of points (x, y) such that $D_n(x, y) \geqq 0$ is a closed convex region bounded by a parabola. By (10) $D_n \subset D_{n-1}$, $1 < n < m$, and the boundaries of D_n and of D_{n-1} have exactly one point (possibly infinity) in common.

LEMMA 1. *Let $\{c_n\}_{n=0}^{\infty}$ be a positive definite H sequence. For each positive integer n, the two dimensional section*

$$(11)\qquad D_n = \{(x, y): (x, y, c_2, c_3, \cdots, c_{2m}) \in \mathfrak{M}_{2m+1}\}$$

is a closed convex region bounded by the parabola

$$(12)\qquad x - x_n = F_n(y - y_n)^2 ,$$

where $x_n = H_n - G_n^2/F_n$, $y_n = -G_n/F_n$ and

$$(13)\qquad F_n = \sum_{j=0}^{n}[P_j(0)]^2,\ G_n = \sum_{j=0}^{n} P_j(0)Q_j(0),\ H_n = \sum_{j=0}^{n}[Q_j(0)]^2 .$$

Furthermore, $D_n \subset D_{n-1}$, $\bigcap_{n=1}^{\infty} D_n \neq \varnothing$, and the sequences $\{x_n\}$, $\{y_n\}$ converge to finite limits.

Proof. The boundary (12) is obtained from (5), (6), and (7) when the indices are augmented by 2 by a straight-forward calculation (see also [9]). The fact that the parabolic regions D_n are nested follows immediately from (9) and (10). One consequence of these observations is that the sequence $\{x_n\}$ is nondecreasing. Since $(c_0, c_1) \in D_n$ for all n, we have $x_n \leqq c_0$ and, hence, $\lim_{n\to\infty} x_n \leqq c_0$. To prove $\{y_n\}$ also converges, first note that $\{F_n\}$ is a nondecreasing sequence of positive numbers. Hence F_n tends to a positive limit or ∞ as $n \to \infty$. From the nesting of the regions, we have $(x_m, y_m) \in D_n$ whenever $m \geqq n$. By (12) this implies

$$(x_m - x_n)/F_n \geqq (y_m - y_n)^2 .$$

It follows that $\{y_n\}$ tends to a finite limit as $n \to \infty$.

Let $x^0 = \lim x_n$, $y^0 = \lim y_n (n \to \infty)$ under the hypothesis of Lemma 1. If $F_n \to F < \infty$, then $\bigcap_{n=1}^{\infty} D_n$ is a closed region bounded by the parabola $x - x^0 = F(y - y^0)^2$. If $F_n \to \infty$, then the length of the latus rectum $1/F_n$ of the nth parabola (12) tends to zero as $n \to \infty$. In this case $\bigcap_{n=1}^{\infty} D_n$ is a ray: $x \geqq x^0$, $y = y^0$.

When $\Delta_n > 0$ for $n < m$, we have $\Delta_{n-1,2} > 0$ and we define $\underline{c}_{0,n}$ by

$$(14) \qquad \begin{vmatrix} \underline{c}_{0,n} & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix} = 0 .$$

Then $c_0 - \underline{c}_{0,n} = \Delta_n / \Delta_{n-1,2}$ so $c_0 > \underline{c}_{0,n}$, $n < m$. The projection $(\underline{c}_{0,n}, c_1, \cdots, c_{2n})$ is a boundary point of $\mathfrak{M}_{2n+1}$.

LEMMA 2. *Let $\{c_n\}_{n=0}^{\infty}$ be a positive semidefinite H sequence for which $\Delta_n > 0$ when $n < m$ and $\Delta_n = 0$ when $n \geqq m$. If $\Delta_{m-1,2} > 0$, then the* mth *region* (11) *is bounded by a proper parabola that contains the point (c_0, c_1). The $m + 1$st region* (11) *degenerates in this case to a ray $x \geqq c_0$, $y = c_1$. If $\Delta_{m-1,2} = 0$, the* mth *region* (11) *is a ray $x \geqq \underline{c}_{0,m-1}$, $y = c_1$.*

Proof. Suppose $\Delta_{m-1,2} > 0$. Then $\Delta_{m-2,4} > 0$ and by (9) the length of the latus rectum $\Delta_{m-1,2}/\Delta_{m-2,4}$ of the mth parabola $D_m(x, y) = 0$ is not zero. Furthermore $D_m(c_0, c_1) = \Delta_m = 0$ so (c_0, c_1) is on the mth parabola. We can define $\underline{c}_{0,m}$ by (14) in this case and $\underline{c}_{0,m} = c_0$. We show next that the $m + 1$st parabolic region $D_n(x, y) \geqq 0 (n = 1, 2, \cdots, m + 1)$ is a ray $x \geqq c_0$, $y = c_1$ when $\Delta_{m-1,2} > 0$. Since $\Delta_n > 0$, $n < m$ and $\Delta_n = 0$, $n \geqq m$, the moment problem for the sequence $\{c_n\}$ is determinate. This implies there is a unique representation

$$(15) \qquad c_n = \sum_{j=1}^{m} \lambda_j t_j^n \qquad (n = 0, 1, 2, \cdots) ,$$

where $\lambda_j > 0 (j = 1, 2, \cdots, m)$ and $t_1 < t_2 < \cdots < t_m$. Since $\underline{c}_{0,m} = c_0$, we have $t_j \neq 0 (j = 1, 2, \cdots, m)$. By replacing each c_n in $\Delta_{m,2}$ with its representation (15), we can express the determinant $\Delta_{m,2}$ as a linear combination of determinants of the form

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ t_{k_1} & t_{k_2} & \cdots & t_{k_{m+1}} \\ \vdots & \vdots & & \vdots \\ t_{k_1}^m & t_{k_2}^m & \cdots & t_{k_{m+1}}^m \end{vmatrix} ,$$

where the indices k_j are between 1 and m. Since each of these last determinants is of order $m+1$, each contains at least two identical columns. We conclude that $\Delta_{m,2} = 0$ and, therefore, $D_{m+1}(x, y)$ is independent of x. Now $D_{m+1}(x, y) \not\equiv 0$ since the coefficient of y^2 is $\Delta_{m-1,4} \neq 0$. To prove this last assertion, assume $\Delta_{m-1,4} = 0$. This implies there is a representation

$$c_{n+4} = \sum_{j=1}^{p} \mu_j s_j^n \qquad (n = 0, 1, 2, \cdots), \tag{16}$$

where $s_1 < s_2 < \cdots < s_p$, $\mu_j > 0 (j = 1, 2, \cdots, p)$, $p \leqq m - 1$. Since $t_j \neq 0 (j = 1, 2, \cdots, m)$ in (15), there are two distinct representations, (15) and (16), for the semidefinite H sequence $\{c_n\}_{n=4}^{\infty}$. This is contrary to the fact that this sequence is determinate and, thus, $\Delta_{m-1,4} \neq 0$. Finally $D_{m+1}(x, y) = 0$ has a unique double root at $y = c_1$ by known results on symmetric determinants [1, p. 139].

Next assume $\Delta_{m-1,2} = 0$. This implies $D_m(x, y)$ is independent of x and, as in the previous case, we conclude $D_m(x, y) = -\Delta_{m-2,4}(y - c_1)^2$. Since $D_{m-1}(x, y) \geqq 0$ forces x to be not smaller than $\underline{c}_{0,m-1}$ when $y = c_1$, we have $D_m = \{(x, y) : x \geqq \underline{c}_{0,m-1}, y = c_1\}$. This completes the proof of the Lemma.

Using the representation (15) we can easily prove $D_{m+j}(x, y) = 0$ for all choices of (x, y) when $j \geqq 2$ and $\{c_n\}_{n=0}^{\infty}$ is a positive semidefinite H sequence with $\Delta_{m-1} > 0$, $\Delta_m = 0$. This means that the conditions $D_n(x, y) \geqq 0 (n = 1, 2, \cdots)$ add no new restrictions on (x, y) once $n > m + 1$, that is, once the parabolic regions degenerate to a ray. It is meaningful, therefore, to speak of this ray as the limit region $\bigcap_{n=1}^{\infty} D_n$ for a positive semidefinite H sequence.

THEOREM 1. *Let $\{c_n\}_{n=0}^{\infty}$ be an H sequence. This sequence is indeterminate if and only if (c_0, c_1) is an interior point of the limit parabolic region.*

Proof. If $\{c_n\}$ is positive semidefinite, then the limit region is a ray by Lemma 2. The limit region has no interior points in this case and the sequence is determinate.

Suppose $\{c_n\}$ is positive definite. By (5) and (12) the length of the latus rectum of the nth parabola is $1/F_n = \Delta_{n-1,2}/\Delta_{n-2,4}$. If $\lim \Delta_{n-1,2}/\Delta_{n-2,4} = 0$ as $n \to \infty$, then the limit parabolic region is a ray and by (2) the sequence is determinate. Suppose therefore $\Delta_{n-1,2}/\Delta_{n-2,4}$ has a nonzero limit as $n \to \infty$. By (14) we have $(\underline{c}_{0,n}, c_1)$ is on the nth parabola and $c_0 - \underline{c}_{0,n} = \Delta_n/\Delta_{n-1,2} > 0$. Since the parabolas are nested by Lemma 1, the sequence $\{\underline{c}_{0,n}\}$ is nondecreasing. Let $\underline{c}_{0,n} \to \underline{c}_0$ as $n \to \infty$. Then $\underline{c}_0 \leqq c_0$ and equality holds if and only if $\Delta_n/\Delta_{n-1,2} \to 0$

as $n \to \infty$. The last condition implies $\{c_n\}$ is determinate by (2). Furthermore, $c_0 = \underline{c}_0$ implies (c_0, c_1) is on the boundary of the limit parabolic region. Now if $\underline{c}_0 < c_0$, then (c_0, c_1) is interior to the limit parabolic region and by (2) the sequence $\{c_n\}$ is indeterminate.

4. S sequences. Every S sequence is an H sequence so the existence of the limit parabolic region is assured by Lemma 1 and Lemma 2. Furthermore a sequence $\{c_n\}_{n=0}^{\infty}$ is an indeterminate S sequence if and only if $\{p_n\}_{n=0}^{\infty}$, where $p_{2n} = c_n$, $p_{2n+1} = 0 (n = 0, 1, 2, \cdots)$, is an indeterminate H sequence. Replacing $\{c_n\}$ by $\{p_n\}$ in (2), we obtain the following criteria for an indeterminate S sequence.

LEMMA 3. *An S sequence is indeterminate if and only if the sequences $\{\Delta_n/\Delta_{n-1,2}\}$ and $\{\Delta_{n,1}/\Delta_{n-1,3}\}$ each have nonzero limits as $n \to \infty$.*

The following determinant identity is needed.

LEMMA 4. $\Delta_{n-1,2}\Delta_{n-1} - \Delta_{n-2,2}\Delta_n = \Delta_{n-1,1}^2$, $n \geqq 1$, $(\Delta_{-1,2} = 1)$.

Proof. For a sequence $\{c_j\}_{j=0}^{2n+1}$ set

$$\Delta_n(t) = \begin{vmatrix} 1 & t & \cdots t^n \\ c_1 & c_2 & \cdots c_{n+1} \\ \vdots & \vdots & \vdots \\ c_n & c_{n+1} & \cdots c_{2n} \end{vmatrix} = (-1)^n \Delta_{n-1,1} t^n + \cdots + \Delta_{n-1,2} .$$

Define the linear functional M on the linear space of polynomials of degree not exceeding $2n+1$ by the condition $M[t^j] = c_j (j = 0, 1, 2, \cdots, 2n+1)$. Using elementary properties of determinants, we have

$$M[t^j \Delta_n(t)] = \begin{cases} \Delta_n & \text{if} \quad j = 0 \\ 0 & \text{if} \quad 0 < j \leqq n , \\ (-1)^n \Delta_{n,1} & \text{if} \quad j = n+1 . \end{cases}$$

Therefore,

$$\begin{aligned} M[\Delta_{n-1}(t)\Delta_n(t)] &= M[\{(-1)^{n-1}\Delta_{n-2,1}t^{n-1} + \cdots + \Delta_{n-2,2}\}\Delta_n(t)] \\ &= \Delta_{n-2,2}\Delta_n \end{aligned}$$

and

$$\begin{aligned} M[\Delta_{n-1}(t)\Delta_n(t)] &= M[\Delta_{n-1}(t)\{(-1)^n\Delta_{n-1,1}t^n + \cdots + \Delta_{n-1,2}\}] \\ &= (-1)^n\Delta_{n-1,1}M[t^n\Delta_{n-1}(t)] + \Delta_{n-1,2}M[\Delta_{n-1}(t)] \\ &= -\Delta_{n-1,1}^2 + \Delta_{n-1,2}\Delta_{n-1} . \end{aligned}$$

The identity follows from these equalities.

Let $\{c_n\}_{n=0}^{\infty}$ be an S sequence such that $\Delta_{n,1} > 0$ for $n \leqq m$. We define $\underline{c}_{1,m}$ by

$$\begin{vmatrix} \underline{c}_{1,m} & c_2 & \cdots & c_{m+1} \\ c_2 & c_3 & \cdots & c_{m+2} \\ \vdots & \vdots & & \vdots \\ c_{m+1} & c_{m+2} & \cdots & c_{2m+1} \end{vmatrix} = 0 .$$

Since $\Delta_{m-1,3} > 0$ in this case, we have $c_1 - \underline{c}_{1,m} = \Delta_{m,1}/\Delta_{m-1,3}$ and $c_1 > \underline{c}_{1,m}$. If $\Delta_{m+1,1} = 0$, define $\underline{c}_{1,m+1}$ to be c_1.

LEMMA 5. *Let $\{c_n\}_{n=0}^{\infty}$ be an S sequence. The sequence $\{\underline{c}_{1,n}\}$ is nondecreasing and $\underline{c}_1 = \lim \underline{c}_{1,n} \leqq c_1$.*

Proof. Since $\underline{c}_{1,n}$ equals c_1 from some index n onward when $\{c_n\}$ is positive semidefinite, we assume $\Delta_n > 0$, $\Delta_{n,1} > 0$ for $n \leqq m$. Then

$$\underline{c}_{1,m} - \underline{c}_{1,m-1} = \frac{\Delta_{m-1,1}}{\Delta_{m-2,3}} - \frac{\Delta_{m,1}}{\Delta_{m-1,3}} = \frac{\Delta_{m-1,2}^2}{\Delta_{m-2,3}\Delta_{m-1,3}} ,$$

where the last equality is obtained by the identity of Lemma 4 applied to the sequence $\{c_n\}_{n=1}^{\infty}$. It follows that $\{\underline{c}_{1,n}\}$ is is nondecreasing and bounded by c_1.

Let $\mathfrak{N}_{2m+1}$ denote the subset of $\mathfrak{M}_{2m+1}$ consisting of S sequences. If $\{c_n\}_{n=0}^{\infty}$ is an S sequence and $\Delta_n > 0$, $\Delta_{n,1} > 0$ for $n \leqq m$, then the vector $(x, y, c_2, \cdots, c_{2m}) \in \mathfrak{N}_{2m+1}$ if and only if $(x, y) \in D_m$ and $y \geqq \underline{c}_{1,m}$. Since $\underline{c}_{1,m} \to \underline{c}_1$ as $m \to \infty$, the limit region corresponding to an S sequence is either a ray $x \geqq \underline{c}_0$, $y = c_1 = \underline{c}_1$ or the intersection of a proper limit parabolic region and the half plane $y \geqq \underline{c}_1$.

THEOREM 2. *Let $\{c_n\}_{n=0}^{\infty}$ be an S sequence. Then $\{c_n\}$ is indeterminate if and only if (c_0, c_1) is interior to the limit parabolic region and $c_1 > \underline{c}_1$.*

Proof. If $\{c_n\}_{n=0}^{\infty}$ is positive semidefinite, the sequence is determinate and the limit parabolic region has no interior points. Suppose, therefore, $\{c_n\}$ is positive definite. If the limit parabolic region is a ray $x \geqq \underline{c}_0, y = c_1$, then $c_1 = \underline{c}_1$ and, hence, $\Delta_{n,1}/\Delta_{n-1,3} \to 0$ as $n \to \infty$. The sequence $\{c_n\}$ is determinate in this case by Lemma 3. If the limit parabolic region has interior points, then (c_0, c_1) is on its boundary if and only if $c_0 - \underline{c}_{0,n} = \Delta_n/\Delta_{n-1,2} \to 0$ as $n \to \infty$. Again by Lemma 3, the sequence $\{c_n\}$ is determinate when the last condition holds. Finally let (c_0, c_1) be an interior point of the limit parabolic region.

Then $\{\Delta_n/\Delta_{n-1,2}\}$ does not tend to zero as $n \to \infty$. In this case, $\{c_n\}$ is determinate if and only if $c_1 - \underline{c}_{1,n} = \Delta_{n,1}/\Delta_{n-1,3} \to 0$ as $n \to \infty$ by Lemma 3. The last condition is equivalent to the condition $c_1 = \underline{c}_1$ and the proof is complete.

COROLLARY. *Let $\{c_n\}_{n=0}^{\infty}$ be a positive definite S sequence. This sequence is a determinate S sequence and an indeterminate H sequence if and only if (c_0, c_1) is an interior point of the limit parabolic region and $c_1 = \underline{c}_1 = \lim_{n\to\infty} \underline{c}_{1,n}$.*

5. A geometric interpretation of the continued fraction coefficients. For $n \geqq 1$, let Δ^*_{n-1} denote the minor of the element c_{2n-1} in the determinant Δ_n. From the algorithm for expanding a power series into a J-fraction [8, p. 196] it is easily proved that the coefficients of (3) are determined by

$$a_n = \frac{\sqrt{\Delta_{n-2}\Delta_n}}{\Delta_{n-1}},\ b_{n+1} = \frac{\Delta^*_{n-1}\Delta_n - \Delta_{n-1}\Delta^*_n}{\Delta_{n-1}\Delta_n}(0 \leqq n \leqq m)\,, \tag{17}$$

where $\Delta^*_{-1} = 0$, $\Delta_{-1} = \Delta_{-2} = 1$, and $\{c_n\}_{n=0}^{\infty}$ is an H sequence such that $\Delta_n > 0$ when $n \leqq m$. To obtain a geometric interpretation of these coefficients, we introduce the two dimensional sections of the cone $\mathfrak{M}_{2n+1}$ defined by

$$E_n = \{(x, y)\colon (c_0, c_1, \cdots, c_{2n-2}, x, y) \in \mathfrak{M}_{2n+1}\}(n \leqq m)\ .$$

Note that E_n is a closed region bounded by the parabola

$$\begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} & c_n \\ c_1 & c_2 & \cdots & c_n & c_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} & x \\ c_n & c_{n+1} & \cdots & x & y \end{vmatrix} = 0\ .$$

Let $\underline{c}_{2n}$ denote the y-coordinate of the point on the parabola for which $x = c_{2n-1}$. A simple computation proves that the axis of this parabola is $x = c^*_{2n-1}$, where Δ^*_{2n-1} is obtained by replacing c_{2n-1} in Δ^*_{n-1} with c^*_{2n-1} and setting the resulting determinant equal to zero.

THEOREM 3. *Let $\{c_n\}_{n=0}^{\infty}$ be an H sequence and let $\Delta_n > 0$ for $n \leqq m$. Let e_n and e^*_n respectively denote the distance in the vertical direction from (c_{2n-1}, c_{2n}) to the boundary of E_n and the distance in the horizontal direction from (c_{2n-1}, c_{2n}) to the axis of the parabolic boundary of E_n. Then the coefficients in* (3) *are*

$$a_n^2 = \frac{e_n}{e_{n-1}},\ b_{n+1} = \frac{e_n^*}{e_{n-1}} - \frac{e_{n+1}^*}{e_n} \tag{18}$$
$$(e_0^* = 0,\ e_{-1} = 1,\ e_0 = c_0,\ 0 \leqq n \leqq m)\ .$$

Furthermore $e_n = c_{2n} - \underline{c}_{2n},\ e_{n+1}^* = c_{2n+1} - c_{2n+1}^*$ *for* $0 \leqq n \leqq m$.

Proof. From the definitions of $\underline{c}_{2n}$ and $\underline{c}_{2n+1}^*$, we have for $0 \leqq n \leqq m$, $e_n = c_{2n} - \underline{c}_{2n} = \Delta_n/\Delta_{n-1}$, $e_{n+1}^* = c_{2n+1} - c_{2n+1}^* = \Delta_n^*/\Delta_{n-1}$. The results in (18) are a consequence of these identities and (17).

The two dimensional sections $\mathfrak{N}_{2n+1}$ for an S sequence $\{c_n\}_{n=0}^{\infty}$ corresponding to E_n are given by

$$\{(x, y): (c_0, c_1, \cdots, c_{2n-2}, x, y) \in \mathfrak{N}_{2n+1}\}\ .$$

If $\Delta_n > 0$, $\Delta_{n,1} > 0$ for $n \leqq m$, these sections are the common part of the parabolic regions E_n and the half plane $x \geqq \underline{c}_{2n-1}$, where $\underline{c}_{2n-1}$ is obtained from $\Delta_{n-1,1}$ by replacing c_{2n-1} with $\underline{c}_{2n-1}$ and setting the resulting determinant equal to zero. The coefficients in the S-fraction (8) are determined by

$$d_{2n} = \frac{\Delta_{n-2,1}\Delta_n}{\Delta_{n-1}\Delta_{n-1,1}},\ d_{2n+1} = \frac{\Delta_{n-1}\Delta_{n,1}}{\Delta_{n-1,1}\Delta_n}(n = 0, 1, 2, \cdots, m)\ , \tag{19}$$

where $\Delta_{-2,1} = \Delta_{-1,1} = \Delta_{-1} = 1$ and $\{c_n\}_{n=0}^{\infty}$ is an S sequence for which $\Delta_n > 0$, $\Delta_{n-1,1} > 0$ when $n \leqq m$.

THEOREM 4. *Let* $\{c_n\}_{n=0}^{\infty}$ *be an S sequence such that* $\Delta_n > 0$, $\Delta_{n-1,1} > 0$ *for* $n \leqq m$. *Then the coefficients in the S-fraction* (8) *corresponding to this sequence are determined by*

$$d_{2n} = \frac{e_n}{\varepsilon_n},\ d_{2n+1} = \frac{\varepsilon_{n+1}}{e_n}$$

where $e_n = c_{2n} - \underline{c}_{2n}$, $\varepsilon_n = c_{2n-1} - \underline{c}_{2n-1}$, $0 \leqq n \leqq m$.

The result is a consequence of (19) and the identities $e_n = \Delta_n/\Delta_{n-1}$, $\varepsilon_n = \Delta_{n-1,1}/\Delta_{n-2,1}$.

REFERENCES

1. S. Barnard and J. M. Child, *Higher Algebra*, MacMillan, 1936.
2. C. Carathéodory, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die Gegebene Werte Nicht Annehmen*, Math. Ann., **64** (1907), 95-115.
3. H. Hamburger, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, Math. Ann., **82** (1920), 168-187.
4. S. Karlin and L. S. Shapley, *Geometry of moment spaces*, Mem. Amer. Math. Soc.,

no 12, 1953.
5. S. Karlin and W. J. Studden, *Tchebycheff Systems: with Applications in Analysis and Statistics*, Interscience, 1966.
6. M. G. Krein, *The ideas of P. L. Čebyšev and A. A. Markov in the theory of limiting values of integrals and their further developments*, Amer. Math. Soc. Trans. Ser. 2, **12** (1951), 1-122.
7. J. A. Shohat and J. D. Tamarkin, *The problem of moments*, Mathematical Surveys, no 1, Amer. Math. Soc., 1951.
8. H. S. Wall, *Analytic Theory of Continued Fractions*, Van Nostrand, 1948.
9. F. M. Wright, *On the backward extension of positive definite Hamburger moment sequences*, Proc. Amer. Math. Soc., **7** (1956), 413-422.

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