PACIFIC JOURNAL OF MATHEMATICS
Vol 68, No 1, 1977

ON A THEOREM OF APOSTOL CONCERNING MÖBIUS FUNCTIONS OF ORDER k

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In 1970, Tom M. Apostol introduced a class of arithmetical functions $\mu_k(n)$ for all positive integral k, as a generalization of the Möbius function $\mu(n) = \mu_1(n)$ and established the following theorem: For $k \geqq 2$, $M_k(x) = \Sigma_{n \leqq x}\mu_k(n) = A_k x + O(x^{1/k} \log x)$, where A_k is a positive constant. In this paper we improve the above O-estimate to $O(x^{4k/(4k^2+1)}\omega(x))$ on the assumption of the Riemann hypothesis, where $\omega(x) = \exp\{A \log x (\log\log x)^{-1}\}$, A being a positive absolute constant.

1. Introduction. T. M. Apostol [1] introduced the following generalization of the Möbius function $\mu(n)$. Let k be a fixed positive integer. Let μ_k, the Möbius function of order k be defined by $\mu_k(1) = 1$, $\mu_k(n) = 0$ if $p^{k+1} | n$ for some prime p, $\mu_k(n) = (-1)^r$ if $n = p_1^k p_2^k \cdots p_r^k \Pi_{i>r} p_i^{a_i}$, $0 \leqq a_i < k$, $\mu_k(n) = 1$ otherwise. In other words, $\mu_k(n)$ vanishes if n is divisible by the $(k+1)$st power of some prime; otherwise, $\mu_k(n)$ is 1 unless the prime factorization of n contains the kth powers of exactly r distinct primes, in which case $\mu_k(n) = (-1)^r$. When $k = 1$, $\mu_k(n)$ is the usual Möbius function, $\mu_1(n) = \mu(n)$.

He established the following asymptotic formula (cf. [1], Theorem 1) for the summatory function $M_k(x) = \Sigma_{n \leqq x}\mu_k(n)$: For $k \geqq 2$ and $x \geqq 2$

$$\sum_{n \leqq x} \mu_k(n) = A_k x + O(x^{1/k} \log x), \tag{1}$$

where A_k is the constant given by

$$A_k = \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right), \tag{2}$$

the product being extended over all primes p.

In this note we improve the O-estimate of the error term in (1) above on the assumption of the Riemann hypothesis by proving the following: For $x \geqq 3$,

$$\sum_{n \leqq x} \mu_k(n) = A_k x + O\left(x^{4k/(4k^2+1)}\omega(x)\right), \tag{3}$$

where $\omega(x)$ is given by (5) below.

2. Lemmas. The proof of (3) is based on the following four lemmas.

LEMMA 1. (cf. **[5]**, Theorem 14–26 (A), p. 316). *If the Riemann hypothesis is true, then for* $x \geqq 3$,

$$M(x) = \sum_{n \leqq x} \mu(n) = O(x^{1/2}\omega(x)), \tag{4}$$

where

$$\omega(x) = \exp\{A \log x (\log\log x)^{-1}\}, \tag{5}$$

A being an absolute positive constant.

LEMMA 2 (cf. **[3]**, Lemma 2.5). *If the Riemann hypothesis is true, then for* $x \geqq 3$, *and* $s > 1$,

$$\sum_{n \leqq x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O(x^{\frac{1}{2}-s}\omega(x)), \tag{6}$$

where $\zeta(s)$ *is the Riemann Zeta function defined by* $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ *for* $s > 1$.

In order to state the other two lemmas we need to recall the following terminology and notation established by E. Cohen **[2]**: Let k be a fixed integer $\geqq 2$. A positive integer n is called *unitarily* k*-free*, if the multiplicity of each prime divisor of n is not a multiple of k; or equivalently, if n is not divisible unitarily by the kth power of any integer > 1. By a *unitary* divisor of n, we mean as usual a divisor $d > 0$ of n such that $(d, n/d) = 1$. The integer 1 is also considered to be unitarily k-free. Let Q_k^* denote the set of unitarily k-free integers and let q_k^* denote the characteristic function of the set Q_k^*; that is, $q_k^*(n) = 1$ or 0 according as $n \in Q_k^*$ or $n \notin Q_k^*$. Let $Q_k^*(x) = \sum_{n \leqq x} q_k^*(n)$. In other words, $Q_k^*(x)$ is the number of unitarily k-free integers $\leqq x$. Then we have

LEMMA 3 (cf. **[4]**, Theorem 3.2). *If the Riemann hypothesis is true, then for* $x \geqq 3$,

$$Q_k^*(x) = A_k \zeta(k) x + O(x^{2/(2k+1)}\omega(x)), \tag{7}$$

where $\omega(x)$ *is given by* (5) *and* A_k *is given by* (2).

LEMMA 4 (cf. [1], eq. (10)). $\mu_k(n) = \sum_{d^k\delta = n} \mu(d) q_k^*(\delta)$.

3. Proof of (3). Using Lemma 4, we obtain

$$M_k(x) = \sum_{n \leqq x} \mu_k(n) = \sum_{n \leqq x} \sum_{d^k\delta = n} \mu(d) q_k^*(\delta) = \sum_{d^k\delta \leqq x} \mu(d) q_k^*(\delta), \tag{8}$$

the summation being taken over all ordered (d, δ) such that $d^k\delta \leqq x$.

Let $z = x^{1/k}$. Further, let $0 < \rho = \rho(x) < 1$, where the function $\rho(x)$ will be suitably chosen later.

If $d^k\delta \leqq x$, then both $d > \rho z$ and $\delta > \rho^{-k}$ can not simultaneously hold and so from (8), we have

$$\begin{aligned} M_k(x) &= \sum_{\substack{d^k\delta \leqq x \\ d \leqq \rho z}} \mu(d) q_k^*(\delta) + \sum_{\substack{d^k\delta \leqq x \\ \delta \leqq \rho^{-k}}} \mu(d) q_k^*(\delta) - \sum_{\substack{d \leqq \rho z \\ \delta \leqq \rho^{-k}}} \mu(d) q_k^*(\delta). \\ &= S_1 + S_2 - S_3, \quad \text{say.} \end{aligned} \tag{9}$$

Applying Lemma 3, we obtain

$$\begin{aligned} S_1 &= \sum_{d \leqq \rho z} \mu(d) \sum_{\delta \leqq x/d^k} q_k^*(\delta) = \sum_{d \leqq \rho z} \mu(d) Q_k^*\left(\frac{x}{d^k}\right) \\ &= \sum_{d \leqq \rho z} \mu(d) \left\{ A_k \zeta(k) \frac{x}{d^k} + O\left(\left(\frac{x}{d^k}\right)^{2/(2k+1)} \omega\left(\frac{x}{d^k}\right)\right)\right\} \\ &= A_k \zeta(k) x \sum_{d \leqq \rho z} \frac{\mu(d)}{d^k} + O\left(x^{2/(2k+1)} \omega(x) \sum_{d \leqq \rho z} d^{-2k/(2k+1)} \right), \end{aligned} \tag{10}$$

since $\omega(x)$ is monotonic increasing. We have

$$\sum_{d \leqq \rho z} d^{-2k/(2k+1)} = O\left((\rho z)^{1-2k/(2k+1)}\right) = O\left((\rho z)^{1/(2k+1)}\right),$$

so that the O-term in (10) is $O(\rho^{1/(2k+1)} z \omega(x))$.

Now, applying Lemma 2 we obtain from (10),

$$\begin{aligned} S_1 &= A_k \zeta(k) x \left\{ \frac{1}{\zeta(k)} + O\left((\rho z)^{-k+1/2} \omega(\rho z)\right)\right\} + O(\rho^{1/(2k+1)} z \omega(x)) \\ &= A_k x + O(\rho^{-k+1/2} z^{\frac{1}{2}} \omega(x)) + O(\rho^{1/(2k+1)} z \omega(x)), \end{aligned} \tag{11}$$

since $\omega(\rho z) \leqq \omega(z) < \omega(x)$.

We have by Lemma 1,

$$S_2 = \sum_{\delta \leqq \rho^{-k}} q_k^*(\delta) \sum_{d \leqq (x/\delta)^{1/k}} \mu(d) = \sum_{\delta \leqq \rho^{-k}} q_k^*(\delta) M\left(\left(\frac{x}{\delta}\right)^{1/k}\right)$$

$$(12) \qquad = O\left(\sum_{\delta \leqq \rho^{-k}} q_k^*(\delta)\left(\frac{x}{\delta}\right)^{1/2k} \omega\left(\left(\frac{x}{\delta}\right)^{1/k}\right)\right)$$

$$= O\left(x^{1/2k}\omega(x) \sum_{\delta \leqq \rho^{-k}} q_k^*(\delta)\delta^{-1/2k}\right)$$

Now, by Lemma 3 and partial summation, we obtain

$$\sum_{\delta \leqq \rho^{-k}} q_k^*(\delta)\delta^{-1/2k} = O\left((\rho^{-k})^{1-1/2k}\right) = O\left(\rho^{-k+1/2}\right).$$

Hence by (12), we have

$$(13) \qquad S_2 = O\left(\rho^{-k+1/2} z^{1/2} \omega(x)\right).$$

Also, by Lemmas 1 and 3, we obtain

$$S_3 = \left(\sum_{d \leqq \rho z} \mu(d)\right)\left(\sum_{\delta \leqq \rho^{-k}} q_k^*(\delta)\right) = O\left(\rho^{1/2} z^{1/2} \omega(\rho z)\rho^{-k}\right)$$

$$(14) \qquad = O\left(\rho^{-k+1/2} z^{1/2} \omega(x)\right).$$

Hence by (9), (11), (13) and (14), we obtain

$$(15) \qquad M_k(x) = A_k x + O\left(\rho^{-k+1/2} z^{1/2} \omega(x)\right) + O\left(\rho^{1/(2k+1)} z \omega(x)\right)$$

Now, choosing $\rho = z^{-(2k+1)/(4k^2+1)}$, we see that $0 < \rho < 1$ and $\rho^{-k+1/2} z^{1/2} = \rho^{1/(2k+1)} z = z^{4k^2/(4k^2+1)} = x^{4k/(4k^2+1)}$, so that the first and second O-terms in (15) are both equal to $O(x^{4k/(4k^2+1)}\omega(x))$. Hence (3) follows from (15).

In conclusion we would like to make the following two remarks:

Remark 1. The O-estimate in (3) is uniform in x and k.

Remark 2. Since we have obtained improvement in Apostol's Theorem (1) on the assumption of the Riemann hypothesis by making use of Lemma 3 or Theorem 3.2 of **[4]**, it might appear that it is possible to obtain improvement in (1) even without any hypothesis, by making use of Theorem 3.1 of **[4]**. However, this does not seem possible, at least by our method.

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Received November 2, 1976.

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