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NTH ORDER OSCILLATIONS WITH MIDDLE TERMS OF ORDER $N-2$

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Equations of the form

$$(*)\qquad x^{(n)}+p(t)x^{(n-2)}+H(t,x)=0\,,\qquad n\geqq 3$$

are studied here, where $p(t)\geqq 0$ and $uH(t,u)\geqq 0$.

It is shown, among other things, that for n even all solutions of (*) oscillate if H is superlinear, satisfying the usual integral conditions, and $u''+p(t)u=0$ is nonoscillatory with $p(t)$ "small" and decreasing. The case of n odd is also covered and some of the recent results of Waltman, Heidel are taken as special cases.

As far as the author knows, the only references concerning the nonlinear (*) for $p(t)\not\equiv 0$ are those of Liossatos [9] and the author [8]. In [9] Liossatos extended several of the results of Heidel [2] to the nth order case. Liossatos however, considered mainly the case $p(t)\leqq 0$, $uH(t,u)\leqq 0$ (actually a special case of $H(t,u)$) and established one theorem concerning the case $p(t)\geqq 0$, $uH(t,u)\geqq 0$. One of the results of Liossatos was improved by the author in [8], where a result is also given for a forced equation.

In Theorem 1 below we establish a result for $n=3$ which improves a result of Waltman [13, Th. 1]. The proof of this theorem contains a simpler proof of Theorem 3.8 in Heidel's paper [2]. In Theorem 2 we provide conditions ensuring the oscillation or convergence to zero of all bounded solutions of (*). Theorem 3 ensures that every solution of (*) with a zero is oscillatory if n is odd. In Theorem 4 the middle term is treated as a small perturbation, and Theorem 5 ensures oscillation or convergence to zero for functions H which are "large" compared to the middle term. For several results concerning nth order nonlinear equations the reader is referred to the survey paper [8].

2. **Preliminaries.** In what follows, T will denote a fixed nonnegative number, $R_T=[T,\infty)$, $R_+=[0,\infty)$, $R=(-\infty,\infty)$. For the equation

$$(*)\qquad x^{(n)}+p(t)x^{(n-2)}+H(t,x)=0\,,\qquad n\geqq 3$$

we shall assume that the functions $p\colon R_T\to R_+$, $H\colon R_T\times R\to R$ are continuous on their domains and such that $uH(t,u)\geqq 0$ for every $(t,u)\in R_T\times R$. A "solution" of (*) will be any function $x(t)$ which is defined, n times continuously differentiable, and satisfies (*) on some infinite subinterval of R_T. Only nontrivial solutions will be considered here, i.e., solutions which do not become identically equal to zero for all large t. A function $f: [a, \infty) \to R$, $a \geqq T$, is said to be "oscillatory" if it has an unbounded set of zeros in $[a, \infty)$. The equation (*) is said to be "D-oscillatory", if, for n even, every solution of (*) is oscillatory, and, for n odd, every solution of (*) is oscillatory, or tends monotonically to zero as $t \to \infty$. The equation (*) is said to be "BD-oscillatory" if the above definition holds true, but for the bounded solutions of (*). A stronger result than the following auxiliary theorem was proven by the author in [6] but for n even.

THEOREM A. *Consider the equation*

$$x^{(n)} + H(t, x) = 0\,, \quad t \geqq T \tag{2.1}$$

and the inequality

$$u^{(n)} + H(t, u) \leqq 0\,, \quad t \geqq T\,, \tag{2.2}$$

where $H(t, u)$ is nonnegative and increasing in u for every $u \in R$. Then the existence of a positive (bounded and positive) *solution of* (2.2) *implies the existence of a positive* (bounded and positive) *solution of* (2.1). *Furthermore, if all positive* (bounded and positive) *solutions of* (2.1) *tend monotonically to zero, then the same is true for all positive* (bounded and positive) *solutions of* (2.2).

Proof. For n even the theorem is a particular case of Lemma 2.1 in [6], where $H(t, u) > 0$ was considered but its proof carries over to the present case without modifications. For n odd, assume that all positive solutions of (2.1) tend monotonically to zero, and let $u(t)$ be a positive solution of (2.2) such that $u(t) > 0$, $u'(t) > 0$ for $t \geqq t_1 \geqq T$. Then, from the proof of Lemma 2.1 in [6], it follows immediately, that (2.1) has also a solution $x(t)$ with $x(t_1) = u(t_1)$ and $x'(t) > 0$, $t \geqq t_1$. This is however a contradiction to the assumption $\lim_{t\to\infty} x(t) = 0$.

For a functional version of the above theorem, the reader is referred to Onose [11].

THEOREM B. *In the eqution* (*) *assume that $x(t)$ is a nonoscillatory solution and that $u'' + p(t)u$ is nonoscillatory. Then $x(t)x^{(n-2)}(t) \geqq 0$ or $x(t)x^{(n-2)}(t) \leqq 0$ for all large t. If, moreover, $uH(t, u) > 0$ for every $(t, u) \in R_T \times R$ with $u \neq 0$, then these inequalities are strict.*

REMARK. The second assertion of this theorem is slightly stronger than the corresponding result of Heidel [2, Th. 3.6].

Proof. We partly follow the steps of Heidel's Theorem 3.6 in [2]. Let $x(t)$ be a nonoscillatory solution of (*) and assume that $x(t) > 0$, $t \geqq t_1 \geqq T$. A similar proof covers the case $x(t) < 0$. Now let $x^{(n-2)}(t)$ be oscillatory, let $x^{(n-2)}(t_2) = 0$ for some $t_2 > t_1$, and assume that $x^{(n-2)}(t_3) < 0$ for some $t_3 > t_2$. Now let

$$L = \sup\{t \in [t_2, t_3); x^{(n-2)}(t) = 0\} ,$$
$$M = \inf\{t \in (t_3, \infty); x^{(n-2)}(t) = 0\} .$$

Then, by continuity, $x^{(n-2)}(L) = 0$, $x^{(n-2)}(M) = 0$, $L < t_3 < M$ and $x^{(n-2)}(t) < 0$, $t \in (L, M)$. Now multiply (*) by $x^{(n-2)}(t)$ and integrate between L, M to obtain

$$\int_L^M p(s)[x^{(n-2)}(s)]^2 ds + \int_L^M H(s, x(s))x^{(n-2)}(s)ds = \int_L^M [x^{(n-1)}(s)]^2 ds . \tag{2.3}$$

An application of Nehari's lemma [10, p. 431] now yields:

$$\int_L^M p(s)[x^{(n-2)}(s)]^2 ds < \int_L^M [x^{(n-1)}(s)]^2 ds , \tag{2.4}$$

which implies

$$\int_L^M H(s, x(s))x^{(n-2)}(s)ds > 0 , \tag{2.5}$$

a contradiction to the negativeness of $x^{(n-2)}(t)$ on (L, M). Consequently, if $x^{(n-2)}(t)$ is oscillatory, we must have $x^{(n-2)}(t) \geqq 0$ for all large t. This proves our assertion. To show the second conclusion, it suffices to observe that if $H(t, x(t)) > 0$ for $t \geqq t_1$, then $x^{(n)}(t) < 0$ at each zero of $x^{(n-2)}(t)$. This concavity property of $x^{(n-2)}(t)$ implies that either $x^{(n-2)}(t)$ crosses the axis—impossible by the above argument—or $x^{(n-2)}(t)$ has a cusp at each one of its zeros—impossible, because it is differentiable there. Consequently, $x^{(n-2)}(t)$ cannot be oscillatory in this case. This completes the proof.

3. Main results.

The following theorem improves Theorem 1 in Waltman's paper [13].

THEOREM 1. *Let $n = 3$. Moreover, let $p(t)$ be decreasing and H be increasing in u. Then if*

$$\int_T^\infty H(t, \pm k)dt = \pm\infty$$

for every $k > 0$, every solution of (*) with a zero is oscillatory.

Proof. Let $x(t)$ be a solution of (*) which has a zero at $t_1 \geqq T$ and is such that $x(t) > 0$, $t \in (t_1, \infty)$. Assume further that $x(t) \geqq L > 0$, $t \geqq t_2 > t_1$, where L is a fixed constant. Then integration of (*) from t_2 to $t \geqq t_2$ gives

$$x''(t) - x''(t_2) + p(t)x(t) - p(t_2)x(t_2) - \int_{t_2}^{t} x(s)d[p(s)]$$
$$\leqq - \int_{t_2}^{t} H(s, L)ds \longrightarrow -\infty \quad \text{as} \quad t \longrightarrow \infty .$$

Thus, $x''(t) \to -\infty$ as $t \to \infty$, a contradiction to the positivity of $x(t)$. It follows that $\lim_{t\to\infty} \inf x(t) = 0$. The proof now follows exactly as in Waltman's theorem in [13], because we have just obtained that since $x(t_1) = 0$ and $\lim_{t\to\infty} \inf x(t) = 0$, there is a number $t_2 > t_1$ such that $x'(t_2) = 0$. We omit the rest of the proof.

Waltman assumed in [13] that $p'(t)$ exists and is nonpositive and

$$A + Bt - \int_{t_1}^{t}\int_{t_1}^{s} q(u)duds < 0 \tag{3.1}$$

eventually, for any constants A, B and any number $t_1 \geqq T$, where $H(t, u) \equiv q(t)u^r$ with $Q(t) \geqq 0$ and $r =$ the quotient of two odd positive integers. Obviously, (3.1) implies the integral condition

$$\int_{t_1}^{\infty} q(s)ds = +\infty ,$$

which is actually equivalent to the integral assumption in Theorem 1 if H is of the above type. For another improvement of Waltman's theorem see Heidel's Corollary 3.4 in [2].

THEOREM 2. *Let $uH(t, u) > 0$ for all $(t, u) \in R_T \times R$ with $u \neq 0$, and let H be increasing in u. Let the equation $u'' + p(t)u = 0$ be nonoscillatory with $[t^\alpha p(t)]$ decreasing in R_T, where $0 \leqq \alpha \leqq n - 1$. Then if*

$$\int_{T}^{\infty} t^\alpha H(t, \pm k)dt = \pm\infty$$

for every $k > 0$, Equation (*) is BD-oscillatory.

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (*). Assume without loss of generality that $x(t) > 0$, $t \geqq t_1 \geqq T$. Then Theorem B implies that $x(t)x^{(n-2)}(t) \leqq 0$ or $x(t)x^{(n-2)}(t) \geqq 0$ for all large t. Assume that $x^{(n-2)}(t) \geqq 0$, $t \geqq t_2 \geqq t_1$. Then from (*) we obtain

$$x^{(n)}(t) + H(t, x(t)) \leqq 0 , \quad t \geqq t_2 . \tag{3.2}$$

Since

$$\int_T^\infty t^{n-1}H(t, \pm k)dt = \pm\infty ,$$

it follows from Theorem 2.9 of the author in [8], and the analogous theorem for n odd, that the equation (2.1) is BD-oscillatory. Theorem A implies now that $x(t)$ cannot be eventually positive for n even, and that $\lim_{t\to\infty} x(t) = 0$ for n odd. However, this last conclusion is also impossible. In fact, since $x^{(n-2)}(t) \geqq 0$, $t \geqq t_2$ and $n-2$ is an odd integer, we must have $x'(t) \geqq 0$ eventually. This contradicts $\lim_{t\to\infty} x(t) = 0$. Consequently, $x^{(n-2)}(t) \leqq 0$ for all large t, say for every $t \geqq t_2 \geqq \max\{1, t_1\}$. Now we multiply (*) by t^α and integrate from t_2 to t to obtain

$$\begin{aligned} t^\alpha x^{(n-1)}(t) - t_2^\alpha x^{(n-1)}(t_2) - \alpha\int_{t_2}^t s^{\alpha-1}x^{(n-1)}(s)ds \\ + t^\alpha p(t)x^{(n-3)}(t) - t_2^\alpha p(t_2)x^{(n-3)}(t_2) \\ - \int_{t_2}^t x^{(n-3)}(s)d[s^\alpha p(s)] = -\int_{t_2}^t s^\alpha H(s, x(s))ds . \end{aligned} \tag{3.3}$$

Now if $x^{(n-3)}(t_3) < 0$ for some $t_3 \geqq t_2$, then $x^{(n-3)}(t) \leqq x^{(n-3)}(t_3) < 0$ for every $t \geqq t_3$. This implies $\lim_{t\to\infty} x(t) = -\infty$, a contradiction. Thus, $x^{(n-3)}(t) \geqq 0$ for $t \geqq t_2$. Now since $x(t)$ is bounded, it follows that $(-1)^j x^{(j)}(t) \leqq 0$ for n even and $(-1)^j x^{(j)}(t) \geqq 0$ for n odd, and for $j = 1, 2, \cdots, n-2$, $t \in [t_2, \infty)$. Assume that for some constant $L > 0$, $x(t) \geqq L$, $t \in [t_2, \infty)$. Then from (3.3) we obtain

$$t^\alpha x^{(n-1)}(t) - \alpha\int_{t_2}^t s^{\alpha-1}x^{(n-1)}(s)ds \leqq C - \int_{t_2}^t t^\alpha H(s, L)ds . \tag{3.4}$$

Taking limits as $t \to \infty$ in (3.4) we obtain

$$\lim_{t\to\infty}\left[t^\alpha x^{(n-1)}(t) - \alpha\int_{t_2}^t s^{\alpha-1}x^{(n-1)}(s)ds \right] = -\infty . \tag{3.5}$$

Now let

$$\varphi(t) \equiv \int_{t_2}^t s^{\alpha-1}x^{(n-1)}(s)ds , \quad t \geqq t_2 . \tag{3.6}$$

Then we have

$$t\varphi'(t) - \alpha\varphi(t) = q(t) , \quad t \geqq t_2 \tag{3.7}$$

where $\lim_{t\to\infty} q(t) = -\infty$. Now we apply Lemma 1 of Staikos and Sficas [12] to obtain

$$\int_{t_2}^\infty s^{\alpha-1}x^{(n-1)}(s)ds = \pm\infty . \tag{3.8}$$

The above integral cannot be $-\infty$ because then (3.5) would imply $x^{(n-1)}(t) \leqq 0$ for all large t_1 which combined with $x^{(n-2)}(t) \leqq 0$ gives $\lim_{t\to\infty} x(t) = -\infty$, a contradiction. Consequently, since $t_2 \geqq 1$, we get

$$\int_{t_2}^{\infty} s^{\alpha-1}x^{(n-1)}(s)ds \leqq \int_{t_2}^{\infty} s^{n-2}x^{(n-1)}(s)ds = +\infty \,. \tag{3.9}$$

Now the argument follows the steps of Theorem 1 in [4] and we omit the rest of it, which leads to the contradictions

$$\int_{t_2}^{\infty} x'(t)dt = \begin{cases} +\infty & n \text{ even} \\ -\infty & n \text{ odd}\,. \end{cases} \tag{3.10}$$

Consequently, no positive $x(t)$ exists for n even and $x(t) \to 0$ monotonically as $t \to \infty$ for n odd. Dual arguments cover the assumption $x(t) < 0$. This completes the proof.

The following theorem characterizes further the solutions of (*) under slightly stronger assumptions than those of Theorem 2 and for n odd.

THEOREM 3. *Let the assumptions of Theorem 2 be satisfied with $p(t)$ decreasing in R_T and $uH(t, u) > 0$ for $u \neq 0$. Then, for n odd, every bounded solution of* (*) *with a zero is oscillatory.*

Proof. Let $x(t)$ be a bounded solution of (*) such that $x(t_1) = 0$ for $t_1 \geqq T$ and $x(t) > 0$, $t \in (t_1, \infty)$. Theorem B implies that $x(t)x^{(n-2)}(t) > 0$ or $x(t)x^{(n-2)}(t) < 0$ for all large t, and from the proof of Theorem 2 it follows that the second of these inequalities holds. Now since $x^{(n-2)}(t) < 0$ for all large t, say $t \geqq t_2 \geqq t_1$, we must have $x^{(n-3)}(t) > 0$ for all $t \geqq t_2$. In fact, if $x^{(n-3)}(\bar{t}) \leqq 0$ for some $\bar{t} \geqq t_2$, then $x^{(n-3)}(t) < x^{(n-3)}(\bar{t})$ for all $t \geqq \bar{t}$, which implies the contradiction $x(t) \to -\infty$ as $t \to \infty$. Now $x^{(n-3)}(t)$ must have a zero to the right of t_1. If this was not true then $x^{(n-3)}(t) > 0$ for $t \geqq t_1$. This, along with $(-1)^j x^{(j)}(t) \geqq 0$, $j = 1, 2, \cdots, n-2$ (cf. proof of Theorem 2), shows that none of the derivatives $x^{(j)}(t)$, $j = 1, 2, \cdots, n-3$ can have a zero after t_1. Thus, $x'(t) < 0$ for $t \in (t_1, \infty)$, a contradiction to the fact that $x(t_1) = 0$ and $x(t) > 0$, $t \in (t_1, \infty)$. Consequently, $x^{(n-3)}(t_3) = 0$ for some $t_3 > t_1$. We are planning to show that $x^{(n-3)}(t)$ is oscillatory. Suppose this is not true. Then let $t_4 \geqq t_3$ be the last zero of $x^{(n-3)}(t)$. We must now have $x^{(n-3)}(t) > 0$, $t > t_4$. Now integrate (*) from t_4 to $t \geqq t_4$, after multiplication by $x^{(n-3)}(t)$, to obtain

$$x^{(n-1)}(t)x^{(n-3)}(t) + \frac{[x^{(n-2)}(t_4)]^2}{2} - \frac{[x^{(n-2)}(t)]^2}{2} \tag{3.11}$$

$$+ p(t)\frac{[x^{(n-3)}(t)]^2}{2} - \int_{t_4}^{t}\frac{[x^{(n-3)}(s)]^2}{2}d[p(s)]$$

$$+ \int_{t_4}^{t} H(s, x(s))x^{(n-3)}(s)ds = 0 .$$

Now we either have $x^{(n-1)}(t) > 0$ for all large t, or $x^{(n-1)}(t)$ is oscillatory. The first of these possibilities combined with (3.11) implies that $\lim_{t\to\infty} x^{(n-2)}(t) = L < 0$, which is impossible because it implies $x^{(n-3)}(t) < 0$ for all large t. Now if $x^{(n-1)}(t)$ is oscillatory, then for some sequence of maxima $x^{(n-2)}(t_m)$ (with $t_m \to \infty$ as $m \to \infty$) we have $\lim_{m\to\infty} x^{(n-2)}(t_m) = 0$ and $x^{(n-1)}(t_m) = 0$. Letting $t = t_m$ in (3.11) we obviously get a contradiction because the limit on the left exists and is finite positive or $+\infty$ ($p(t)$ is bounded and $\lim_{t\to\infty} x^{(n-3)}(t) = 0$). Consequently, $x^{(n-3)}(t)$ has to be oscillatory. However, we have already established that $x^{(n-2)}(t) < 0$ for all large t. This obviously implies a contradiction. Consequently, $x(t)$ cannot be positive after it has a zero. A similar argument holds true if we assume that $x(t)$ is negative for all large t. This completes the proof.

Results are expected of course to hold for all solutions of (*) under stronger assumptions on the function $H(t, u)$. It turns out however that the method developed by the author in [4] does not apply in the present case due to the fact that we do not have enough information concerning the sign of $x^{(n-1)}(t)$ for a positive or negative solution of (*). Nevertheless, if the function $p(t)$ is sufficiently small, we can treat the term $p(t)x^{(n-2)}(t)$ as a small perturbation because we can show that, under the conditions of Theorem B with $uH(t, u) > 0$ for $u \neq 0$, the decrease of $p(t)$ for all large t suffices to ensure that $\lim_{t\to\infty} x^{(n-1)}(t) = 0$ for any nonoscillatory solution $x(t)$ of (*). This is the content of the following

THEOREM 4. *Let the equation $u'' + p(t)u = 0$ by nonoscillatory. Moreover, let $p(t)$ be decreasing in R_T and such that*

$$\int_T^\infty t^n p(t)dt < +\infty .$$

Then (*) *is D-oscillatory* (*BD*-oscillatory) *if the same is true for* (2.1).

Proof. Let us first remark that $\lim_{t\to\infty} p(t) = 0$. In fact, if this is not true, then

$$\int_T^\infty p(t)dt = +\infty ,$$

which, by the classical Wintner's criterion [14], implies that all solutions of $u'' + p(t)u = 0$ oscillate, a contradiction. Now let $x(t)$ be an eventually positive solution of (*). Then the conclusion of Theorem B implies the existence of some $t_1 \geqq T$ such that $x^{(n-2)}(t)x(t) > 0$ or $x^{(n-2)}(t)x(t) < 0$ for $t \geqq t_1$. Now let the first of these inequalities hold, (2.1) be D-oscillatory and n even. Then it follows that $x(t)$ is an eventually positive solution of

$$x^{(n)} + H(t, x) \leqq 0 \tag{3.12}$$

which, according to Theorem A, is a contradiction. Consequently, $x^{(n-2)}(t) < 0$ and $x(t) > 0$, $t \geqq t_1$. Thus, we must have $x^{(n-3)}(t) > 0$, $t \geqq t_1$. This happens because $x^{(n-3)}(t) < 0$ for all large t implies $x(t) \to -\infty$ as $t \to \infty$, and if $x^{(n-3)}(\tilde{t}) = 0$ for some $\tilde{t} \geqq t_1$, then $x^{(n-3)}(\tilde{t}) < 0$ for all $t \geqq \tilde{t}$, a contradiction again. Now integrate (*) from t_1 to $t \geqq t_1$ to obtain

$$x^{(n-1)}(t) - x^{(n-1)}(t_1) + \int_{t_1}^{t} p(s)x^{(n-2)}(s)ds + \int_{t_1}^{t} H(s, x(s))ds = 0 ,$$

or

$$\begin{aligned} x^{(n-1)}(t) - x^{(n-1)}(t_1) = & -p(t)x^{(n-3)}(t) + p(t_1)x^{(n-3)}(t_1) \\ & + \int_{t_1}^{t} x^{(n-3)}(s)d[p(s)] - \int_{t_1}^{t} H(s, x(s))ds . \end{aligned} \tag{3.13}$$

Now, as we noticed above, $\lim_{t\to\infty} p(t) = 0$, and $x^{(n-3)}(t)$ is bounded. Consequently, the first term on the right of (3.13) tends to zero as $t \to \infty$, and the two last terms are nonpositive and decreasing to finite limits, otherwise $x^{(n-1)}(t) \to -\infty$ as $t \to \infty$, a contradiction. It follows that $\lim_{t\to\infty} x^{(n-1)}(t) = L$ exists. If $L > 0$, then $x^{(n-2)}(t) \to +\infty$ as $t \to \infty$, a contradiction. If $L < 0$, then $x^{(n-2)}(t) \to -\infty$ as $t \to \infty$, a contradiction again to the positiveness of $x(t)$. Thus, $L = 0$. Consequently,

$$\begin{aligned} |x^{(n-2)}(t)| & \leqq |x^{(n-2)}(t_1)| + \int_{t_1}^{t} |x^{(n-1)}(s)|ds \\ & \leqq Mt \end{aligned} \tag{3.14}$$

for all $t \geqq t_1$ and some positive constant M. Now let $q(t) \equiv -p(t)x^{(n-2)}(t)$. Then the integral

$$-\int_{t}^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} q(s)ds < +\infty ,$$

and this follows easily from the integral assumption on $p(t)$. If we let

$$P(t) \equiv -\int_{t}^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} q(s)ds ,$$

then $P^{(n)}(t) = q(t)$. Using the transformation $W(t) \equiv x(t) - P(t)$, $t \geqq t_1$, we obtain

$$W^{(n)} + H(t, W + P(t)) = 0 , \quad t \geqq t_1 .$$

Now the rest of the proof follows as in [6, Th. 2.1] for n even and we omit it. The case of odd n can be treated similarly and its proof is omitted. Similar considerations cover the case of bounded solutions under the assumption of BD-oscillation of (2.1), and we omit these arguments as well. This completes the proof.

The above theorem is intimately related to but does not contain any of the results of the third section of Heidel's paper [2]. We remark that Theorem 3 does not make use of all the hypotheses of Theorem 3, and that the conclusions in Theorem 3 can be deduced for any solution of (*) that has a zero if (2.1) is D-oscillatory. The proof of this last statement follows exactly as in Theorem 3 and we omit it.

In the following result we actually take into consideration the growth of the function $H(t, u)$ with respect to the middle term.

THEOREM 5. *Let* (*) *satisfy the following assumptions*:

(i) $u'' + p(t)u = 0$ *is nonoscillatory and* $p(t)$ *is decreasing in* R_T;

(ii) *For every* $\lambda > 0$, $\mu \geqq 0$, *there are no eventually positive* (negative) *solutions* $u(t)$ *of the equation*

$$u^{(n)} + H(t, u) - \mu t p(t) = 0 \tag{3.15}$$
$$(u^{(n)} + H(t, u) + \mu t p(t) = 0)$$

such that $u(t) \geqq \lambda$ $(u(t) \leqq -\lambda)$ *and* $H(t, u(t)) - \mu t p(t) \geqq 0$ $(H(t, u(t)) - \mu t p(t) \leqq 0)$ *for all large* t. *Moreover, if for a sequence* $\{t_m\}$, $m = 1, 2, \cdots$ *we have* $\lim_{m\to\infty} t_m = \infty$ *and* $H(t_m, u_m) < \mu t_m p(t_m)(H(t_m, u_m) > -\mu t_m p(t_m))$ *where* $u_m > 0$ $(u_m < 0)$ $m = 1, 2, \cdots$, *then* $\lim_{m\to\infty} \inf |u_n| = 0$. *Then* (*) *is D-oscillatory.*

Proof. Assume that there exists a solution $x(t)$ of (*) which is eventually bounded below by a constant $\lambda > 0$. Then $x^{(n-2)}(t) \leqq 0$ for all large t, and $\lim_{t\to\infty} x^{(n-1)}(t) = 0$. Consequently, there exists $t_1 \geqq T$ such that

$$x(t) \geqq \lambda > 0 , \quad x^{(n-2)}(t) \leqq 0 , \quad -x^{(n-2)}(t) \leqq \mu t$$

for every $t \geqq t_1$. Consequently,

$$\begin{aligned} 0 &= x^{(n)}(t) + H(t, x(t)) + p(t)x^{(n-2)}(t) \\ &\geqq x^{(n)} + H(t, x(t)) - \mu t p(t) . \end{aligned} \tag{3.16}$$

It follows from our assumptions that we cannot have $H(t, x(t)) \geqq \mu t p(t)$ for all large t. If this was true, then Theorem A would imply that (3.15) has a positive solution such that $u'(t) \geqq 0$ for all large t (again, this last assertion follows from Lemma 2.1 in [6] taking into consideration the class $B(T, k)$ considered there). This is a contradiction. Thus, for some sequence $\{t_m\}$, $n = 1, 2, \cdots$ with $\lim_{m\to\infty} t_m = \infty$, we must have

$$H(t_m, x(t_m)) \leqq \mu t_m p(t_m)\ , \quad m = 1, 2, \cdots\ ,$$

which, by our assumptions, implies

$$\liminf_{m\to\infty} x(t_m) = \liminf_{t\to\infty} x(t) = 0\ . \tag{3.17}$$

Now, taking into consideration that $x^{(n-2)}(t) \leqq 0$, it follows that (3.17) is impossible for n even, and it implies $\lim_{t\to\infty} x(t) = 0$ for n odd. Similar consideration cover the case of a negative $x(t)$. This completes the proof.

The conditions of the above theorem seem rather stringent at first glance. They are not however, and they imply oscillation criteria for all three important cases; sublinear, superlinear, linear, as the following corollary indicates. The method of proof of the above theorem was employed by the author in [7] where oscillations with perturbations $Q(t, x)$ have been studied. A similar argument for a similar problem can be found in the paper [1] of Graef and Spikes.

COROLLARY 1. *Consider the differential equation*

$$x^{(n)} + p(t)x^{(n-2)} + q(t)|x|^\alpha \operatorname{sgn} x = 0\ , \quad t \in R_T\ , \tag{3.18}$$

where $p(t)$ is decreasing and such that the equation $u'' + p(t)u = 0$ is nonoscillatory. Moreover, $q(t) > 0$, $t \in R_T$ and

$$\lim_{t\to\infty} [tp(t)/q(t)] = 0\ . \tag{3.19}$$

Then if α is a positive constant (3.18) *is D-oscillatory provided one of the following conditions holds*:

(i) $\int_T^\infty t^{n-1-\varepsilon} q(t)dt = +\infty$ *for $\alpha = 1$ and some $\varepsilon > 0$*;

(ii) $\int_T^\infty t^{\alpha(n-1)} q(t)dt = +\infty$ *for $\alpha < 1$*;

(iii) $\int_T^\infty t^{n-1} q(t)dt = +\infty$ *for $\alpha > 1$.*

One can easily see now that the conditions of Theorem 5 are satisfied, in connection with the result of the author in [3]. Another corollary of the proof of Theorem 5 provides an appraisal of the nonoscillatory solutions of (*):

COROLLARY 2. *Let the assumptions of Corollary* 1 *hold with* (3.19) *replaced by*

$$[tp(t)/q(t)] \leqq L \quad \text{(constant)}$$

and the integral conditions replaced by the same conditions where $\pm q(t) - ktp(t)$ (for any $k > 0$) *replaces* $q(t)$ *and* $\pm\infty$ *replaces the second members. Then for every nonoscillatory solution* $x(t)$ *of* (*) *we have*

$$\lim_{t\to\infty} |x(t)| = \liminf_{t\to\infty} |x(t)| = K \leqq (kL)^{1/\alpha}$$

for some constant $k > 0$ *such that* $|x^{(n-2)}(t)| \leqq kt$ *eventually.*

In Theorems 2-5 we made the basic assumption that $u'' + p(t)u = 0$ is nonoscillatory. Let us first notice that if this equation is oscillatory, then $x^{(n-2)}(t)x(t) \geqq 0$ ($x^{(n-2)}(t)x(t) \leqq 0$) is impossible for a positive (negative) nonoscillatory solution of (*) and all large t. In fact, assume that $x(t) > 0$ for all large t. Then from (*) it follows that

$$u'' + p(t)u \leqq 0 \tag{3.20}$$

for all large t, where $u(t) \equiv x^{(n-2)}(t)$. If we assume the first of the above inequalities, then (3.20) has a positive solution. By Theorem A this is impossible. Similarly, the second of the above inequalities is impossible for a negative $x(t)$. Results can be now formulated analogous to several third order results in the third section of Heidel's paper [2]. We shall undertake this task in a future paper.

It would be very interesting to have some results ensuring the fact that $x^{(n-1)}(t)$ is of fixed sign for any nonoscillatory solution $x(t)$ of (*). Such a result would definitely permit relaxation of the conditions imposed in Theorem 5 and would make possible the extension of Theorem 2 to all solutions of (*).

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