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ON EXTREME POINTS OF THE JOINT NUMERICAL RANGE OF COMMUTING NORMAL OPERATORS

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Let $W(T) = \{\langle Tx, x\rangle : \|x\| = 1; x \in H\}$ denote the numerical range of a bounded normal operator T on a complex Hilbert space H. S. Hildebrandt has proved that if λ is an extreme point of $\overline{W(T)}$, the closure of $W(T)$, and $\lambda \in W(T)$ then λ is in the point spectrum of T. In this note, we shall prove an analogous result for an n-tuple of commuting bounded normal operators on H.

2. Notations and terminology. Let $A = (A_1, \cdots, A_n)$ be an n-tuple of commuting bounded operators on H and $\mathscr{U}$, the double commutant of $\{A_1, \cdots, A_n\}$. Then $\mathscr{U}$ is a commutative Banach algebra with identity, containing the set $\{A_1, \cdots, A_n\}$. We shall need the following definitions [3] and [4].

A point $z = (z_1, \cdots, z_n)$ of $\mathscr{C}^n$ is in the joint spectrum $\sigma(A)$ of A relative to $\mathscr{U}$ if for all $B_1, \cdots, B_n$ in $\mathscr{U}$

$$\sum_{j=1}^{n} B_j(A_j - z_j) \neq I .$$

The joint numerical range of A is the set of all points $z = (z_1, \cdots, z_n)$ of $\mathscr{C}^n$ such that for some x in H with $\|x\| = 1$, $z_j = \langle A_j x, x\rangle$ i.e.,

$$W(A) = \{\langle Ax, x\rangle = (\langle A_1x, x\rangle, \cdots, \langle A_nx, x\rangle)\} .$$

We say that $z = (z_1, \cdots, z_n)$ is in the joint point spectrum $\sigma_p(A)$ if there exists some $0 \neq x \in H$ such that

$$A_j x = z_j x , \qquad j = 1, \cdots, n ,$$

and that z is in the joint approximate point spectrum $\sigma_\pi(A)$ if there exists a sequence $\{x_n\}$ of unit vectors in H such that $\|(A_j - z_j)x_n\| \to 0$ as $n \to \infty$, $j = 1, \cdots, n$.

Bunce [2] has proved that $\sigma_\pi(A)$ is a nonempty compact subset of $\mathscr{C}^n$.

If $A = (A_1, \cdots, A_n)$ is an n-tuple of commuting normal operators, then the extreme points of $\overline{W(A)}$ are in the joint approximate point spectrum $\sigma_\pi(A)$. This is immediate from the fact that for such A_j's,

$$\begin{aligned}\overline{W(A)} &= \text{closed convex hull of } \sigma(A)\\ &= \text{closed convex hull of } \sigma_\pi(A) ,\end{aligned}$$

and that every compact set contains the extreme points of its closed convex hull [1, Cor. 36. 11, p. 144]. We show in the following theorem that something more can be said about the extreme points of $\overline{W(A)}$, see Hildebrandt [5].

THEOREM. *Let* $A = (A_1, \cdots, A_n)$ *be an n-tuple of commuting normal operators on H. If* $\lambda = (\lambda_1, \cdots, \lambda_n)$ *is an extreme point of* $\overline{W(A)}$ *and* $\lambda \in W(A)$, *then* $\lambda \in \sigma_p(A)$.

Proof. Firstly, we shall prove the result for commuting self-adjoint operators.

It is sufficient to show that if $(0, \cdots, 0)$ is an extreme point of $\overline{W(A)}$ and $(0, \cdots, 0) \in W(A)$, then $(0, \cdots, 0) \in \sigma_p(A)$.

Since $(0, \cdots, 0)$ is an extreme point of $\overline{W(A)}$, we may assume that

$$\overline{W(A)} \subset \{z = (\alpha_1, \cdots, \alpha_n) \in \mathscr{C}^n;\ \mathrm{Re}\,\alpha_n \geqq 0\}\,. \tag{1}$$

As $A_1, \cdots, A_n$ are commuting self-adjoint operators, there exists a measure space (X, μ) and a set of bounded measurable functions $\varphi_1, \cdots, \varphi_n$ in $L^\infty(X, \mu)$ such that each A_j is unitarily equivalent to multiplication by φ_j on $L^2(X, \mu)$. Thus

$$A_j f = \varphi_j f, \quad \text{for all} \quad f \in L^2(X, \mu)$$

and for each $j = 1, 2, \cdots, n$ [3].

Because of the assumption (1), and since $\sigma(A) \subset \overline{W(A)}$, we have

$$\sigma(A) \subset \{z = (\alpha_1, \cdots, \alpha_n) \in \mathscr{C}^n;\ \mathrm{Re}\,\alpha_n \geqq 0\}\,.$$

It follows that $A_n \geqq 0$ and so $\varphi_n(x) \geqq 0$ a.e. Let, if possible, $(0, \cdots, 0) \notin \sigma_p(A_1, \cdots, A_n)$. Then $|\varphi_j(x)| > 0$ a.e. for at least one $j = 1, 2, \cdots, n$. Let

$$E_1 = \{x \in X;\ \mathrm{Im}\,\varphi_j(x) \geqq 0\}$$

and

$$E_2 = \{x \in X;\ \mathrm{Im}\,\varphi_j(x) < 0\}\,.$$

Since $(0, \cdots, 0) \in W(A_1, \cdots, A_n)$, for some $f \in H$ with $\|f\| = 1$, $\langle A_j f, f\rangle = 0$, $j = 1, 2, \cdots, n$ and

$$\begin{aligned} 0 = \langle A_j f, f\rangle &= \int_X \varphi_j(x)|f(x)|^2 d\mu \\ &= \int_{E_1} \varphi_j |f|^2 d\mu + \int_{E_2} \varphi_j |f|^2 d\mu \\ &= \int_X \varphi_j |\chi_{E_1} f|^2 d\mu + \int_X \varphi_j |\chi_{E_2} f|^2 d\mu \end{aligned}$$

$$= \int_X \varphi_j |g_1|^2 d\mu + \int_X \varphi_j |g_2|^2 d\mu$$
$$= \langle A_j g_1, g_1\rangle + \langle A_j g_2, g_2\rangle ,$$

where $g_k x = (\chi)_{E_k}(x) f(x)$, $k = 1, 2$, χ denotes the characteristic function.

As $A_n \geqq 0$ and $\langle A_n g_1, g_1\rangle + \langle A_n g_2, g_2\rangle = 0$, it follows that $\langle A_n g_1, g_1\rangle = 0$ and $\langle A_n g_2, g_2\rangle = 0$.

(i) Suppose that $|\varphi_n(x)| > 0$ a.e. Then $\langle A_n g_1, g_1\rangle = 0$ implies that f and φ_n have complementary support which is a contradiction to the fact that $\|f\| = 1$ and $|\varphi_n(x)| > 0$ a.e.

(ii) If $|\varphi_j(x)| > 0$ a.e for $j \neq n$, then $\langle A_j g_1, g_1\rangle \neq 0$ for if $\langle A_j g, g_1\rangle = 0$, then $\langle A_j g_2, g_2\rangle = 0$ which means that f and φ_j have complementary support which is again not possible as argued in (i). Thus $\langle A_j g_1, g_1\rangle \neq 0$, $\langle A_j g_2, g_2\rangle \neq 0$. We write $h_k(x) = g_k(x)/\|g_k\|$, $k = 1, 2$ and

$$\lambda = \{\langle A_1 h_1, h_1\rangle, \cdots, \langle A_n h_1, h_1\rangle\}$$

and

$$\mu = \{\langle A_1 h_2, h_2\rangle, \cdots, \langle A_n h_2, h_2\rangle\} .$$

Thus λ and μ are two points in the joint numerical range with $(0, \cdots, 0)$ as an interior point of the line segment joining these two, which is a contradiction. This proves the result for commuting self-adjoint A_j's.

Now, we consider A_j's to be commuting normal operators on H. Since each A_j has a unique decomposition

$$A_j = A_{j1} + iA_{j2} , \qquad j = 1, 2, \cdots, n ,$$

where A_{j_1} and A_{j2} are self-adjoint, the $2n$-tuple

$$\{A_{11}, A_{21}, \cdots, A_{n1}, A_{12}, \cdots, A_{n2}\}$$

is of commuting self-adjoint operators. Similarly if

$$\lambda_j = \lambda_{j1} + i\lambda_{j2} , \qquad j = 1, 2, \cdots, n$$

then $\lambda' = \{\lambda_{11}, \cdots, \lambda_{n1}, \lambda_{12}, \cdots, \lambda_{n2}\}$ is an extreme point of $\overline{W(A_{11}, \cdots, A_{n2})}$ and $\lambda' \in W(A_{11}, \cdots, A_{n2})$. Thus $\lambda' \in \sigma_p(A_{11}, \cdots, A_{n2})$. Hence $\lambda = (\lambda_1, \cdots, \lambda_n) \in \sigma_p(A_1, \cdots, A_n)$ and the result is proved.

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