ON EXTREME POINTS OF THE JOINT NUMERICAL RANGE OF COMMUTING NORMAL OPERATORS

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Let \( W(T) = \{ \langle Tx, x \rangle : \| x \| = 1; x \in H \} \) denote the numerical range of a bounded normal operator \( T \) on a complex Hilbert space \( H \). S. Hildebrandt has proved that if \( \lambda \) is an extreme point of \( \overline{W(T)} \), the closure of \( W(T) \), and \( \lambda \in W(T) \) then \( \lambda \) is in the point spectrum of \( T \). In this note, we shall prove an analogous result for an \( n \)-tuple of commuting bounded normal operators on \( H \).

2. Notations and terminology. Let \( A = (A_1, \cdots, A_n) \) be an \( n \)-tuple of commuting bounded operators on \( H \) and \( \mathfrak{Z} \), the double commutant of \( \{A_1, \cdots, A_n\} \). Then \( \mathfrak{Z} \) is a commutative Banach algebra with identity, containing the set \( \{A_1, \cdots, A_n\} \). We shall need the following definitions [3] and [4].

A point \( z = (z_1, \cdots, z_n) \) of \( \mathbb{C}^n \) is in the joint spectrum \( \sigma(A) \) of \( A \) relative to \( \mathfrak{Z} \) if for all \( B_1, \cdots, B_n \) in \( \mathfrak{Z} \)

\[ \sum_{j=1}^{n} B_j(A_j - z_j) \neq I. \]

The joint numerical range of \( A \) is the set of all points \( z = (z_1, \cdots, z_n) \) of \( \mathbb{C}^n \) such that for some \( x \) in \( H \) with \( \| x \| = 1 \),

\[ \langle Ax, x \rangle = \langle A_jx, x \rangle, \cdots, \langle A_nx, x \rangle \] 

We say that \( z = (z_1, \cdots, z_n) \) is in the joint point spectrum \( \sigma_p(A) \) if there exists some \( 0 \neq x \in H \) such that

\[ A_jx = z_jx, \quad j = 1, \cdots, n, \]

and that \( z \) is in the joint approximate point spectrum \( \sigma_s(A) \) if there exists a sequence \( \{x_n\} \) of unit vectors in \( H \) such that

\[ \| (A_j - z_j)x_n \| \to 0 \]

as \( n \to \infty \), \( j = 1, \cdots, n \).

Bunce [2] has proved that \( \sigma_s(A) \) is a nonempty compact subset of \( \mathbb{C}^n \).

If \( A = (A_1, \cdots, A_n) \) is an \( n \)-tuple of commuting normal operators, then the extreme points of \( \overline{W(A)} \) are in the joint approximate point spectrum \( \sigma_s(A) \). This is immediate from the fact that for such \( A_j \)'s,

\[ \overline{W(A)} = \text{closed convex hull of } \sigma(A) = \text{closed convex hull of } \sigma_s(A), \]

and that every compact set contains the extreme points of its closed
convex hull [1, Cor. 36.11, p. 144]. We show in the following theorem that something more can be said about the extreme points of \( \overline{W(A)} \), see Hildebrandt [5].

**Theorem.** Let \( A = (A_1, \cdots, A_n) \) be an \( n \)-tuple of commuting normal operators on \( H \). If \( \lambda = (\lambda_1, \cdots, \lambda_n) \) is an extreme point of \( \overline{W(A)} \) and \( \lambda \in W(A) \), then \( \lambda \in \sigma_s(A) \).

**Proof.** Firstly, we shall prove the result for commuting self-adjoint operators.

It is sufficient to show that if \((0, \cdots, 0)\) is an extreme point of \( \overline{W(A)} \) and \((0, \cdots, 0) \in W(A) \), then \((0, \cdots, 0) \in \sigma_s(A) \).

Since \((0, \cdots, 0)\) is an extreme point of \( \overline{W(A)} \), we may assume that

\[
\overline{W(A)} \subset \{ z = (\alpha_1, \cdots, \alpha_n) \in \mathbb{C}^n; \text{Re} \alpha_n \geq 0 \}.
\]

As \( A_1, \cdots, A_n \) are commuting self-adjoint operators, there exists a measure space \((X, \mu)\) and a set of bounded measurable functions \( \varphi_1, \cdots, \varphi_n \) in \( L^\infty(X, \mu) \) such that each \( A_j \) is unitarily equivalent to multiplication by \( \varphi_j \) on \( L^2(X, \mu) \). Thus

\[
A_j f = \varphi_j f, \quad \text{for all} \quad f \in L^2(X, \mu)
\]

and for each \( j = 1, 2, \cdots, n \) [3].

Because of the assumption (1), and since \( \sigma(A) \subset \overline{W(A)} \), we have

\[
\sigma(A) \subset \{ z = (\alpha_1, \cdots, \alpha_n) \in \mathbb{C}^n; \text{Re} \alpha_n \geq 0 \}.
\]

It follows that \( A_n \geq 0 \) and so \( \varphi_n(x) \geq 0 \) a.e. Let, if possible, \((0, \cdots, 0) \in \sigma_s(A_1, \cdots, A_n) \). Then \( |\varphi_j(x)| > 0 \) a.e. for at least one \( j = 1, 2, \cdots, n \). Let

\[
E_1 = \{ x \in X; \text{Im} \varphi_j(x) \geq 0 \}
\]

and

\[
E_2 = \{ x \in X; \text{Im} \varphi_j(x) < 0 \}.
\]

Since \((0, \cdots, 0) \in W(A_1, \cdots, A_n) \), for some \( f \in H \) with \( \| f \| = 1 \), \( \langle A_j f, f \rangle = 0, j = 1, 2, \cdots, n \) and

\[
0 = \langle A_j f, f \rangle = \int_X \varphi_j(x) |f(x)|^2 \mu(x)
\]

\[
= \int_{E_1} \varphi_j f^2 \mu + \int_{E_2} \varphi_j f^2 \mu
\]

\[
= \int_X \varphi_j \chi_{E_1} f^2 \mu + \int_X \varphi_j \chi_{E_2} f^2 \mu.
\]
where \( g_k = (\chi)^{a_k}/a_k \), \( f_c = 1, 2 \), \( \chi \) denotes the characteristic function.

As \( A_n \geq 0 \) and \( \langle A_n g_1, g_1 \rangle + \langle A_n g_2, g_2 \rangle = 0 \), it follows that \( \langle A_n g_1, g_1 \rangle = 0 \) and \( \langle A_n g_2, g_2 \rangle = 0 \).

(i) Suppose that \( |\varphi_n(x)| > 0 \) a.e. Then \( \langle A_n g_1, g_1 \rangle = 0 \) implies that \( f \) and \( \varphi_n \) have complementary support which is a contradiction to the fact that \( \|f\| = 1 \) and \( |\varphi_n(x)| > 0 \) a.e.

(ii) If \( |\varphi_j(x)| > 0 \) a.e for \( j \neq n \), then \( \langle A_j g_1, g_1 \rangle \neq 0 \) for if \( \langle A_j g_2, g_2 \rangle = 0 \), then \( \langle A_j g_2, g_2 \rangle = 0 \) which means that \( f \) and \( \varphi_j \) have complementary support which is again not possible as argued in (i). Thus \( \langle A_j g_1, g_1 \rangle = 0, \langle A_j g_2, g_2 \rangle = 0 \). We write \( h_k(x) = g_k(x)/\|g_k\|, k = 1, 2 \) and

\[
\lambda = \{\langle A, h_1, h_1 \rangle, \ldots, \langle A, h_i, h_i \rangle\}
\]

and

\[
\mu = \{\langle A, h_2, h_2 \rangle, \ldots, \langle A, h_i, h_i \rangle\}.
\]

Thus \( \lambda \) and \( \mu \) are two points in the joint numerical range with \((0, \ldots, 0)\) as an interior point of the line segment joining these two, which is a contradiction. This proves the result for commuting self-adjoint \( A_j \)'s.

Now, we consider \( A_j \)'s to be commuting normal operators on \( H \). Since each \( A_j \) has a unique decomposition

\[ A_j = A_{j_1} + iA_{j_2}, \quad j = 1, 2, \ldots, n, \]

where \( A_{j_1} \) and \( A_{j_2} \) are self-adjoint, the \( 2n \)-tuple

\[ \{A_{11}, A_{21}, \ldots, A_{ni}, A_{12}, \ldots, A_{n2}\} \]

is of commuting self-adjoint operators. Similarly if

\[
\lambda_j = \lambda_{j_1} + i\lambda_{j_2}, \quad j = 1, 2, \ldots, n
\]

then \( \lambda' = (\lambda_{11}, \ldots, \lambda_{ni}, \lambda_{12}, \ldots, \lambda_{n2}) \) is an extreme point of \( W(A_{11}, \ldots, A_{n2}) \) and \( \lambda' \in W(A_{11}, \ldots, A_{n2}) \). Thus \( \lambda' \in \sigma_p(A_{11}, \ldots, A_{n2}) \). Hence \( \lambda = (\lambda_{11}, \ldots, \lambda_n) \in \sigma_p(A_1, \ldots, A_n) \) and the result is proved.

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REFERENCES


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