PACIFIC JOURNAL OF MATHEMATICS
Vol. 66, No. 1, 1976

THE BOUNDARY BEHAVIOR OF HENKIN'S KERNEL

PATRICK AHERN AND ROBERT SCHNEIDER

In this paper, the boundary behaviour of a reproducing kernel, introduced by Henkin, for strictly pseudoconvex domains is studied. As an application, an improved version of a known result about generators of certain maximal ideals is given.

The boundary behaviour of the Bergmann kernel $B(z,\zeta)$ for a strictly pseudoconvex domain has been studied by Bergmann **[1]** and Hörmander **[5]**. Among other things, they determine the rate at which $B(z,z)$ goes to infinity as z approaches a boundary point of the domain. Another type of reproducing kernel has been introduced by Henkin **[3]** for bounded strictly pseudoconvex domains D, in $\mathbf{C}^n$. Henkin's kernel is of the form $K(\zeta,z)/\Phi^n(\zeta,z)$, where K and Φ are holomorphic in a neighborhood of $\bar{D}$ for each ζ in ∂D, the boundary of D. The denominator Φ has the properties that $\Phi(\zeta,\zeta)=0$ for all $\zeta\in\partial D$ and that $\Phi(\zeta,z)\neq 0$ if $z\in\bar{D}\backslash\{\zeta\}$. For z near ζ, Φ is given explicitly (up to a nonvanishing factor) in terms of the plurisubharmonic function ρ that defines the domain D. Precise statements about the way $\Phi(\zeta,z)$ approaches zero as z approaches ζ from inside D are given in Henkin's paper **[3]**. We show that this determines the behaviour of the kernel K/Φ^n by showing that $K(\zeta,\zeta)\neq 0$.

It has been proven in **[4]**, **[6]**, **[7]** and **[9]**, that if f is in the space $A(D)$ of functions continuous on $\bar{D}$ and holomorphic in D and if $a\in D$ then there exist functions $g_1,\cdots,g_n\in A(D)$ such that

$$f(z)-f(a)=\sum_{j=1}^{n}(z_j-a_j)g_j(z).$$

This is a solution to a problem originally posed by Gleason **[2]** for the unit ball in $\mathbf{C}^n$. Using Henkin's integral formula and our result on the behaviour of Henkin's kernel we can improve the result just stated in two ways. Firstly, we show that the g_i can be chosen in such a way that the association between f and the n-tuple of functions $(g_1,\cdots,g_n)$ is linear, and secondly we show that the g_i may be also chosen to depend analytically on a as well as on z.

1. Notation. D will always denote a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ defined as $D=\{z:\rho(z)<0\}$, where ρ is defined and strictly plurisubharmonic in a neighborhood U of $\bar{D}$, such that the gradient of ρ is not zero on the boundary of D. For $\epsilon>0$ we let

$D_\epsilon = \{z \in U: \rho(z) < \epsilon\}$ and if V is a neighborhood of ∂D we let $V_\epsilon = V \cap D_\epsilon$. We denote by $C^k(V_\epsilon, H(D_\epsilon))$ the space of C^k functions on V_ϵ with values in the space $H(D_\epsilon)$ of functions holomorphic in D_ϵ. In other words, functions that are C^k on $V_\epsilon \times D_\epsilon$ and holomorphic in D_ϵ for each fixed $\zeta \in V_\epsilon$. Finally, we let $S_{z,\delta} = \{\zeta : |\zeta - z| < \delta\}$.

2. The work of Henkin **[3]**, modified slightly by Øvrelid **[8]**, shows that if D has a C^3 boundary then there are functions K and Φ and a neighborhood V of ∂D and an $\epsilon > 0$ such that:

2.1. (a) $K \in C^1(V_\epsilon, H(D_\epsilon))$ and

$$\Phi \in C^2(V_\epsilon, H(D_\epsilon)).$$

(b) $\Phi(\zeta, z) \neq 0$ if $z \in \bar{D} \setminus \{\zeta\}$.

2.2. If $f \in A(D)$ then

$$f(z) = \int_{\partial D} f(\zeta) \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \, d\sigma(\zeta), \qquad \text{for all } z \in D,$$

where $d\sigma$ is $2n - 1$ dimensional volume measure on ∂D.

2.3. There are constants γ, $\delta_0 > 0$ such that for all $z \in \bar{D}$ and $0 < \delta < \delta_0$,

$$\int_{\partial D \cap S_{z,\delta}} \frac{|\zeta - z|}{|\Phi^n(\zeta, z)|} \, d\sigma(\zeta) \leqq \gamma\delta \log \frac{1}{\delta}.$$

THEOREM A. *Suppose K and Φ satisfy properties* 2.1, 2.2, *and* 2.3, *then $K(\zeta_0, \zeta_0) \neq 0$ for any $\zeta_0 \in \partial D$.*

Proof. We assume that $K(\zeta_0, \zeta_0) = 0$ and arrive at a contradiction. If $K(\zeta_0, \zeta_0)$ were zero then, from property 2.1, there would be a constant M such that

(a) $|K(\zeta, \zeta_0)| \leqq M|\zeta - \zeta_0|$,
(b) $|K(\zeta, z) - K(\zeta_0, z)| \leqq M|\zeta - \zeta_0|$,
(c) $|K(\zeta_0, z)| \leqq M|z - \zeta_0|$.

Now it follows from (a) and 2.3 that

$$\int_{\partial D} \frac{|K(\zeta, \zeta_0)|}{|\Phi^n(\zeta, \zeta_0)|} \, d\sigma(\zeta) < \infty.$$

We will show that if $f \in A(D)$, then

2.4. $$f(\zeta_0) = \int_{\partial D} f(\zeta) \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} \, d\sigma(\zeta).$$

Due to the remark just made, the right hand side of 2.4 is well-defined. To prove 2.4 we show that as z approaches ζ_0 in a certain way, the expression,

2.5. $$f(z) - \int_{\partial D} f(\zeta) \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} \, d\sigma(\zeta)$$

converges to 0. Now by 2.2 we have,

$$\begin{aligned} f(z) - \int_{\partial D} f(\zeta) \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} \, d\sigma(\zeta) &= \int_{\partial D} f(\zeta) \left[\frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} - \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \right] d\sigma(\zeta) \\ &= \int_{\partial D \setminus S_{\zeta_0, \delta}} f(\zeta) \left[\frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} - \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \right] d\sigma(\zeta) \\ &\quad + \int_{\partial D \cap S_{\zeta_0, \delta}} f(\zeta) \left[\frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} - \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \right] d\sigma(\zeta). \end{aligned}$$

Now for any fixed $\delta > 0$, the first integral above approaches zero as z approaches ζ_0, since we can take the limit under the integral sign. As for the second integral, its absolute value is not greater than

$$\int_{\partial D \cap S_{\zeta_0, \delta}} |f(\zeta)| \frac{|K(\zeta, \zeta_0)|}{|\Phi^n(\zeta, \zeta_0)|} \, d\sigma(\zeta) + \int_{\partial D \cap S_{\zeta_0, \delta}} |f(\zeta)| \frac{|K(\zeta, z)|}{|\Phi^n(\zeta, z)|} \, d\sigma(\zeta).$$

Now by (a) and 2.3, the first of these integrals is majorized by $M \|f\|_\infty \gamma \delta \log 1/\delta$. To estimate the second of these integrals we let z approach ζ_0 along the inward normal to ∂D. Now if z lies on this normal and if δ is sufficiently small then there is a constant C such that $|z - \zeta_0| \leqq C|z - \zeta|$ and $|\zeta - \zeta_0| \leqq C|z - \zeta|$ as long as $|z - \zeta_0| < \delta$ and $|\zeta - \zeta_0| < \delta$, and hence $|K(\zeta, z)| \leqq |K(\zeta, z) - K(\zeta_0, z)| + |K(\zeta_0, z)| \leqq M|\zeta - \zeta_0| + M|z - \zeta_0| \leqq 2MC|\zeta - z|$. So with these assumptions,

$$\begin{aligned} \int_{\partial D \cap S_{\zeta_0, \delta}} |f(\zeta)| \frac{|K(\zeta, z)|}{|\Phi^n(\zeta, z)|} \, d\sigma(\zeta) &\leqq \|f\|_\infty 2MC \int_{\partial D \cap S_{\zeta_0, \delta}} \frac{|\zeta - z|}{|\Phi^n(\zeta, z)|} \, d\sigma(\zeta) \\ &\leqq 2MC \|f\|_\infty \int_{\partial D \cap S_{z, 2\delta}} \frac{|\zeta - z|}{|\Phi^n(\zeta, z)|} \, d\sigma(\zeta) \\ &\leqq 2MC \|f\|_\infty \gamma 2\delta \log \frac{1}{2\delta}, \quad \text{if } 2\delta < \delta_0. \end{aligned}$$

So now if we first choose δ sufficiently small and then let z approach ζ_0

along the inward normal we see that 2.5 approaches zero. This proves 2.4. Now it is easy to finish the proof of the theorem. We take $f \in A(D)$ such that $f(\zeta_0) = 1$ and $|f(\zeta)| < 1$ for $\zeta \in \bar{D} \backslash \{\zeta_0\}$. Applying 2.4 to f^N we get

$$1 = f^N(\zeta_0) = \int_{\partial D} f^N(\zeta) \frac{K(\zeta, \zeta_0)}{\Phi^n(\zeta, \zeta_0)} d\sigma(\zeta).$$

However the right hand side approaches zero, by the bounded convergence theorem. This contradiction completes the proof of Theorem A.

We now apply Theorem A to obtain

THEOREM B. *Suppose D is a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with a C^3 boundary. There is a linear mapping $T: A(D) \rightarrow H(D \times D)^n$ such that $(Tf)_i \in C[(\bar{D} \times \bar{D}) \backslash \{(z, z): z \in \partial D\}]$ for every $f \in A(D)$ and such that*

$$f(z) - f(\omega) = \sum (z_i - \omega_i)(Tf)_i(z, \omega).$$

Proof. From Henkin's integral formula we see that

$$f(z) - f(\omega) = \int f(\zeta) \frac{\Phi^n(\zeta, \omega)K(\zeta, z) - \Phi^n(\zeta, z)K(\zeta, \omega)}{\Phi^n(\zeta, z)\Phi^n(\zeta, \omega)} d\sigma(\zeta).$$

If $L(\zeta, z, \omega) = \Phi^n(\zeta, \omega)K(\zeta, z) - \Phi^n(\zeta, z)K(\zeta, \omega)$, then $L \in C^1(V_\epsilon, H(D_\epsilon \times D_\epsilon))$ and $L(\zeta, z, z) \equiv 0$, so by the argument given as a remark on page 148 of **[8]** there are functions $L_i \in C^1(V_{\epsilon'}, H(D_{\epsilon'} \times D_{\epsilon'}))$ (for some $\epsilon' < \epsilon$) such that

$$L(\zeta, z, \omega) = \sum_{i=1}^{n} (z_i - \omega_i) L_i(\zeta, z, \omega).$$

Hence, we have

$$f(z) - f(\omega) = \sum_{i=1}^{n} (z_i - \omega_i) \int f(\zeta) \frac{L_i(\zeta, z, \omega)}{\Phi^n(\zeta, z)\Phi^n(\zeta, \omega)} d\sigma(\zeta).$$

So it remains to show that

$$f_i(z, \omega) = \int f(\zeta) \frac{L_i(\zeta, z, \omega)}{\Phi^n(\zeta, z)\Phi^n(\zeta, \omega)} d\Phi(\zeta)$$

satisfies the statement of the theorem. Certainly $f_i \in H(D \times D)$ so we need only show that $f_i \in C[(\bar{D} \times \bar{D}) \backslash \{(z, z): z \in \partial D\}]$. Suppose $(z, \omega) \in D \times D$ and $(z, \omega) \to (\zeta_0, \omega_0) \in \bar{D} \times \bar{D} \backslash \{(z, z): z \in \partial D\}$. We wish to show that $f_i(z, \omega)$ has a limit. We will assume that $\zeta_0 \in \partial D$ and $\omega_0 \in \partial D$ and $\zeta_0 \neq \omega_0$. The other possibilities are treated in a similar fashion (and are easier). By Theorem A, $K(\zeta_0, \zeta_0) \neq 0$, and $K(\omega_0, \omega_0) \neq 0$. Hence there is a $\delta > 0$ such that if $|z - \zeta_0| \leqq 2\delta$ and $|\zeta - \zeta_0| \leqq 2\delta$ then $K(z, \zeta) \neq 0$, and if $|z - \omega_0| \leqq 2\delta$ and $|\zeta - \omega_0| \leqq 2\delta$ then $K(z, \zeta) \neq 0$. We also assume $4\delta < |\zeta_0 - \omega_0|$. Let $\varphi(z)$ be a C^∞ function that is identically equal to 1 if $|z| \leqq \delta^2$ and identically 0 if $|z| \geqq (2\delta)^2$. Now we write

$$\begin{aligned} f_i(z, \omega) = & \int f(\zeta) \frac{L_i(\zeta, z, \omega)\varphi(|z - \zeta|^2)}{K(\zeta, z)\Phi^n(\zeta, \omega)} \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \, d\sigma(\zeta) \\ & + \int f(\zeta) \frac{L_i(\zeta, z, \omega)\varphi(|\omega - \zeta|^2)}{K(\zeta, \omega)\Phi^n(\zeta, z)} \frac{K(\zeta, \omega)}{\Phi^n(\zeta, \omega)} \, d\sigma(\zeta) \\ & + \int f(\zeta) \frac{L_i(\zeta, z, \omega)}{\Phi^n(\zeta, z)\Phi^n(\zeta, \omega)} [1 - \varphi(|z - \zeta|^2) - \varphi(|\omega - \zeta|^2)] \, d\sigma(\zeta), \end{aligned}$$

for $|z - \zeta_0| < \delta$ and $|\omega - \omega_0| < \delta$. The third term has a limit as $(z, \omega) \to (\zeta_0, \omega_0)$ since we may take the limit under the integral sign. We write the first term as

$$\text{2.6.} \qquad \int f(\zeta)\chi(\zeta, z, \omega) \frac{K(\zeta, z)}{\Phi^n(\zeta, z)} \, d\sigma(\zeta),$$

where all we need to know about χ is that it is continuous on $\partial D \times \bar{D} \times S_{\omega_0, \delta}$ and that there is a constant C such that $|\chi(\zeta, z, \omega) - \chi(\zeta', z, \omega) \leqq C|\zeta - \zeta'|$, for all z, ω, ζ, and ζ'. Now we just imitate the proof of Lemma 4.3 of **[3]** to see that 2.6 has a limit as $(z, \omega) \to (\zeta_0, \omega_0)$. The second term is handled in the same way as the first. This completes the proof.

Note that if

$$f(z) - f(\omega) = \sum_{i=1}^{n} (z_i - \omega_i) g_i(z, \omega) \quad \text{then} \quad \frac{\partial f}{\partial z_i}(z) = g_i(z, z),$$

so so that g_i need not be in $A(D \times D)$ when $f \in A(D)$.

References

1. S. Bergman, *Über die Kernfunktion eines Bereiches und ihr Verhalten am Rande*, I–II., J. Reine Angew. Math., **169** (1933), 1–42, and **172** (1935), 89–128.

2. A. Gleason, *Finitely generated ideals in Banach algebras*, J. Math. Mech., **13** (1964), 125–132.

3. G. Henkin, *Integral representations of functions holomorphic in strictly pseudoconvex domains and some applications*, Mat. Sb., **78** (120) (1969), 597–616.
4. ———, *The approximation of functions in pseudoconvex domains and a Theorem of Z,* L. Leibenzan. Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys., **19** (1971), 37–42.
5. L. Hörmander, *L^2 estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Math., **113** (1965), 89–152.
6. N. Kerzman and A. Nagel, *Finitely generated ideals in certain function algebras*, J. Functional Analysis, **7** (1971), 212–215.
7. I. Lieb, *Die Cauchy-Riemannischen Differential-gleichungen auf streng pseudokonvexen Gebieten, Beschrankte Lösungen*, Math. Ann., **190** (1970), 6–44.
8. N. Øvrelid, *Integral representation formulas and L^p estimates for the $\bar{\partial}$ equation*, Math. Scand., **29** (1971), 137–160.
9. ———, *Generators of the maximal ideals of $A(\bar{D})$*, Pacific J. Math., **39**, no. 1, (1971), 219–223.

Received March 11, 1976. The first author was partially supported by an NSF grant. The second author was partially supported by a grant from the research foundation of CUNY.

UNIVERSITY OF WISCONSIN–MADISON
AND
HERBERT LEHMAN COLLEGE OF CUNY