PACIFIC JOURNAL OF MATHEMATICS
Vol. 67, No. 2, 1976

INTEGRALS OF FOLIATIONS ON MANIFOLDS WITH A GENERALIZED SYMPLECTIC STRUCTURE

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Let M be a C^∞ manifold of dimension m and E an integrable subbundle (foliation) of the tangent bundle TM. We are interested in structures on the set of all local integrals of E. For example, if M is a symplectic manifold then the Poisson brackets operation on the set C^∞_{loc} of all local functions of M defines an algebraic structure on C^∞_{loc}. Earlier authors have called such structures "function groups." In particular, if X_H is a nonvanishing Hamiltonian vector field, then X_H defines a foliation E of M and the set of all local integrals of E is also a function group.

The Poisson brackets operation can be defined on manifolds with somewhat less restrictive requirements than that of being symplectic. Other authors such as S. Lie and C. Carathéodory [4] have studied this more general notion of Poisson brackets in the classical local setting. Hermann [9, p. 31] has indicated how to extend the definition of Poisson brackets to functions on manifolds having a closed 2-form ω of constant rank (Recall that M is called symplectic if ω_p has rank m for each $p \in M$).

The paper is largely self-contained, but does require the use of the following basic identities:

$$L_X Y = [X, Y], \quad L_X = i_X d + d i_X, \quad L_X i_Y - i_Y L_X = i_{[X,Y]}.$$

The proofs of these identities may be found in Chapter IV of the first volume of [7]. Other undefined terms appear either in [1] or [7].

1. Generalized symplectic structures on manifolds.

Let M be a C^∞ manifold of dimension m and let ω be a closed 2-form on M. Recall that the kernel of a 2-form ω can be defined at each point $p \in M$ by

$$\begin{aligned}\ker \omega_p &= \{v \in M_p \mid \omega(v, M_p) = 0\} \\ &= \{v \in M_p \mid \omega(M_p, v) = 0\}.\end{aligned}$$

The rank of ω at p is defined to be the rank of the bilinear map $\omega_p: M_p \times M_p \to R$. Of course, since ω_p is a skew-symmetric bilinear map its rank is the even integer $m - \dim(\ker \omega_p)$.

Let Γ denote the set of sections of TM and Γ^* the set of sections of T^*M. Define $\alpha: \Gamma \to \Gamma^*$ by

$$\alpha_X = i_X\omega \ .$$

Let $\Gamma_\omega = \{X \in \Gamma \mid i_X\omega = 0\} = \ker \alpha$.

If we fix $p \in M$ then we may regard $\alpha = \alpha_p$ as a map from T_pM into T_p^*M. Since T_pM is finite dimensional, $T_pM \cong T_p^{**}M$ and we may apply the standard duality theorems of linear algebra. Thus, if we use the usual pairing between T_pM and T_p^*M we have, for $x, y \in T_pM$,

$$\langle \alpha(y), x\rangle = \alpha(y)(x) = \omega_p(y, x) = -\omega_p(x, y) = \langle y^{**}, -\alpha(x)\rangle \ .$$

Thus α is skew adjoint: $\alpha^* = -\alpha$, and

$$\operatorname{im}(\alpha^*) = \operatorname{im}(\alpha) = \ker(\alpha)^\perp$$

where $\ker(\alpha)^\perp$ is the annihilator of $\ker(\alpha)$ in T_p^*M.

From this we see that if $\Gamma_\omega^* \equiv \{\beta \in \Gamma^* \mid \beta(\Gamma_\omega) = 0\}$, then $\Gamma_\omega^* = \ker(\alpha)^\perp \subseteqq \Gamma^*$. From these remarks it follows that $\Gamma_\omega^* = \operatorname{im}(\alpha)$.

If $\operatorname{inv}(\Gamma)$ is defined by $\operatorname{inv}(\Gamma) = \{X \in \Gamma \mid L_X\Gamma_\omega \subseteqq \Gamma_\omega\}$ then $\operatorname{inv}(\Gamma)$ is the normalizer of Γ_ω in Γ and thus is a Lie subalgebra of Γ. Moreover, it is immediate from the definitions any subalgebra of a Lie algebra is always an ideal in its normalizer, thus Γ_ω is an ideal in $\operatorname{inv}(\Gamma)$. We summarize all these remarks as a proposition.

PROPOSITION 1.1. *The image of the map* $\alpha: \Gamma \rightarrow \Gamma^*$ *is precisely*

$$\Gamma_\omega^* \equiv \{\beta \in \Gamma^* \mid \beta(\Gamma_\omega) = 0\} \ .$$

Moreover, $\operatorname{inv}(\Gamma) \equiv \{X \in \Gamma \mid L_X\Gamma_\omega \subseteqq \Gamma_\omega\}$ *is a Lie subalgebra of* Γ *which contains* Γ_ω *as an ideal.*

We now want to show that $\alpha|\operatorname{inv}(\Gamma)$ is a Lie algebra antihomomorphism from $\operatorname{inv}(\Gamma)$ onto the set $\operatorname{inv}(\Gamma_\omega^*) \subseteqq \Gamma^*$ where $\operatorname{inv}(\Gamma_\omega^*)$ is defined by

$$\operatorname{inv}(\Gamma_\omega^*) \equiv \{\beta \in \Gamma_\omega^* \mid L_Z\beta = 0 \quad \text{for all} \quad Z \in \Gamma_\omega\} \ .$$

Before doing this we need to define a Lie algebra structure on $\operatorname{inv}(\Gamma_\omega^*)$. For this we need a lemma.

LEMMA 1.2. *If* $Z \in \Gamma_\omega$, *then* $L_Z\Gamma_\omega^* \subseteqq \Gamma_\omega^*$. *In fact,* $L_Z\alpha_X = \alpha_{L_ZX}$ *for each* $X \in \Gamma$.

Proof. Since $L_Z\omega = (i_Zd + di_Z)\omega = 0$, $L_Z\alpha_X = L_Zi_X\omega = i_XL_Z\omega + i_{[X,Z]}\omega = \alpha_{[Z,X]} = \alpha_{L_ZX}$.

COROLLARY 1.3. $\alpha(\operatorname{inv}\Gamma) = \operatorname{inv}(\Gamma_\omega^*)$.

Proof. From Proposition 1.1, we know that $\operatorname{inv}(\Gamma_\omega^*)$ is contained in im (α). By the lemma above, for $Z \in \Gamma_\omega$, $L_Z\alpha_X = \alpha_{L_Z X} = -\alpha_{L_X Z}$; thus $\alpha_X \in \operatorname{inv}(\Gamma_\omega^*)$ iff $L_X Z \in \Gamma_\omega$ for all $Z \in \Gamma_\omega$. It follows that $\alpha(\operatorname{inv}\Gamma) = \operatorname{inv}(\Gamma_\omega^*)$.

The map α is a linear transformation from $\operatorname{inv}(\Gamma)$ onto $\operatorname{inv}(\Gamma_\omega^*)$ with kernel Γ_ω. Thus $\operatorname{inv}(\Gamma_\omega^*) \cong \operatorname{inv}(\Gamma)/\Gamma_\omega$ as vector spaces. Since Γ_ω is a Lie ideal in $\operatorname{inv}(\Gamma)$, the quotient $\operatorname{inv}(\Gamma)/\Gamma_\omega$ is a Lie algebra. We impose this Lie structure on $\operatorname{inv}(\Gamma_\omega^*)$ via the vector space isomorphism induced by α.

PROPOSITION 1.4. *The set* $\operatorname{inv}(\Gamma_\omega^*)$ *of all invariant elements of* Γ_ω^* *is a Lie algebra under* $\{\ ,\ \}$ *where* $\{\ ,\ \}$ *is defined by*

$$\{\alpha_X, \alpha_Y\} = -\alpha_{[X,Y]} .$$

The map $\alpha\colon \operatorname{inv}(\Gamma) \to \operatorname{inv}(\Gamma_\omega^*)$ is a Lie algebra antihomomorphism with kernel Γ_ω, thus the sequence

$$0 \longrightarrow \Gamma_\omega \longrightarrow \operatorname{inv}(\Gamma) \xrightarrow{\ \alpha\ } \operatorname{inv}(\Gamma_\omega^*) \longrightarrow 0 ,$$

is an exact sequence of Lie algebras.

REMARK. It is easy to see that for $\alpha, \beta \in \operatorname{inv}(\Gamma_\omega^*)$ one has

$$\{\alpha, \beta\}|_U = \{\alpha|_U, \beta|_U\}$$

for open subsets U of M.

REMARK. We now call attention to certain identities which have proven useful in our work. If β and γ are closed 1-forms in Γ_ω^* and X and Y are vector fields such that $\beta = \alpha_X$, $\gamma = \alpha_Y$, then

$$\{\beta, \gamma\} = -i_{[X,Y]}\omega = -L_X Y = L_Y\beta = d(2\omega(X, Y)) .$$

Note, in particular, that $\{\beta, \gamma\}$ is *exact*.

To see that the above identities hold, observe that

$$\begin{aligned}\{\beta, \gamma\} &= \{\alpha_X, \alpha_Y\} = -\alpha_{[X,Y]} \\ &= -i_{[X,Y]}\omega = -L_X i_Y\omega + i_Y L_X\omega \\ &= -L_X\alpha_Y + i_Y(di_X + i_X d)\omega = -L_X\alpha_Y + i_Y(d\alpha_X) \\ &= -L_X\gamma = -(di_X + i_X d)\gamma = -d(i_X\gamma) \\ &= 2d(\omega(X, Y)) .\end{aligned}$$

Let $C^\infty(\omega)$ denote the set of all invariant functions of $\ker\omega$, i.e.

$$C^\infty(\omega) = \{f \mid L_Z f = df(Z) = 0 \quad \text{for all} \quad Z \in \Gamma_\omega\} .$$

We now define the Poisson bracket $\{\ ,\ \}$ for pairs of invariant functions of $\ker\omega$:

$$\{f, g\} = 2\omega(X_f, X_g)$$

where X_f and X_g are any two vector fields such that

$$dh = i_{X_h}\omega$$

for $h = f, g$. Clearly $\{\ ,\ \}$ is well-defined.

PROPOSITION 1.5. *If $f, g \in C^\infty(\omega)$ the following statements are true*:

(1) $\{f, g\} = -L_{X_f}(g) = L_{X_g}(f)$

(2) $d\{f, g\} = \{df, dg\}$.

Moreover, $C^\infty(\omega)$ is a Lie algebra with respect to $\{\ ,\ \}$ and

(3) $X_{\{f,g\}} + [X_f, X_g] \in \Gamma_\omega$.

Proof. If $f, g \in C^\infty(\omega)$ then (1) follows from $\{f, g\} = 2\omega(X_f, X_g) = df(X_g) = L_{X_g}(f)$. By the above remark we have $d\{f, g\} = d(2\omega(X_f, X_g)) = \{df, dg\}$ and thus (2) follows. The statement (3) is immediate from definitions.

PROPOSITION 1.6. *If $f, g \in C^\infty(\omega)$ and $dg = i_{X_g}\omega$ then f is constant on integral curves of X_g iff $\{f, g\} = 0$.*

Proof. $X_g(f) = L_{X_g}(f) = \{f, g\} = 0$.

2. Function groups. Let M be a connected C^∞-manifold of dimension m with a 2-form ω of constant rank $\rho \leqq m$. In this case $\ker\omega$ is locally trivial, i.e., $\ker\omega$ is a subbundle of TM. Moreover, $\ker\omega$ is actually an integrable subbundle of TM and thus is a foliation of M. To see this observe that for $X \in \Gamma_\omega$,

$$L_X\omega = i_X(d\omega) + d(i_X\omega) = 0 .$$

Thus for X, Y in Γ_ω,

$$i_{[X,Y]}\omega = L_X(i_Y\omega) - i_Y(L_X\omega) = 0 .$$

A function f is called a local C^∞ function on M iff the domain $U = \operatorname{dom}(f)$ of f is an open subset of M and $f \in C^\infty(U)$. Let $C^\infty_{\text{loc}} = C^\infty_{\text{loc}}(M)$ denote the set of all local C^∞ functions of M. Let $C^\infty_{\text{loc}}(\omega)$ denote the set of all local integrals of the foliation $\ker\omega$, i.e.,

$$C^\infty_{\text{loc}}(\omega) = \{f \in C^\infty_{\text{loc}} \mid df(\ker(\omega_p)) = 0 \quad \text{for all} \quad p \in \operatorname{dom} f\} .$$

Note that in the symplectic case $C^\infty_{\text{loc}}(\omega) = C^\infty_{\text{loc}}$.

Recall that a function $f \in C^\infty_{\text{loc}}$ is said to be C^∞-dependent on $f_1, f_2, \cdots, f_r \in C^\infty_{\text{loc}}$ at $p \in M$ provided that there is a neighborhood U

of p and a function $F \in C^\infty_{\mathrm{loc}}(\boldsymbol{R}^r)$ such that

(1) the functions $f, f_1, f_2, \cdots, f_r$ are all defined on U, and

(2) $f(x) = F(f_1(x), f_2(x), \cdots, f_r(x))$ for each $x \in U$.

If $f, g \in C^\infty_{\mathrm{loc}}(\omega)$ and $U = \operatorname{dom} f \cap \operatorname{dom} g \neq \varnothing$, then U can be regarded as a manifold with $\omega|_U$ a 2-form of constant rank on U. Thus $\{f, g\} = \{f \mid U, g \mid U\}$ is a well-defined element of $C^\infty(\omega \mid U)$. It follows that X_f and X_g have domains dom f and dom g respectively and thus $[X_f, X_g]$ and $X_{\{f,g\}}$ are well-defined vector fields on U. Similarly, $\{df, dg\}$ is a well-defined 1-form on U.

DEFINITION 2.1. A nonvoid subset $\mathscr{S}$ of $C^\infty_{\mathrm{loc}}(\omega)$ is called a *function group* iff the following conditions hold:

(1) $M = \bigcup_{f \in \mathscr{S}} \operatorname{dom}(f)$,

(2) if $f \in \mathscr{S}$ and U is an open subset of dom f then $f \mid U \in \mathscr{S}$,

(3) if $f, g \in \mathscr{S}$ and $\operatorname{dom}(f) \cap \operatorname{dom}(g) \neq \phi$, then $\{f, g\} \in \mathscr{S}$,

(4) if $f_1, f_2, \cdots, f_k$ are elements of $\mathscr{S}$ and f is C^∞-dependent on $f_1, f_2, \cdots, f_k$ then $f \in \mathscr{S}$,

(5) Let $U = \bigcup_j U_j$ where U_j is an open subset of M for each j. If $f \in C^\infty(U)$ and $f \mid U_j \in \mathscr{S}$, for each j, then $f \in \mathscr{S}$.

A function group is said to be of rank r at a point $p \in M$ provided that there are r functions $f_1, f_2, \cdots, f_r$ in $\mathscr{S}$ such that

(1) there is a neighborhood U of p contained in the domain of each of the functions $f_1, f_2, \cdots, f_r$ such that for each $q \in U$

$$df_{1q}, df_{2q}, \cdots, df_{rq}$$

are independent elements of M^*_q, and

(2) for each $f \in \mathscr{S}$, with $p \in \operatorname{dom} f$, f is C^∞-dependent on $f_1, f_2, \cdots, f_r$ on some neighborhood of p.

In case $f_1, f_2, \cdots, f_r$ satisfy (1) and (2) we say that $f_1, f_2, \cdots, f_r$ *generate* $\mathscr{S}$ at p.

REMARK. If $f_1, f_2, \cdots, f_r$ generate $\mathscr{S}$ at p and $g_1, g_2, \cdots, g_s$ generate $\mathscr{S}$ at p, then $r = s$. To see this observe that the definition implies that there exists functions $F_i \in C^\infty_{\mathrm{loc}}(\boldsymbol{R}^r)$, $G_j \in C^\infty_{\mathrm{loc}}(\boldsymbol{R}^r)$ such that for $i = 1, 2, \cdots, s$ and $j = 1, 2, \cdots, r$

$$g_i = F_i(f_1, \cdots, f_r) \quad \text{and} \quad f_j = G_j(g_1, \cdots, g_s) .$$

Then the chain rule applied to the equalities

$$g_i = F_i(G_1(g_1, \cdots, g_s), \cdots, G_r(g_1, \cdots, g_s))$$
$$f_j = G_j(F_1(f_1, \cdots, f_r), \cdots, F_s(f_1, \cdots, f_r))$$

implies that $(\partial F_i/\partial f_j)$ and $(\partial G_k/\partial g_l)$ are inverse matrices. Hence $r = .s$

REMARK. If $\mathscr{S}$ is a function group of rank r at $p \in M$, then one can easily show that if $h_1, h_2, \cdots, h_r$ are elements of $\mathscr{S}$ such that $dh_{1p}, dh_{2p}, \cdots, dh_{rp}$ are independent in M_p^* then they generate $\mathscr{S}$ at p.

A function group is said to be of rank r iff it is of rank r at each point of M.

The following is an example to show that a function group may not have the same rank at each point of M. Let $M = \boldsymbol{R}^2$ and $\omega = dx \wedge dy$. Let $f \in C^\infty(\boldsymbol{R})$ such that

$$f(x) = 0, x \leqq 0 \quad \text{and} \quad f(x) > 0, x > 0 .$$

Define functions F and G on $\boldsymbol{R}^2$ by $F(x, y) = x$ and $G(x, y) = f(x)y$. Let $\mathscr{S}$ denote the set of all functions of the form

$$(x, y) \longrightarrow \Phi(F(x, y), G(x, y))$$

where Φ is any element of $C^\infty_{\mathrm{loc}}(\boldsymbol{R}^2)$. Then $\mathscr{S}$ is a function group which has rank 2 at points (x, y) where $x > 0$ and rank 1 at points (x, y) where $x < 0$.

We describe the relation between function groups of rank r and foliations.

THEOREM 2.2. *Let $\mathscr{S}$ be a function group of rank r and let $E_p = \{X_p \mid 2\omega_p(X_p, \cdot) = df(\cdot)$ for $f \in \mathscr{S}\}$ for each $p \in M$. Then $E = \bigcup_{p \in M} E_p \subseteqq TM$ is an integrable subbundle of TM which contains* $\ker(\omega)$.

Proof. We show E is locally trivial. Choose $p \in M$, U a neighborhood of p, and $f_1, \cdots, f_r$ in $\mathscr{S}$ as in the definition of a generating set for $\mathscr{S}$ at p. Let $X_i = X_{f_i}$. If $q \in U$ and $v \in E_q$ then $v = (X_h)_q$ for some $h \in \mathscr{S}$. Since $df_{1q}, \cdots, df_{rq}$ are independent we know that there exists $F \in C^\infty_{\mathrm{loc}}(\boldsymbol{R}^r)$ such that

$$h = F(f_1, \cdots, f_r)$$

on a neighborhood V of q. One sees that

$$X_h - \sum_1^r \frac{\partial F}{\partial x_i} X_i \in \Gamma(\ker(\omega \mid V))$$

and thus $v = (X_h)_q \in \langle X_{1q}, \cdots, X_{rq} \rangle + \ker(\omega_q)$. Therefore E is a subbundle of TM.

We show E is integrable. Let X, Y belong to $\Gamma(E)$ and let $p \in M$. On a neighborhood U of p both X and Y are of the form

$$\Sigma \lambda_i X_i + Z$$

for $\lambda_i \in C^\infty(U)$, $Z \in \Gamma(\omega|_U)$, and $X_i = X_{f_i}$. Then $[X, Y]$ will be in $\Gamma(E)$ provided that for $1 \leqq i, j \leqq r$, $[X_i, X_j] \in \Gamma(E)$ and for $Z \in \Gamma_{(\omega|U)}$, $[X_i, Z] \in \Gamma(E)$. Since $\mathscr{S}$ is a function group, $\{f_i, f_j\} \in \mathscr{S}$ and $X_{\{f_i, f_j\}} \in \Gamma(E \mid U)$. By (3) of Proposition 1.5 it follows that $[X_i, X_j] \in \Gamma(E \mid U)$. Moreover, $2\omega([Z, X_j], Y) = (i_{[Z, X_j]}\omega)(Y) = L_Z(i_{X_j}\omega)(Y) = L_Z(df_j)(Y) = d(i_Z df_j)(Y) = 0$ for all $Y \in \Gamma$. Thus $[Z, X_j] \in \Gamma_\omega$ for each $Z \in \Gamma_\omega$ and consequently E is integrable.

Hereafter the foliation E described above will be called the foliation determined by $\mathscr{S}$.

If $\mathscr{S}$ is a function group then the reciprocal of $\mathscr{S}$ is defined to be the set of all $g \in C^\infty_{\text{loc}}(\omega)$ such that $\{f, g\} = 0$ for all $f \in \mathscr{S}$ such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \phi$. We denote the reciprocal of $\mathscr{S}$ by $\mathscr{S}'$. The fact that $\mathscr{S}'$ is a function group is somewhat trivial. To see that $\mathscr{S}'$ is closed under $\{\ ,\ \}$ one uses the Jacobi identity. To see that (4) of Definition 2.1 holds we need an identity which is useful in subsequent sections of our paper: for arbitrary $h_1, h_2, \cdots, h_n \in C^\infty_{\text{loc}}(\omega)$ and $F \in C^\infty_{\text{loc}}(\boldsymbol{R}^n)$, then

$$\{f, F(h_1, h_2, \cdots, h_n)\} = \sum_i \frac{\partial F}{\partial h_i}(h_1, h_2, \cdots, h_n)\{f, h_i\} . \tag{2.4}$$

Part (4) follows immediately from this identity. To prove 2.4 observe that

$$\begin{aligned}\{f, F(h_1, h_2, \cdots, h_n)\} &= -2\omega(X_F, X_f) = -dF(X_f) \\ &= -\sum_i \frac{\partial F}{\partial h_i}\{h_i, f\} = \sum_i \frac{\partial F}{\partial h_i}\{f, h_i\}.\end{aligned}$$

REMARK. It is obvious that $\mathscr{S} \subseteqq \mathscr{S}''$ for any function group $\mathscr{S}$. Observe that if $\mathscr{S}$ has rank r, then $\mathscr{S} = \mathscr{S}''$.

If $\mathscr{S}$ is a function group then $\mathscr{T}$ is a subgroup of $\mathscr{S}$ iff $\mathscr{T}$ is a function group such that $\mathscr{T} \subseteqq \mathscr{S}$.

Observe that every function group is a subgroup of the function group $C^\infty_{\text{loc}}(\omega)$. Also the intersection of two subgroups is a subgroup. In particular $\mathscr{S} \cap \mathscr{S}'$ is a subgroup of both $\mathscr{S}$ and $\mathscr{S}'$.

PROPOSITION 2.6. *Let $\mathscr{S}$ be a function group of rank r at p. Then its reciprocal has rank $\rho - r$ at p.*

Proof. Let $p \in M$ and let $f_1, \cdots, f_r$ be generators of $\mathscr{S}$ at p. Choose coordinates $x_1, \cdots, x_m$ at p such that $X_i = X_{f_i} = \partial/\partial x_i$ for $1 \leqq i \leqq r$ and such that $\{\partial/\partial x_{r+j}\}$ $1 \leqq j \leqq m - \rho$ generate Γ_ω near p. Then any integral of the integrable system $X_1, \cdots, X_r, \partial/\partial x_{r+1}, \cdots, \partial/\partial x_{r+m-\rho}$ depends only on the last coordinates. Since each

$f \in \mathcal{S}'$ is an integral of this system it follows that $x_{m+r-\rho+1}, \cdots x_m$ generates $\mathcal{S}'$ at p.

Using arguments similar to those above we obtain the following corollary.

COROLLARY 2.7. *Let $\mathcal{S}$ be a function group of rank r, $\mathcal{S}'$ the reciprocal of $\mathcal{S}$, and E the foliation determined by $\mathcal{S}$. Then*

(1) *$E_p = \cap \{\ker dg_p \mid g \in \mathcal{S}'\}$, for each $p \in M$,*

(2) *if $g_1, g_2, \cdots, g_{\rho-r}$ generate $\mathcal{S}'$ at $p \in M$, then there is a neighborhood U of p such that the map $x \to (g_1(x), g_2(x), \cdots, g_{\rho-r}(x))$ is constant on each leaf of the foliation $E \mid U$ of U.*

We say that a subbundle E of TM is locally *Hamiltonian* iff $\ker(\omega) \subseteq E$ and for each $p \in M$ there is a neighborhood U of p such that $\Gamma(E \mid U)$ is spanned by vector fields X which satisfy $df = i_X\omega$ for some $f \in C^\infty_{\text{loc}}(\omega)$.

PROPOSITION 2.8. *An integrable subbundle E is the foliation determined by some function group iff E is locally Hamiltonian. Moreover, the function group which determines such an E is unique.*

Proof. Clearly if E is determined by some function group, then E is locally Hamiltonian.

Conversely, suppose that E is locally Hamiltonian and consider the set $\mathcal{I}$ of all local integrals of E. We now show that $\mathcal{I}$ is a function group and that E is determined by the reciprocal, $\mathcal{I}'$, of $\mathcal{I}$. Let $f, g \in \mathcal{I}$, $p \in M$, and $X \in \Gamma(E)$. There is no loss of generality in assuming that there is an $H \in C^\infty_{\text{loc}}(\omega)$ such that $2\omega(X, \cdot) = dH(\cdot)$ in a neighborhood of p. It follows that

$$\begin{aligned} d\{f, g\}(X) &= L_{X_H}(\{f, g\}) = \{f, \{g, H\}\} + \{g, \{H, f\}\} \\ &= \{L_X g, f\} + \{L_X f, g\} = 0 \end{aligned}$$

by Proposition 1.5, the Jacobi identity, and the fact that $X \in \Gamma(E)$. Thus $\{f, g\} \in \mathcal{I}$ and it follows that $\mathcal{I}$ is a function group with constant rank. Since $\mathcal{I} = \mathcal{I}''$ it follows from Corollary 2.7 that

$$E = \cap \{\ker df \mid f \in \mathcal{I}'' = \mathcal{I}\}\,.$$

REMARK. If $\mathcal{S}$ is any function group then $\mathcal{S}$ determines a unique integrable locally Hamiltonian subbundle E of TM and conversely. If E is determined by $\mathcal{S}$ then the reciprocal of $\mathcal{S}$ is precisely the set of all local integrals of E. If E is an integrable locally Hamiltonian subbundle of TM then the set of all local integrals of E is a function group. The foliation determined by the reciprocal of this function group is precisely E.

Let $\mathscr{S}$ be a function group of rank r. We say that a set $S \subseteqq C^\infty(M)$ globally generates $\mathscr{S}$ provided that for each $p \in M$ there exist functions $f_1, f_2, \cdots, f_r \in S$ and a neighborhood U of p such that $\{f_1 | U, f_2 | U, \cdots, f_r | U\}$ generates $\mathscr{S}$ at p. We say that a set $T \subseteqq \Gamma^*$ of closed 1-forms globally generates $\mathscr{S}$ provided that for each $p \in M$ there exist forms $\beta_1, \cdots, \beta_r \in T$, a neighborhood U of p, functions $f_1, f_2, \cdots, f_r$ satisfying $df_i = \beta_i$ on U for $i = 1, 2, \cdots, r$ such that $\{f_1|U, f_2|U, \cdots, f_r|U\}$ generates $\mathscr{S}$ at p.

PROPOSITION 2.9. *Suppose that there exist closed* 1*-forms* $\beta_1, \beta_2, \cdots, \beta_n$ *in* Γ^*_ω *and* $r > 0$ *such that*

(i) $\beta_1(p), \beta_2(p), \cdots, \beta_n(p)$ *span an r-dimensional subspace of* M^*_p *for each* $p \in M$,

(ii) *there exist functions* $a_{ijk} \in C^\infty(M)$ *such that*

$$\{\beta_i, \beta_j\} = \sum_{k=1}^{n} a_{ijk}\beta_k .$$

Then there exists a unique function group $\mathscr{S}$ *of rank r which is globally generated by* $\{\beta_1, \beta_2, \cdots, \beta_n\}$. *Conversely, if* $\mathscr{S}$ *is a function group of rank r which is globally generated by* $\beta_1, \beta_2, \cdots, \beta_n$ *then conditions* (i) *and* (ii) *are satisfied.*

Proof. The details of this proof are much like those of Theorem 2.2 and are left to the reader.

Recall that $\mathrm{inv}(\Gamma^*_\omega)$ is a Lie algebra under $\{\ ,\ \}$. Observe that if $\alpha_1, \alpha_2, \cdots, \alpha_n$ are elements of $\mathrm{inv}(\Gamma^*_\omega)$ they span a finite dimensional subalgebra of $\mathrm{inv}(\Gamma^*_\omega)$ iff

$$\{\alpha_i, \alpha_j\} = \sum_k c_{ijk}\alpha_k$$

for constants $c_{ijk} \in \boldsymbol{R}$.

We now give an application of function groups which is a slight generalization of certain well-known theorems.

THEOREMS 2.10. *Let M be a symplectic manifold* ($\rho = m = 2N$) *and* $\mathscr{S}$ *a function group of rank r on M. Suppose that the closed* 1*-forms* $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ *globally generate* $\mathscr{S}$ *and that they span an n-dimensional subalgebra* $\mathscr{L}$ *of* $\mathrm{inv}(\Gamma^*_\omega) = \Gamma(T^*M)$. *If the vector field* X_{α_i} *is complete for each* $i = 1, 2, \cdots, n$, *then each leaf of the foliation determined by* $\mathscr{S}$ *is diffeomorphic to a homogeneous space*

G/H where G is the unique simply connected Lie group with Lie algebra $\mathscr{L}$ and H is a closed subalgebra of G.

Proof. This is a consequence of a well-known theorem due to Palais [11] (see also Loos [10]). The details of the proof of Theorem 2.10 are similar to those of Theorem 1 of [2].

REMARK. Note that if we take $r = 2N - 1$ we obtain a part of Theorem 1 of Andrié and Simms [2]. Note that if we take $r = N$ and assume that $\mathscr{L}$ is commutative we obtain a part of a theorem of Arnold [1] in which the leaves of the foliation turn out to be cylinders or tori (see, for example, Abraham [1, page 113]).

3. Invariant metrics and transverse structures. Let M be a connected C^∞-manifold of dimension m and let E be an integrable subbundle of TM of dimension r. The normal bundle TM/E of E will be denoted by Q and its dual Q^* will be identified with the bundle E^0 where, for each $x \in M$, E_x^0 is the annihilator of E_x in T_x^*M, i.e.,

$$E_x^0 = \{\beta \in T_x^*M \mid \beta(E_x) = 0\}\ .$$

Define a connection ∇^* on $\Gamma(E^0)$ along the leaves of E by $\nabla_X^*\beta = L_X\beta$ for $\beta \in \Gamma(E^0)$ and $X \in \Gamma(E)$.

Observe that if f is any local integral of E then $\nabla_X^*(df) = L_X(df) = df(X) = 0$ and thus df is covariant constant along leaves of E. Also, if $f_1, f_2, \cdots, f_{m-r}$ are independent local integrals of E defined on an open set $U \subseteq M$, then $df_1, df_2, \cdots, df_{m-r}$ span E^0 on U.

LEMMA 3.1. *If $\beta \in \Gamma(E^0)$ is closed, then β is parallel along the leaves of E,* i.e., *$\nabla_X^*\beta = 0$ for all $X \in \Gamma(E)$.*

Proof. $\nabla_X^*\beta = L_X\beta = (i_X d)\beta + (d i_X)\beta = 0$ for all $X \in \Gamma(E)$ and $\beta \in \Gamma(E^0)$.

COROLLARY 3.2. *If $\beta_1, \beta_2, \cdots, \beta_{m-r}$ are global, independent, closed elements of $\Gamma(E^0)$, then E^0 is parallelizable,* i.e., *it has m-r global, independent, parallel sections.*

If σ is a Riemannian metric on M, then Q may be identified with the orthogonal complement of E in TM. Let $\sigma_Q = \sigma \mid (Q \times Q)$ be the induced metric on Q. If $\beta \in \Gamma(E^0)$, then grad β is that unique vector field in $\Gamma(Q)$ such that

$$\sigma(\operatorname{grad} \beta, \cdot) = \beta$$

and, for $\xi \in \Gamma(Q)$, β_ξ is that element of $\Gamma(E^0)$ defined by

$$\beta_\xi = \sigma(\xi, \cdot)$$

We define the dual connection ∇ of ∇^* to be that connection on $\Gamma(Q)$ along leaves of E such that

$$\nabla_X(\xi) = \operatorname{grad}(\nabla_X^* \beta_\xi)$$

for $X \in \Gamma(E)$ and $\xi \in \Gamma(Q)$. Another connection $\tilde{\nabla}$ for $\Gamma(Q)$ along the leaves of E is defined by

$$\tilde{\nabla}_X(\xi) = [L_X \xi]_Q$$

where $X \in \Gamma(E)$, $\xi \in \Gamma(\xi)$ and where $[Y]_Q$ denotes the component of Y in Q.

LEMMA 3.3. *If σ_Q is invariant with respect to $\tilde{\nabla}$ then $\tilde{\nabla} = \nabla$.*

Proof. For $\xi, \eta \in \Gamma(Q)$ we have: $(\nabla_X^* \beta_\xi)(\eta) = (L_X \beta_\xi)(\eta) = i_\eta(L_X \beta_\xi) = L_X(i_\eta \beta_\xi) - i_{[X,\eta]}(\beta_\xi) = L_X(\sigma_Q(\xi, \eta)) - \sigma(\xi, [X, \eta]) = [\sigma_Q(\tilde{\nabla}_X \xi, \eta) + \sigma_Q(\xi, \tilde{\nabla}_X \eta)] - \sigma(\xi, [X, \eta]_Q) = \sigma_Q(\tilde{\nabla}_X \xi, \eta)$. Thus $\tilde{\nabla}_X \xi = \operatorname{grad}(\nabla_X^* \beta_\xi) = \nabla_X \xi$.

We say that σ is invariant when σ_Q is invariant with respect to the connection $\tilde{\nabla}$ in which case $\nabla = \tilde{\nabla}$. Observe that a metric σ satisfies this property iff it is "bundle-like" in the sense of Reinhart [12]. Also the connection $\tilde{\nabla}_X$ can be defined for all $X \in \Gamma(TM)$ in such a way that $\tilde{\nabla}$ is a "basic connection" (see Conlon [5]). Moreover the last result is a reflection of the fact that restrictions of basic connections to $\Gamma(E)$ are unique.

LEMMA 3.4. *If σ is an invariant metric, then β is parallel with respect to ∇^* iff* grad β *is parallel with respect to $\tilde{\nabla}$.*

Proof. It is a standard result that β is ∇^*-parallel iff grad β is parallel relative to the dual connection ∇ (see [7], Vol. II, page 342). Since $\nabla = \tilde{\nabla}$ the result follows.

REMARK. If σ is an invariant metric the usual one-to-one correspondence between $\Gamma(Q)$ and $\Gamma(E^0)$ induces a one-to-one correspondence between $\tilde{\nabla}$-parallel sections of Q and ∇^*-parallel sections to E^0.

REMARK. If ξ and η are $\tilde{\nabla}$-parallel along leaves of E then the invariance of σ implies that $\sigma(\xi, \eta)$ is an integral of E. Thus if β is a closed element of $\Gamma^*(E)$ we conclude that $\sigma(\operatorname{grad}\beta, \operatorname{grad}\beta)$ is constant on leaves of E. If σ is complete as well as invariant then the vector field

$$\frac{1}{\sigma(\operatorname{grad}\beta,\ \operatorname{grad}\beta)}\cdot\operatorname{grad}\beta$$

is a complete vector field for nonvanishing closed β in $\Gamma(E^0)$.

The foliation E is *transversally* parallelizable iff there exist m-r independent elements of ΓQ each of which is $\tilde{\nabla}$-parallel along the leaves of E.

THEOREM 3.5. *Suppose there exist $m-r$ everywhere independent closed 1-forms $\beta_1, \beta_2, \cdots, \beta_{m-r}$ such that*

$$\beta_i(\Gamma(E)) = 0 \quad \textit{for} \quad i = 1, 2, \cdots, m-r\,.$$

Then E is transversally parallelizable.

Proof. If we show that there exists an invariant metric on E, then the theorem will be a consequence of Lemmas 3.1 and 3.4. Let Q be the orthogonal complement of E in TM relative to an arbitrary Riemannian τ on TM. Define σ on TM by

$$\sigma = \tau \mid (E \times E) \oplus \sum_{i=1}^{m-r} (\beta_i \otimes \beta_i)\,.$$

Clearly σ is a Riemannian on TM. We show that σ is invariant. First observe that for $\xi, \eta \in \Gamma(Q)$ and $X \in \Gamma(E)$,

$$L_X(\sigma_Q(\xi, \eta)) = \sum_{i=1}^{m-r} L_X(\beta_i(\xi)\beta_i(\eta)) = \sum_{i=1}^{m-r} [\beta_i(\xi)L_X(\beta_i(\eta)) + \beta_i(\eta)L_X(\beta_i(\xi))]\,.$$

But

$$\begin{aligned} L_X(\beta_i(\eta)) &= L_X(i_\eta\beta_i) = i_{[X,\eta]}\beta_i + i_\eta(L_X\beta_i) \\ &= \beta_i([X, \eta]) + i_\eta([i_X d + d i_X](\beta_i)) = \beta_i([X, \eta]_Q) = \beta_i(\tilde{\nabla}_X(\eta))\,. \end{aligned}$$

Thus

$$\begin{aligned} L_X(\sigma_Q(\xi, \eta)) &= \Sigma[\beta_i(\xi)\beta_i(\tilde{\nabla}_X(\eta)) + \beta_i(\eta)\beta_i(\tilde{\nabla}_X(\xi))] \\ &= \sigma_Q(\xi, \tilde{\nabla}_X\eta) + \sigma_Q(\tilde{\nabla}_X\xi, \eta) \end{aligned}$$

as required. The theorem follows.

REMARK. In the proof of the preceding theorem we have introduced a new metric $\sigma = \tau|_E \oplus \sum_{i=1}^{m-r} (\beta_i \otimes \beta_i)$. Observe that the orthogonal complement of E relative to σ is the same as for τ, namely Q. The gradient vector fields of the 1-forms $\beta_1, \beta_2, \cdots, \beta_{m-r}$ with respect to this metric are parallel along the leaves of E. In the following we will use these vector fields without specific references to the metric σ. Thus grad β_i is the unique section of Q satisfying

$$\sum_{j=1}^{m-r} \beta_j(\text{grad}\,\beta_i)\beta_j(Y) = \beta_i(Y) \tag{3.6}$$

for all $Y \in \Gamma(Q)$.

We make a few remarks regarding completeness. First note that if the metric τ is complete then the metric σ will also be complete if there exist numbers l and L such that

$$l\tau_p(X_p, X_p) \leqq \sum_{i=1}^{m-r} \beta_i(X_p)^2 \leqq L\tau_p(X_p, X_p)$$

for all $p \in M$ and $X \in \Gamma(Q)$. If this is the case then the vector fields $[1/\beta_i(\text{grad}\,\beta_i)]\,\text{grad}\,\beta_i$ are complete vector fields. In any case (assuming τ is complete) the vector fields grad β_i will be complete if they are bounded in the metric τ. Moreover, in this case, every linear combination in the grad β_i is complete.

COROLLARY 3.7. *If in addition to the hypothesis of Theorem* 3.5 *we require that every linear combination of the vector fields grad* β_i (see 3.6) *be complete, then*

(1) *any two leaves of E are diffeomorphic and if any leaf of E is closed in M they all are,*

(2) *if E admits a closed leaf then there is a fibre bundle p: $M \to N$ where N is parallelizable and E is the foliation of M whose leaves are the fibres of p.*

Proof. The corollary follows immediately from Theorem 3.5 above and Propositions 4.3 and 4.4 of Conlon [5].

We now apply the results of this section to function groups.

As an example consider the case where M is symplectic and suppose there is a Hamiltonian function $H \in C^\infty(M)$ such that $dH(p) \neq 0$ for each $p \in M$. Clearly $\{H\}$ globally generates a function group $\mathscr{H}$ of rank 1. This leads to a foliation E which is generated by the unique Hamiltonian vector field $X_H = X_{dH}$. The reciprocal function group $\mathscr{H}'$, which consists of all local integrals of E, also determines a foliation E'. Thus by Theorem 3.5, E' is transversally parallelizable. Indeed, if the vector field grad(dH) is complete then each two leaves of E' are diffeomorphic. Also since the leaves are the components of the level surfaces of H, they are closed and hence, by Corollary 3.7, they fibre M over a parallelizable manifold. The following theorem generalizes this example where

M is not necessarily symplectic and the grad β_i are as defined by 3.6.

THEOREM 3.8. *Let ω be a closed 2-form of constant rank ρ on M. Let $\beta_1, \beta_2, \cdots, \beta_r$ be closed 1-forms which globally generate a function group $\mathscr{S}$ of rank r. Then the foliation E' determined by the reciprocal function group of $\mathscr{S}$ is transversally parallelizable. Moreover, if every linear combination of the vector fields* grad *β_i, $i = 1, 2, \cdots, r$ is complete then E' is a complete transversally parallelizable foliation and each two leaves of E' are diffeomorphic. Furthermore, if one of the leaves of E' is closed then they all are and M is a fibre bundle over a parallelizable manifold in which the fibres are the leaves of E'.*

Proof. The theorem is an immediate consequence of what it means for $\{\beta_1, \beta_2, \cdots, \beta_r\}$ to globally generate $\mathscr{S}$, Theorem 3.5 and Corollary 3.7.

REMARK. Suppose that in the above theorem we have $r = \rho$. In this case $E' = \ker \omega$. Moreover, if some leaf L of the foliation E' is closed then the manifold M is fibered by $\pi: M \to N$ where $\pi^{-1}(x) \cong L$, for each x, and N, the manifold of leaves of ker ω, is a symplectic manifold. This is true since $N_p \cong Q_p$ and $\omega \mid (Q \times Q)$ is nondegenerate.

REMARK. If in the above theorem $r = 1$, then E' is a foliation of codimension 1 and thus by [5, Proposition 5.1] we conclude that either every leaf of E' is closed or else every leaf of E' is dense in M.

REMARK. If in addition to the hypothesis of the above theorem we assume that the 1-forms $\beta_1, \beta_2, \cdots, \beta_r$ are exact, then there exist functions $H_1, H_2, \cdots, H_r$ such that $dH_i = \beta_i$ and the leaves of E', being components of level surfaces of $H_i = h_i$, are necessarily closed. Thus we see that if the functions $\{H_1, H_2, \cdots, H_r\}$ globally generate a function group of rank r and every linear combination of the grad (H_i) is complete then each two components of the level surfaces $H_i = h_i$ are diffeomorphic.

ACKNOWLEDGEMENT. The authors would like to express their gratitude to David J. Simms for his many valuable comments.

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Received December 11, 1975 and in revised form July 6, 1976.

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