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AN UPPER BOUND FOR THE PERIOD OF THE SIMPLE CONTINUED FRACTION FOR $\sqrt{D}$

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Let $p(D)$ denote the length of the period of the simple continued-fraction expansion of $\sqrt{D}$, where D is a positive non-square integer. In this paper, it is shown that

$$p(D) < 0.72D^{1/2} \log D$$

for all squarefree $D > 7$, and an estimate for $p(D)$ is given when D is not squarefree.

1. Introduction. The problem of finding a good upper bound for the length $p(D)$ of the period of the simple continued fraction for $\sqrt{D}$, where D is a positive nonsquare integer, has received relatively little attention. Recently, Hickerson [6] and Hirst [7] have given estimates for $p(D)$; Hickerson's estimate implies that

$$\log p(D) < \log D(1/2 + \log 2(\log \log D)^{-1} + o(\log \log D)^{-1}) , \tag{1.1}$$

where D is nonsquare, and Hirst's implies that

$$p(D) < 2D^{1/2} \log D + 0(D^{1/2}) , \tag{1.2}$$

where D is squarefree. Both authors give more precise error terms, but these are not relevant here. For general nonsquare $D > 0$, Hirst shows that

$$p(D) = O(D^{1/2}s \log D) \tag{1.3}$$

uniformly in s, where s^2 is the largest square factor of D. For sufficiently large squarefree D, (1.2) is clearly better than (1.1). On the other hand, (1.3) is better than (1.1) only when s, regarded as a function of D, is sufficiently small. Pen and Skubenko [14] have given an upper bound for $p(D)$ which we will discuss later; it depends on the size of the least positive solution of $x^2 - Dy^2 = 1$.

The authors [17] have used combinatorial methods to show that

$$p(D) < 0.82D^{1/2} \log D$$

for all squarefree $D > 7$. In this paper, we use a different approach which refines this result to

$$p(D) < 0.72D^{1/2} \log D \tag{1.4}$$

for all squarefree $D > 7$. It is also shown that

$$p(D) < 3.76D^{1/2} \log (D/s^2) \tag{1.5}$$

for all nonsquare $D > 0$, where s is defined in (1.3). The data in [1] suggest that $p(D) = o(D \log D)^{1/2}$.

It is clear that (1.5) is better than (1.1) for all large D. Moreover, (1.5) is an improvement on (1.3) in that it decreases, rather than increases, with s. When D is squarefree, we also obtain a more precise theorem which implies that

$$p(D) < AD^{1/2} \log D.\ 2^{-\nu} \tag{1.6}$$

for $D > 1$, where ν is the number of prime factors of D and A is a computable constant. We conclude the paper by discussing the question of finding functions g such that $p(D) > Cg(D)$ for an infinity of D, where $g(D) \to \infty$ with D and where C is a positive constant.

We use the elementary theory of continued fractions and the theory of the units and class number of a real quadratic field as found, for example, in [2] or [11]. All small Roman letters denote positive integers unless otherwise stated; the phrase "continued fraction" always means "simple continued fraction".

2. A bound for $p(D)$ in terms of $L(1, \chi)$. We first prove a preliminary estimate. Suppose that D is a squarefree integer > 1. Then

$$p(D) < \mu \log \varepsilon_0 / \log \alpha \tag{2.1}$$

where

$$\alpha = (1 + \sqrt{5})/2\ , \tag{2.2}$$

$$\varepsilon_0 = (u_0 + v_0\sqrt{D})/2 \tag{2.3}$$

is the fundamental unit of $Q(\sqrt{D})$, and

$$\mu = 3 \text{ if } 2 \nmid u_0\ , \qquad \mu = 1 \text{ if } 2 \mid u_0\ . \tag{2.4}$$

We note that, since $u_0^2 - Dv_0^2 = \pm 4$, $2 \nmid u_0$ implies $D \equiv 5 \pmod 8$; it follows from (2.4) that

$$\mu = 1 \text{ if } D \not\equiv 5 \pmod 8\ , \qquad \mu \mid 3 \text{ if } D \equiv 5 \pmod 8\ . \tag{2.5}$$

Now let $[q_0, \overline{q_1, \cdots, q_p}]$ be the continued-fraction expansion of $\sqrt{D}$, where $p = p(D)$; we have $q_0 = [\sqrt{D}]$. Further, if we formally define

$$A_{-2} = 0\ , \quad B_{-2} = 1\ , \quad A_{-1} = 1\ , \quad B_{-1} = 0\ ,$$

and use the recursions

$$A_n = q_n A_{n-1} + A_{n-2}\ , \qquad B_n = q_n B_{n-1} + B_{n-2}\ ,$$

for $n \geqq 0$, then A_n/B_n is the n^{th} convergent of the continued fraction for $\sqrt{D}$.

The relations $f_0 = 0$, $f_1 = 1$, $f_m = f_{m-1} + f_{m-2}$ for $m \geqq 2$, define the mth Fibonacci number f_m. Hence we immediately obtain, by induction, the inequalities

$$A_n \geqq f_{n+2}\,, \qquad B_n \geqq f_{n+1}\,, \tag{2.7}$$

for $n \geqq -1$. Since $\alpha^2 = (1 + \sqrt{5})^2/4 = \alpha + 1$, we find that $f_{n+2} \geqq \alpha^n$ for $n \geqq -1$; from (2.7), it follows that

$$\eta = A_{p-1} + B_{p-1}\sqrt{D} > A_{p-1} + B_{p-1} > \alpha^p\,, \tag{2.8}$$

where $p = p(D)$.

A similar induction yields the better estimate

$$A_n \geqq q_0 f_{n+1} + f_n \qquad (n \geqq 0)\,;$$

using the standard formula for f_n in terms of α, this produces

$$\eta > (q_0/\sqrt{5})\alpha^p\,, \tag{2.9}$$

as used in [17]. We later show that this sharper inequality (2.9) only improves Theorem 1 by an amount that is negligible when D is large.

Now the least positive solution (x_1, y_1) of $x^2 - Dy^2 = \pm 1$ is (A_{p-1}, B_{p-1}): here we take the minus sign if $x^2 - Dy^2 = -1$ is solvable; otherwise, we take the plus sign. Then the number η in (2.8) is a unit in $Q(\sqrt{D})$; indeed,

$$\eta = \varepsilon_0^\mu\,, \tag{2.10}$$

where ε_0 is the fundamental unit of $Q(\sqrt{D})$ and μ is either 1 or 3. Then (2.8) and (2.10) give

$$p(D) \log \alpha < \mu \log \varepsilon_0\,,$$

as stated in (2.1).

We now apply a standard class-number formula to get the desired inequality for $p(D)$ in terms of $L(1, \chi)$. For squarefree $D > 1$, the discriminant Δ of $Q(\sqrt{D})$ is given by

$$\Delta = 4D \text{ if } D \not\equiv 1 \pmod 4, \quad \Delta = D \text{ if } D \equiv 1 \pmod 4\,. \tag{2.11}$$

It is known (see, for example, [2]) that

$$\log \varepsilon_0 = \sqrt{\Delta}\, L(1, \chi)/2h\,, \tag{2.12}$$

where h is the class number of $Q(\sqrt{D})$,

$$L(1, \chi) = \sum_{n \geqq 1} (\Delta \mid n) n^{-1} , \tag{2.13}$$

and $(\Delta \mid n)$ is the Kronecker symbol (for a concise discussion of this symbol, see [4]). From (2.1) and (2.12), we obtain the

LEMMA. *Let D be positive and squarefree; then*

$$p(D) < \mu\sqrt{\Delta}\, L(1, \chi)/(2h \log \alpha) , \tag{2.14}$$

where μ and α are given by (2.4) *and* (2.2) *respectively.*

We should here make some remarks about (2.14). It is easily proved by partial summation that $L(1, \chi) < A \log D$; so, by (2.14), we have $p(D) < BD^{1/2} \log D$ (here A and B are constants). In the next section, we will use an inequality due to Hua [8] to obtain an estimate for $L(1, \chi)$ of the above form with a better constant (for large D) than that given by the partial summation method alone.

The Riemann hypothesis for $L(s, \chi)$ implies that $L(1, \chi) = 0(\log \log D)$ [12, p. 367]; this result would give the estimate

$$p(D) = 0(D^{1/2} \log \log D) .$$

On the other hand, it is known that $L(1, \chi) > C \log \log D$ for an infinity of squarefree D, where C is a positive constant (see, for example, [9]). However, we do not know whether there is a positive constant E such that $p(D) > ED^{1/2} \log \log D$ for an infinity of D; more generally, we do not know if (2.14) is sharp since we can not prove if there is a constant $F > 0$ such that

$$p(D) > FD^{1/2} L(1, \chi)/h(D) \tag{2.15}$$

for an infinite sequence of squarefree D. (We shall return to the question of lower bounds for $p(D)$ in § 6.)

It is easily seen that (2.15) can not hold for all nonsquare D. Since the right members of (2.1) and (2.14) are equal, (2.15) is equivalent to $p(D) > G\mu \log \varepsilon_0 = G \log \eta$; so (2.9) implies that $p(D) > H \log D$. Here G and H are positive constants. But $p(D) = 1$ when $D = a^2 + 1$.

If we were to use (2.9) instead of (2.8), (2.14) would be replaced by

$$p(D) < \frac{\mu\sqrt{\Delta}\, L(1, \chi)}{2h \log \alpha} - \frac{\log (q_0/\sqrt{5})}{\log \alpha} ; \tag{2.16}$$

we later show that (2.16) yields no significant improvement in (4.1) for large D.

We conclude this section by noting the estimate, due to Pen and Skubenko [14],

$$p(D) < \log(T + U\sqrt{D})/\log\alpha \, , \tag{2.17}$$

where (T, U) is the least positive solution of $x^2 - Dy^2 = 1$ and D is squarefree. Now, $T + U\sqrt{D} = \eta$ or η^2 according as $x^2 - Dy^2 = -1$ is not or is solvable (see, for example, [11]). Hence, by (2.10), $T + U\sqrt{D} = \varepsilon_0^{\mu}$ or $\varepsilon_0^{2\mu}$ according as the first or second alternative holds. So (2.17) is equivalent to

$$p(D) < \mu \log \varepsilon_0/\log\alpha \quad \text{or} \quad p(D) < 2\mu \log\varepsilon_0/\log\alpha \, ,$$

respectively. Thus (2.1) is always at least as good at (2.17), and is sometimes better than it by a factor of 2. Furthermore, our method is considerably simpler and more straightforward than that of [14].

Pen and Skubenko also give an inequality corresponding to (2.14) but they do not obtain any explicit numerical upper bound for $p(D)$.

3. Bounds for $L(1, \chi)$. Let $L(1, \chi)$ be given by (2.13), where Δ is now any nonsquare positive integer $\equiv 0$ or $1 \pmod 4$. Hua has shown [8] that

$$L(1, \chi) < 1 + \sum_{n=1}^{j} \frac{2\,|S(n)|}{n(n+1)(n+2)} \, , \tag{3.1}$$

where

$$S(n) = \sum_{a=1}^{n} \sum_{m=1}^{a} (\Delta \mid m) \tag{3.2}$$

and

$$j = [\sqrt{\Delta}\,] \, . \tag{3.3}$$

We note that $j \geqq 2$, since $\Delta \geqq 5$.

We first estimate $S(n)$ and consider three cases.

(i) $\Delta \equiv 0 \pmod 4$. Then $(\Delta \mid 2r) = 0$ so that $|(\Delta \mid m)| \leqq (1 - (-1)^m)/2$. It follows at once from (3.2) that

$$|S(n)| < (n+1)^2/4 \, . \tag{3.4}$$

(ii) $\Delta \equiv 1 \pmod 8$. Then we have trivially

$$|S(n)| \leqq n(n+1)/2 \, . \tag{3.5}$$

(iii) $\Delta \equiv 5 \pmod 8$. We estimate $L(1, \chi)$ for such Δ without using a bound for $S(n)$.

The sequence $D_n = 1 + 1/2 + \cdots + 1/n - \log n$ is easily proved to be monotone decreasing to Euler's constant γ; see, for example,

[16], where it is proved that

$$\frac{1}{2n(n+1)} > D_n - D_{n+1} > \frac{1}{(n+1)(2n+1)} .$$

Then

$$-1 + \sum_{n=1}^{m-1} (D_n - D_{n+1}) = -1 + \sum_{n=1}^{\infty} (D_n - D_{n+1}) - \sum_{n=k}^{\infty} (D_n - D_{n+1}) .$$

Since $D_n - D_{n+1} = -1/(n+1) + \log(n+1) - \log n$, we at once obtain

$$\log k - \sum_{k=1}^{k} \frac{1}{n} = \lim_{k\to\infty} \left(\log k - \sum_{n=1}^{k} \frac{1}{n}\right) - \sum_{n=k}^{\infty} (D_n - D_{n+1}) .$$

Thus

$$\begin{aligned} \sum_{n=1}^{k} \frac{1}{n} &= \log k + \gamma + \sum_{n=k}^{\infty} (D_n - D_{n+1}) \\ &< \log k + \gamma + \sum_{n=k}^{\infty} \frac{1}{2k(k+1)} , \end{aligned}$$

that is,

$$\sum_{n=1}^{k} \frac{1}{n} < \log k + \gamma + \frac{1}{2k} . \tag{3.6}$$

We now apply (3.6) and our estimates for $S(n)$ to (3.1). Write L for $L(1, \chi)$, and consider three cases.

Case 1. $\Delta = 4D \equiv 0 \pmod 4$.
Substitute (3.4) into (3.1) to give

$$\begin{aligned} L &< \frac{1}{2}\sum_{n=1}^{j} \frac{1}{n} + 1 - \frac{1}{4}\left(1 + \frac{1}{2} - \frac{1}{j+1} - \frac{1}{j+2}\right) \\ &< \frac{1}{2}\sum_{n=1}^{j} \frac{1}{n} + \frac{5}{8} + \frac{1}{2j} . \end{aligned}$$

Apply (3.6) and (3.3) to give

$$L < \frac{1}{4}\log D + \frac{1}{2}\log 2 + \frac{1}{2}\gamma + \frac{5}{8} + \frac{3}{4j} .$$

Thus

$$L < \frac{1}{4}\log D + 1.28 \quad \text{for} \quad \Delta = 4D,\ D > 1500 . \tag{3.7}$$

(Note that, in this section, Δ is any nonsquare positive integer $\equiv 0$ or 1, mod 4, and so need not satisfy (2.11).)

Case 2. $\Delta = D \equiv 1 \pmod 8$.
By (3.1) and (3.5), we have

$$L < 1 + \sum_{n=1}^{j} \frac{1}{n+2} < \sum_{n=1}^{j} \frac{1}{n} - \frac{1}{2} + \frac{2}{j} ;$$

so we obtain, as before,

$$L < \frac{1}{2} \log D + \gamma - \frac{1}{2} + \frac{5}{2j} .$$

Thus

$$L < \frac{1}{2} \log D + 0.09 \tag{3.8}$$

for $\Delta = D \equiv 1 \pmod 8$ and $D > 64{,}000$.

Case 3. $\Delta = D \equiv 5 \pmod 8$. Here $(\Delta \mid 2) = -1$; hence

$$L = \sum_{i=1}^{\infty} (D \mid 2i-1)(2i-1)^{-1} \sum_{k=0}^{\infty} (-2)^{-k} , \tag{3.9}$$

since the first series is convergent, and the second is absolutely convergent. Thus

$$L = \frac{2}{3} \sum_{n=1}^{\infty} (d \mid n) n^{-1} . \tag{3.10}$$

where $d = 4D$. Now use Case 1 and apply (3.7) to (3.10); this gives

$$L < \frac{2}{3}\left(\frac{1}{4} \log D + 1.28\right) < \frac{1}{6} \log D + 0.86 \tag{3.11}$$

for $\Delta = D \equiv 5 \pmod 8$, $D > 1500$.

4. Upper bounds for $p(D)$ when D is squarefree. We use the preceding estimates for $L(1, \chi)$ to prove

THEOREM 1. *Suppose that D is squarefree and > 1, and let μ and α be given by* (2.4) *and* (2.2), *respectively. Let r be the number of distinct prime factors of Δ, and set*

$$t = r - 1 \quad \textit{if } D \textit{ is a sum of two squares} ,$$
$$t = r - 2, \quad \textit{otherwise} .$$

Thus $t \geqq 0$, and

$$p(D) < \mu D^{1/2}(A \log D + B)(2^t \log \alpha)^{-1} \quad \textit{for} \quad D > 64000 , \tag{4.1}$$

where the constants A *and* B *are given by the following table.*

(4.2)

	A	B
$D \equiv 2, 3 \pmod 4$	1/4	1.28
$D \equiv 1 \pmod 8$	1/4	0.045
$D \equiv 5 \pmod 8$	1/12	0.43

Proof. On combining (2.14) with (3.7), (3.8), and (3.11), respectively, we find that

$$p(D) < \mu D^{1/2}(A \log D + B)/(h \log \alpha) \quad \text{for} \quad D > 64000 ,$$

where A and B are given by (4.2). By a standard theorem on the class number h of a quadratic field [2, p. 225], we have $2^t \mid h$; hence $2^t \leqq h$, which gives (4.1).

We now derive some corollaries; in the remainder of this section, D denotes a squarefree integer > 1.

COROLLARY 1. *For any fixed* $\varepsilon > 0$, *and all sufficiently large* D *we have*

$$p(D) < (A_1 + \varepsilon)D^{1/2} \log D ,$$

where $A_1 = 1/(4 \log \alpha) < 0.52$. *In particular, we have*

$$p(D) < 0.52 D^{1/2} \log D \quad \textit{for} \quad D > D_1 ,$$

where D_1 *is a computable constant.*

Proof. We have $\mu A \leqq 1/4$ by (4.2) and (2.5), and $2^t \geqq 1$. The corollary follows at once.

COROLLARY 2. *We have*

$$p(D) < \mu 2^{-t} C(D) D^{1/2} \log D \quad \textit{for} \quad D > 1.27 \times 10^6 , \tag{4.3}$$

where

$$C(D) = 0.71 \textit{ for } D \equiv 2 \textit{ or } 3 \pmod 4 ,$$
$$C(D) = 0.53 \textit{ for } D \equiv 1 \pmod 8 ,$$
$$C(D) = 0.24 \textit{ for } D \equiv 5 \pmod 8 ,$$

and we have (1.4) *for* $D > 7$.

Proof. W obtain (4.3) from (4.1) by routine computation. This gives (1.4) for $D > 1.27 \times 10^6$ since $\mu C(D) \leqq 0.72$ by (2.5). For $7 < D \leqq 1.27 \times 10^6$, (1.4) can be verified by use of Table 1 in [1].

COROLLARY 3. *The estimate* (1.6) *holds.*

Proof. Immediate by (4.1) or (4.3) and the definition of t.

We remark that it can be verified, in the same way, that

$$p(D) < 0.3D^{1/2}\log D \quad \text{for} \quad 800 < D \leqq 1.27 \times 10^6 ;$$

this result is better than the bounds given by Corollary 1.

We conclude this section by discussing the consequences of using (2.16) instead of (2.14) in deriving Theorem 1. We use A_2 to A_5 to denote positive constants and $E(D)$ to denote the right member of (4.1). It is clear that the use of (2.16) replaces $E(D)$ by $E(D) - A_2 \log q_0 > E(D) - A_3 \log D$ since $q_0 = [\sqrt{D}]$. Now $E(D) > A_4 D^{1/2}\log D\, 2^{-\nu}$ where ν is the number of prime factors of D. By a standard inequality [5, p. 262], we have $\nu < A_5 \log D/\log\log D$. Hence $E(D) > D^{1/2-\varepsilon}$ for any $\varepsilon > 0$ and sufficiently large D. Thus the use of (2.16) produces only a negligible improvement in Theorem 1 for large D.

5. A bound for $p(D)$ when D contains a square factor. We shall employ the preceding sections and elementary congruence arguments to prove the upper bound (1.5) for $p(D)$, which holds for all nonsquare integers $D > 0$. Let D be such an integer and set

$$D = D_0 s^2 , \tag{5.1}$$

where D_0 is a fixed squarefree integer > 1. Let (a_s, b_s) be the least positive solution of $x^2 - Dy^2 = x^2 - D_0 s^2 y^2 = \pm 1$. Put

$$\eta_s = a_s + b_s\sqrt{D} = a_s + sb_s\sqrt{D_0} , \tag{5.2}$$

and for convenience write

$$\eta = \eta_1 = a + b\sqrt{D_0} . \tag{5.3}$$

Now (2.7), with η replaced by η_s, holds for all nonsquare $D > 0$. Hence we have

$$p(D) < \log \eta_s/\log \alpha . \tag{5.4}$$

Since $\eta_s > 1$, it follows from the theory of Pell's equation that, for fixed D_0, there is a function $e(s) > 0$ such that

$$\eta^{e(s)} = \eta_s . \tag{5.5}$$

Moreover, $e(s)$ is the minimum positive k such that η^k is congruent to a rational integer (mod s). Hence, by (5.4) and (2.10), we have

$$p(D) < \mu e(s) \log \varepsilon_0/\log \alpha . \tag{5.6}$$

In §§ 2-4, we showed that

$$p(D_0) < \mu \log \varepsilon_0/\log \alpha < 0.72 D_0^{1/2} \log D_0$$

for $D_0 > 7$. By calculation, we find that

$$\mu \log \varepsilon_0/\log \alpha < 1.88 D_0^{1/2} \log D_0$$

for all $D_0 \leqq 7$. It follows from (5.1) and (5.6) that

$$p(D) < 1.88 s^{-1} e(s) D^{1/2} \log (D/s^2) \tag{5.7}$$

for all nonsquare $D > 1$.

Thus, to prove (1.5), we only need to prove $e(s) \leqq 2s$, where η is a unit of $Z[\sqrt{D_0}]$. Actually, we prove that

$$e(s) \leqq s \quad \text{if} \quad N\eta = 1 \tag{5.8}$$

and

$$e(s) \leqq 2s \quad \text{if} \quad N\eta = -1 \,. \tag{5.9}$$

We first show that (5.8) implies (5.9). Suppose that $N\eta = -1$. Then $N\eta^2 = 1$; so, by (5.5) and (5.8), there is an integer

$$k \leqq s \ni (\eta^2)^k = \eta^{2k} = \eta_s \,.$$

Hence $e(s) \leqq 2s$, which is (5.9).

Now assume that $N\eta = 1$; we prove (5.8) by induction, as follows. First suppose that $s = s_1 s_2$ with $(s_1, s_2) = 1$. Then, by the remark after (5.5), it follows that $e(s) \leqq e(s_1)e(s_2)$. Next, suppose that $\beta \in Z[\sqrt{D_0}]$ and $\beta \equiv g (\text{mod } p^i)$ for a fixed $i > 0$, where p is prime and g is an integer. Then we have $\beta^p \equiv g^p \pmod{p^{i+1}}$. Hence it only remains to prove (5.8) when s is a prime p. We use the fact that $\eta^{-1} \in Z[\sqrt{D_0}]$.

Case 1. $p = 2$. Then $\eta^2 \equiv a^2 + D_0 b^2 \pmod 2$ by (5.3).

Case 2. $p > 2$. We have $a^p \equiv a$ and $D_0^{(p-1)/2} \equiv (D_0 \mid p)$, where $(D_0 \mid p)$ is the Legendre symbol (all congruences modulo p). Hence $\eta^p \equiv a + (D_0 \mid p) b\sqrt{D_0}$; thus $(D_0 \mid p) = 0$ implies $\eta^p \equiv a$ and $e(p) \leqq p$. Next we have $\eta^p \equiv \eta$ or $\eta^p \equiv \eta^{-1}$ according as $(D_0 \mid p) = 1$ or -1. Hence $\eta^{2j} \equiv 1$, where $2j = p - (D_0 \mid p)$. Set $\eta^j = h + k\sqrt{D_0}$. Then we have $h^2 + D_0 k^2 \equiv 1$. But $h^2 - D_0 k^2 = N\eta^j = 1$; hence $p \mid k$ and $\eta^j \equiv h$. Thus we have $e(p) \leqq p$, $(p-1)/2$, $(p+1)/2$ for $(D_0 \mid p) = 0, 1, -1$, respectively. This completes the proof of (5.8).

Actually, Mathews [13, p. 94] gives a formula which yields an explicit multiple of $e(s)$ which is $\leqq s$; a proof is given for the case $s = p$ only, and we have used his argument.

6. A conjectural lower bound for $p(D)$ when D is squarefree. By (1.5), we have $p(D) = 0(D^{1/2} \log D)$ for nonsquare D. It is natural to ask for results in the opposite direction, namely, to ask for functions g such that

$$p(D) > Ag(D) \tag{6.1}$$

for an infinity of D, where A is a positive constant.

The tables in [1] suggest we can take $g(D) = D^{1/2}$; however, the best known result appears to be

$$p(D) > A \log D\ , \tag{6.2}$$

which is obtainable from the fact that

$$p(D) = m \quad \text{for} \quad D = \frac{1}{4}(f_m + 1)^2 + f_{m-1} + 1\ , \tag{6.3}$$

where $m \not\equiv 0 \pmod 3$, and f_m is the mth Fibonacci number; in this case, the period of $\sqrt{D}$ contains $m - 1$ 1's followed by $f_m + 1$. Now (6.3) is easily verified by means of (2.6), and (6.2) follows on applying the inequality $f_n < \alpha^n$ for $n \geqq 0$.

We now use an estimate due to Perron [15, p. 72] and a theorem of Siegel on $L(1, \chi)$ to prove

THEOREM 2. *Suppose there exists an infinite sequence S of squarefree numbers D such that $h(D) = o(D^{\varepsilon/2})$ for D in S and all $\varepsilon > 0$. Then* (6.1) *holds with $g(D) = D^{1/2-\varepsilon}$ for any $\varepsilon > 0$.*

REMARK. There is abundant numerical support for the truth of above hypothesis, in fact for the stronger conjecture that $h(D) = 1$ infinitely often [10].

Proof. We use the following cruder form of Perron's estimate. Let D be a nonsquare > 1, and let (x_1, y_1) be the least positive solution of $x^2 - Dy^2 = 1$. Then we have $x_1 < (\sqrt{AD})^{2p(D)} = (AD)^{p(D)}$, where A is a constant. Suppose now that D is squarefree. Then $\varepsilon_0 \leqq x_1 + y_1\sqrt{D} < 2x_1$, where ε_0 is the fundamental unit of $Q(\sqrt{D})$. Hence there is a constant B such that

$$p(D) > B \log \varepsilon_0 / \log D = Bh \log \varepsilon_0 / h \log D\ .$$

Now fix $\varepsilon > 0$; by Siegel's theorem on the size of $L(1, \chi)$ and (2.12), there exists $D_1(\varepsilon)$ such that, for $D > D_1(\varepsilon)$, we have

$$h \log \varepsilon_0 > D^{(2-\varepsilon)/4}$$

(see [3, p. 130]). Hence

$$p(D) > \frac{BD^{(2-\varepsilon)/4}}{h \log D}$$

for $D > D_1(\varepsilon)$. Theorem 2 follows by taking D so large that $\log D < D^{\varepsilon/4}$.

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