PACIFIC JOURNAL OF MATHEMATICS
Vol. 65, No. 2, 1976

SIDON PARTITIONS AND p-SIDON SETS

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Let Γ be a discrete abelian group, and $\Gamma^\wedge = G$ its compact abelian dual group. $E \subset \Gamma$ is called a p-Sidon set, $1 \leqq p < 2$, if $C_E(G)^\wedge \subseteq l^p(E)$. In this paper, a sufficient condition for p-Sidon is displayed and some applications are given.

The notion of p-Sidon sets was first explored in [2]. One of the highlights of [2] was the observation that if E_1 and E_2 are infinite, mutually disjoint sets, and $E_1 \cup E_2$ is dissociate, then $E_1 + E_2$ is 4/3-Sidon, but not $(4/3 - \varepsilon)$-Sidon for any $\varepsilon > 0$. This was subsequently extended in [4]: Let $E_1, E_2, \cdots, E_N$ be infinite and mutually disjoint sets whose union is dissociate. Then $E_1 + E_2 + \cdots + E_N$ is $(2N/N + 1)$-Sidon, but not $(2N/(N + 1) - \varepsilon)$-Sidon for any $\varepsilon > 0$. The methods in [2] and [4] relied on the theory of tensors, and were based on Littlewood's classical inequality ([6]): Let $(a_{ij})_{i,j=1}^N$ be a complex matrix so that $|\sum a_{ij}s_it_j| \leqq 1$ for any $(s_i)_{i=1}^N$ and $(t_j)_{j=1}^N$ where $|s_i|$, $|t_j| \leqq 1$, $i, j = 1, \cdots, N$. Then, $\sum_i (\sum_j |a_{ij}|^2)^{1/2} \leqq K$, where K is a universal constant (independent of N). In this paper, we do away with the language of tensors, and isolate the ingredients that were essential to the examples of p-Sidon sets constructed thus far (Theorem A in § 1).

In § 2, we give some applications of Theorem A: If $E \subset \Gamma$ is a Sidon set, then $E \times E$ is 4/3-Sidon in $\Gamma \times \Gamma$. We prove also that if $E \subset \Gamma$ is dissociate, then $\underbrace{E \pm E \pm \cdots \pm E}_{N\text{-times}}$ is $(2N/N + 1)$-Sidon. We conclude (§ 3) with some remarks on the connection between harmonic analysis and the metric theory of tensors.

1. Sidon partitions.

DEFINITION 1.1. $\{F_j\}$ is a Sidon partition for $E \subset \Gamma$ if (i) $\cup F_j = E$, and (ii) every bounded function that is constant on F_j can be realized as a restriction to E of a Fourier-Stieltjes transform.

REMARK. A simple category argument shows that there is $C \geqq 1$ so that whenever $\phi \in l^\infty(E)$ is constant on F_j, for all j, and $\|\phi\|_\infty \leqq 1$, then the interpolating measure μ_ϕ in the above definition can be chosen so that $\|\mu_\phi\| \leqq C$.

THEOREM A. *Suppose that $E \subset \Gamma$ can be written as $E =$*

$\{\gamma_{i_1,\ldots,i_N}\}^{\infty}{}_{i_1,\ldots,i_N=1,}$ *where for all* $1 \leqq j \leqq N$

$$\{\{\gamma_{i_1,\ldots,i_N}\}^{\infty}_{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_N=1}\}^{\infty}_{i_j=1}$$

is a Sidon partition for E. Then, E is $2N/(N+1)$*-Sidon.*

We first collect 3 lemmas. The proof of Lemma 1.2 uses standard arguments; Lemma 1.3 is a variation on a well known theme (see [1] or Appendix D in [9]), and Lemma 1.4 is essentially Littlewood's inequality stated in a harmonic analytic language.

Let $\{F_j\}$ be a Sidon partition for E. For $f \in C_E(G)$ (continuous functions with spectrum in E), write $f = \Sigma f_j$, where $\hat{f}_j = \chi_{F_j}\hat{f}$ ($\chi_{F_j} =$ characteristic function of F_j).

LEMMA 1.2. *$\{F_j\}$ is a Sidon partition for E if and only if there is* $C \geqq 1$ *so that*

$$\|\Sigma |f_j|\|_\infty \leqq K\|f\|_\infty ,$$

for all trigonometric polynomials f with spectrum in E.

Proof. ($\Rightarrow$) Let $\|\sum_j |f_j|\|_\infty = |\sum_j f_j(g)e^{i\theta}_j|$, and let $\mu \in M(G)$ be so that $\hat{\mu} = e^{i\theta_j}$ on F_j and $\|\mu\| \leqq C$. Then

$$|\textstyle\sum_j f_j(g)e^{i\theta_j}| \leqq \|\sum_j f_i(\cdot)e^{i\theta_j}\|_\infty \leqq \|\sum f_j(\cdot)\|_\infty \cdot C .$$

($\Leftarrow$) Let $(\alpha_j)_{j=1}^\infty$ be any sequence of complex scalars, $|\alpha_j| \leqq 1$. For any trigonometric polynomials $f \in C_E(G)$, we have

$$|\textstyle\sum f_j(0)\alpha_j| \leqq \sum |f_j(0)| \leqq C\|f\|_\infty .$$

Therefore, there exists $\mu \in M(G)$ so that $\hat{\mu} = \alpha_j$ on F_j.

We recall that $E \subset \Gamma$ is a $\Lambda(q)$ set, for $q > 1$, if $L^1_E(G) = L^q_E(G)$. We set $\beta_E(q) = \sup\{\|f\|_q/\|f\|_1 : f \in L^q_E, f \neq 0\}$.

LEMMA 1.3. *Let E be as in the theorem. Then E is a* $\Lambda(q)$ *set, for all q. Furthermore,* $\beta_E(q)$*, the* $\Lambda(q)$ *constant of E, is* $\mathcal{O}(q^{N/2})$.

Proof. The proof is by induction on N. When $N = 1$, E is a Sidon set, and a Sidon set is $\Lambda(q)$ for all q (see Th. 5.7.7 in [7]). Let $N > 1$, and assume that the lemma is true for $N - 1$. Let f be a trigonometric polynomial with spectrum in E, and write $f = \sum f_j$, where $\hat{f}_j = \chi_{F_j}\hat{f}$ and $\{F_j\}_j$ is a Sidon partition for E. We follow the outline of the proof of "A Sidon set is $\Lambda(q)$, for all q:"

Write

$$f_\alpha(\cdot) = \textstyle\sum f_j(\cdot)r_j(\alpha) ,$$

where (r_j) is the usual basis in $\bigoplus \boldsymbol{Z}_2$, and $\alpha \in \bigotimes \boldsymbol{Z}_2$. Since there is a measure μ_α so tnat $\hat{\mu}_\alpha|_{F_j} = r_j(\alpha)$ and $\|\mu_\alpha\| \leqq C$, it follows that

$$C\,\|f_\alpha\|_q \geqq \|f\|_q\,, \qquad \text{for all} \quad q > 2\,.$$

Therefore, as we integrate over G and over $\bigotimes \boldsymbol{Z}_2$, reverse the order of integration, we obtain

$$\int_G \left(\int_{\otimes Z_2} |\sum f_j(g) r_j(\alpha)|^q \, d\alpha\right) dg \geqq C^q \, \|f\|_q^q\,.$$

But, $(r_j)_j \subset \bigoplus \boldsymbol{Z}_2$ is $\Lambda(q)$ for all q ($\beta_{(r_j)}(q)$ is $\mathcal{O}(q^{1/2})$), and therefore

$$q^{1/2}\left[\int_G (\sum |f_j(g)|^2)^{q/2} dg\right]^{1/q} \geqq C^{-1}\, \|f\|_q\,. \tag{1}$$

Applying Minkowski's inequality to (1) (see Appendix A.1 in [9]), we obtain

$$Cq^{1/2}(\textstyle\sum_j \|f_j\|_q^2)^{1/2} \geqq \|f\|_q\,. \tag{2}$$

But, by induction hypothesis, the spectrum of each f_j is a $\Lambda(q)$ set, for all q (with $\beta(q) = \mathcal{O}(q^{(N-1)/2})$). Therefore, we finally obtain

$$Cq^{N/2}\,\|f\|_2 \geqq \|f\|_q\,.$$

LEMMA 1.4. *Let E be as in Theorem* A, *and f a trigonometric polynomial with spectrum in E. Then*

$$\|f\|_\infty \geqq C_q \sum \|f_j\|_q\,, \qquad \textit{for all} \quad 1 \leqq q < \infty\,,$$

(the C_q's depend only on q, and necessarily tend to 0 as $q \to \infty$).

Proof. By Lemma 1.2,

$$C\,\|f\|_\infty \geqq \sum \|\,|f_j|\,\|_\infty \geqq \sum \|f_j\|_1\,.$$

But, by Lemma 1.3, the spectrum of each f_j is $\Lambda(q)$, for all q ($\Lambda(q)$ constant is independent of (j), and assertion follows.

Proof of Theorem A. Let $q = 2$ in Lemma 1.4. For $N = 2$, the assertion of the theorem follows by the same argument given by Littlewood (p. 169 of [6]). The general case ($N > 2$) is argued in Lemma 3 of [4].

2. Applications.

COROLLARY 2.1. *Let E and F be Sidon sets in Γ so that $gp(E) \cap gp(F) = \{0\}$. Then $E + F$ is 4/3-Sidon.*

Proof. Let $\Gamma_1 = gp(E)$ and $\Gamma_2 = gp(F)$.

We assume that $E \subset \Gamma_1$ and $F \subset \Gamma_2$, and need to show that $E \times F$ is 4/3-Sidon in $\Gamma_1 \times \Gamma_2$. We write $E = \{\lambda_i\}$, and $F = \{\nu_j\}$, and prove that $\{\{\lambda_i\} \times \{\nu_j\}_{j=1}^\infty\}_{i=1}^\infty$ and $\{\{\nu_j\} \times \{\lambda_i\}_{i=1}^\infty\}_{j=1}^\infty$ are Sidon partitions for $E \times F$: Let $f \in C_{E\times F}(G_1 \times G_2)$ where $G_i = \Gamma_i^\wedge$, $i = 1, 2$, and $f(g_1, g_2) = \sum_i \sum_j a_{ij}(\lambda_i, g_1)(\nu_j, g_2)$. Since E and F are Sidon sets in Γ_1 and Γ_2, it follows that for any $(g_1, g_2) \in G_1 \times G_2$

$$C_1 \| f \|_\infty \geqq \sum_j | \sum_i a_{ij}(\lambda_i, g_1) |$$

and

$$C_2 \| f \|_\infty \geqq \sum_i | \sum_j a_{ij}(\nu_j, g_2) | .$$

Our claim now follows from Lemma 1.2.

2.1 extends immediately to the k-fold sum of Sidon sets that are mutually independent. We also note that $p = 4/3$ $(= 2k/k + 1)$ is sharp: This follows from 2.7 in [2] (see also 1.1 in [4]).

The corollary below partly answers the following question raised by Edwards and Ross (Remark 3.4 in [2]):

Let $E \subset \Gamma$ be so that for some $B > 0$, $R_s(E, 0) \leqq B^s$ for all $s > 0$ (see p. 124 of [7]). Is $\underbrace{E \pm E \pm \cdots \pm E}_{k\text{-times}}$ $(2k/(k + 1))$-Sidon?

COROLLARY 2.2. *Le* $E = (\gamma_j) \subset \Gamma$ *be a dissociate set in* Γ. *Then* $\underbrace{\pm E \pm \cdots \pm E}_{k\text{-times}} = E_\kappa$ *is* $2k/k + 1$-*Sidon.*

Proof. We first prove that $E + E$ is 4/3-Sidon, and then indicate how to proceed in the general case.

We claim that we can identify isomorphically $C_{E+E}(G)$ with $\{f \in C_{E\times E}(G \times G): \hat{f}(\gamma_i, \gamma_j) = \hat{f}(\gamma_j, \gamma_i)\}$: Let f be any trigonometric polynomial with spectrum in $E + E$, $f(g) = \sum_{i\leqq j} a_{ij}(\gamma_i, g)(\gamma_j, g)$, and define $f_0 \in C_{E\times E}(G \times G)$ by $\hat{f}_0(\gamma_i, \gamma_j) = \hat{f}_0(\gamma_j, \gamma_i) = a_{ij}$. We need to show that there is a K (independent of f) so that $K \| f \|_\infty \geqq \| f_0 \|_\infty$, for then our assertion will follow from Corollary 2.1. Let g_1 and $g_2 \in G$ be so that $\| f_0 \|_\infty = | f_0(g_1, g_2) |$. By symmetry, it is clear that

$$(2) \qquad | f_0(g_1, g_2) | = 1/2 \Big| \sum_{i,j} \hat{f}_0(\gamma_i, \gamma_j)((\gamma_i, g_1) + (\gamma_i, g_2))((\gamma_j, g_1) + (\gamma_j, g_2)) - \sum_{i,j} \hat{f}_0(\gamma_i, \gamma_j)(\gamma_j, g_1)(\gamma_j, g_1) - \sum_{i,j} \hat{f}_0(\gamma_i, \gamma_j)(\gamma_i, g_2)(\gamma_j, g_2) \Big| .$$

But each term on the right hand side is of the form

$$2 \sum_{i<j} a_{ij}\phi(i)\phi(j)\ , \qquad \text{where} \quad |\phi(i)| \leqq 2 \quad \text{for all} \quad i\ ,$$

and by considering Riesz products given by

$$\mu_\phi = \prod_j \left(1 + \frac{\phi(j)(\gamma_j, g) + \overline{\phi(j)}(-\gamma_j, g)}{4}\right),$$

we obtain that

$$\begin{aligned} |\sum_{i \leqq j} a_{ij}\phi(i)\phi(j)| &= 4\,|f*\mu_\phi(0)| \\ &\leqq 4\,||f||_\infty\ . \end{aligned}$$

Therefore,

$$|f_0(g_1, g_2)| = ||f_0||_\infty \leqq 12\,||f||_\infty\ .$$

To extend the above argument to "$\underbrace{E + \cdots + E}_{k\text{-times}}$ is $2k/k+1$-Sidon", we need to extend (1) above. This is simply done as follows: Let $g_1, \cdots, g_k$ be points in G so that $|f_0(g_1, \cdots, g_k)| = ||f_0||_\infty$ (f and f_0 are as above, $f \in C_{E+\cdots+E}(G)$, and $f_0 \in C_{E\times\cdots\times E}(G^k)$). For a subset $S \subset \{1, \cdots, k\}$, write

$$\psi_S(\gamma) = \sum_{r \in S} (\gamma, g_r)\ .$$

Again, by symmetry, it is clear that

$$|f_0(g_1, \cdots, g_k)| = \left|\frac{1}{k!}\sum_{m=1}^{k}(-1)^m \sum_{|S|=m} \sum \hat{f}_0(\gamma_{i_1}, \cdots, \gamma_{i_k})\psi_S(\gamma_{i_1})\cdots\psi_S(\gamma_{i_k})\right|\ .$$

Again, by appealing to Riesz products, we obtain that the right hand side is dominated by $K\,||f||_\infty$ (K depends only on k).

Our final task is to consider $E_k = \underbrace{\pm E \pm \cdots \pm E}_{k}$. We claim the following: Let $\varepsilon = (\varepsilon_j)_{j=1}^k$ be any (fixed) choice of signs. Then there is $\mu \in M(G)$ so that $\hat{\mu} = 1$ on $\varepsilon_1 E + \cdots + \varepsilon_k E \equiv \varepsilon E$, and $\hat{\mu} = 0$ on $\varepsilon'_1 E + \cdots + \varepsilon'_k E \equiv \varepsilon' E$, where ε' is any other choice of signs so that $|\{j: \varepsilon_j = -1\}| \neq |\{j: \varepsilon'_j = -1\}|$. To verify this claim, we choose $d \in [0, 2\pi)$ so that $md \neq ld (\mathrm{mod}\, 2\pi)$ for $-k \leqq m < l \leqq k$, and write a Riesz product

$$\nu = \prod_j \left(1 + \frac{e^{id}(\gamma_j, g) + e^{-id}(-\gamma_j, g)}{2}\right).$$

It is easy to check that

$$a_\varepsilon = \hat{\nu}\,|_{\varepsilon E} \neq \hat{\nu}\,|_{\varepsilon' E} = a_{\varepsilon'}\ .$$

Now, let P be a polynomial with the property that $P(a_\varepsilon) = 1$ and $P(a_{\varepsilon'}) = 0$, for all ε' as above; $\mu = P(\nu)$ gives us the desired separation. Therefore, by the above separation, to prove $2k/(k+1)$-Sidonicity of E_k, it suffices to show that εE is $2k/k+1$-Sidon for a fixed ε. Slightly modified, the previous arguments now apply.

REMARK. It was brought to our attention that G. Woodward independently obtained (Th. 4 in [10]) a somewhat weaker result than the above Corollary 2.2.

3. Some remarks on the metric theory of tensors. We recall that p-Sidon sets were originally manufactured by a machinary of tensors. Our presentation that avoided that language suggests that tensor analytic results can be produced in the analogous harmonic analytic setting ($E_1 + E_2$, $E_1 \cup E_2$ dissociate), and then translated (via Riesz products) to the language of tensors. Specifically, Lemma 1.4 when $q = 2$, and $E = E_1 + E_2$, $E_1 \cup E_2$ dissociate, is precisely Littlewood's classical inequality. In fact, Grothendieck's inequality, which is an extension of Littlewood's, can also be deduced in this setting, through technically, it is the same proof as the one given by Grothendieck (see pp. 62-64 in [3], and [5]).

The inequality can be stated as follows:

Let $\{x_i\}_{i=1}^N$ and $\{y_j\}_{j=1}^N$ be vectors on the unit sphere S^N in $\boldsymbol{R}^N$, equipped with the Euclidean norm. Let $E = \{\lambda_i\}_{i=1}^\infty$ and $\{\nu_j\}_{j=1}^\infty$ be disjoint subsets in Γ so that $E \cup F$ is dissociate. Set $\phi(\lambda_i + \nu_j) = (x_i, y_j)$ for $i, j = 1, \cdots, N$, and 0 otherwise. ($(\cdot, \cdot)$ denotes the usual inner product in $\boldsymbol{R}^N$.) Then, $\|\phi\|_{B(E+F)} \leqq K_G$, where K_G is independent of $\{x_i\}$, $\{y_j\}$ and N.

The proof is based on the following elementary fact (see [3]):

LEMMA. *Let σ be the normalized rotation invariant measure on S^N. Then, for any $x, y \in S^N$*

$$\int_{S^N} \operatorname{sign}(x, y) \operatorname{sign}(y, u) d\sigma(u) = 1 - \frac{2}{\pi} \operatorname{arc\,cos}(x, y) .$$

Proof of Grothendieck's inequality. For each $u \in S^N$, let μ_u be the Riesz product so that $\hat{\mu}_u(\lambda_i + \nu_j) = \operatorname{sign}(x_i, u) \operatorname{sign}(y_j, u)$, and integrate over S^N the B-valued function μ_u:

$$\mu = \int_{S^N} \mu_u d\sigma(u) \in M(G) .$$

From the above lemma, it is clear that if $\nu = \sin((\pi/2)\mu) \in M(G)$, then $\hat{\nu}(\lambda_i + \nu_j) = (x_i, y_j)$. Furthermore, $\|\nu\| \leqq \sinh(\pi/2)$.

Added in proof. Another proof of Grothendieck's inequality and some of its extensions are given by the author in "A uniformity property for $\Lambda(2)$-sets and Grothendieck's inequality," (to appear).

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Received July 11, 1975 and in revised form March 15, 1976.

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