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NORMS OF COMPACT PERTURBATIONS OF OPERATORS

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Let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on a complex separable Hilbert space. This paper is concerned with reducing the norm of a product of operators by compact perturbations of one or more of the factors. For any T in $\mathscr{B}(\mathscr{H})$, it is well known that the infimum,

$$\|T\|_e = \inf\{\|T+K\|: K \text{ is a compact operator}\}$$

is attained by some compact perturbation $T + K_0$. For T a noncompact product of n operators, $T = T_1 \cdots T_n$, it is proved that this infimum can be obtained by a compact perturbation of any one of the factors. If T is a compact product, so that the infimum is zero, it is shown that there are compact perturbations $T_1 + K_1, \cdots, T_n + K_n$ of the factors of T such that the product $(T_1 + K_1) \cdots (T_n + K_n)$ is zero; furthermore, it may be necessary to perturb every factor of T in order to obtain this zero infimum. These results are applied to an arbitrary operator T to find a compact perturbation $T + K$ with $\|(T+K)^2\| = \|T^2\|_e$ and $\|(T+K)^3\| = \|T^3\|_e$; here the identical factors are perturbed in identical fashion to achieve both infima. Stronger theorems of this latter sort are proved for special classes of operators.

For any T in $\mathscr{B}(\mathscr{H})$, let $\|T\|_e$ as defined above, be called the *essential norm* of T **[7]**. I. C. Gohberg and M. G. Krein first showed in **[4]** that for any T in $\mathscr{B}(\mathscr{H})$ there is a compact perturbation $T + K_0$ which realizes the essential norm (so $\|T + K_0\| = \|T\|_e$). The case $n = 2$ of the theorem stated above for compact products was proved in a different way in **[6]**: for any compact product $T = T_1T_2$ of two factors, a projection E was constructed so that T_1E and $(I - E)T_2$ are both compact (and so that the product of perturbations $T_1(I - E)$ and ET_2 is zero).

This study was motivated partly by questions considered by J. K. Plastiras and the author in **[7]**: if T is a bounded operator on $\mathscr{H}$, is there a compact K with $\|p(T+K)\| = \|p(T)\|_e$ for all complex polynomials p? Less ambitiously, if T and p are both given, is there a compact K_p such that $\|p(T+K_p)\| = \|p(T)\|_e$? We know of no examples where either of these questions has a negative answer.

It follows from the results proved here on perturbations of products that for each T in $\mathscr{B}(\mathscr{H})$, there is a compact K with $\|T + K\| = \|T\|_e$ and $\|(T+K)^2\| = \|T^2\|_e$; and a compact L with $\|(T+L)^2\| = \|T^2\|_e$ and $\|(T+L)^3\| = \|T^3\|_e$. If T^3 is not compact we can take $K = L$, to get one perturbation achieving all three essential norms. There appear to be erious difficulties in passing from T^3 to T^4. The existence of an operator K as above was proved in **[7]** for any partial isometry T, and for certain other operators.

Stronger results are obtainable for special classes of operators. In **[7]** it was shown that for operators T which are subnormal or essentially normal, there is one compact K such that $\|p(T+K)\| = \|p(T)\|_e$, for every complex polynomial p. Here we prove this for n-normal operators. Turning to operators with no normality properties, we show that for any weighted shift T, there is one compact K with $\|(T+K)^n\| = \|T^n\|_e$ for all n. If in addition T is nilpotent, then $\|p(T+K)\| = \|p(T)\|_e$ for every polynomial p. In **[6]** it was shown that for any T in $\mathscr{B}(\mathscr{H})$ with $p(T)$ compact, there is a compact K_p with $\|p(T+K_p)\| = \|p(T)\|_e = 0$.

If it were true that every T in $\mathscr{B}(\mathscr{H})$ could be perturbed by K, to simultaneously obtain $\|p(T+K)\| = \|p(T)\|_e$ for every polynomial p, this would have significant consequences. It would immediately imply the theorem of T. T. West **[11]** that every Riesz operator is a compact perturbation of a quasinilpotent, and would also answer a question of W. Arveson: if $\pi(T)$ is quasialgebraic in the Calkin algebra, so that $\|p_n(\pi(T))\|^{1/\deg p_n} \to 0$, then is there a compact K so that $\|p_n(T+K)\|^{1/\deg p_n} \to 0$, for the same sequence $\{p_n\}_n$ of monic polynomials? A partial answer to this latter question, and further discussion is given in **[7]**. See also the question raised by S. R. Caradus **[3]**.

In a recent communication we have learned that D. Legg, P. Smith, and J. Ward have proved using Banach space techniques, that for any T in $\mathscr{B}(\mathscr{H})$, there is one compact K with $\|T+K+\lambda I\| = \|T+\lambda I\|_e$, for all complex λ. Thus it is possible to simultaneously attain the essential norm for all linear polynomials in T.

A related result in a more general setting has been obtained by G. K. Pedersen **[8]**. In **[7]** it was shown for any T in $\mathscr{B}(\mathscr{H})$ and for any polynomial p, that

$$\|p(T)\|_e = \inf \|p(T+K)\|, \ K \text{ compact}.$$

Pedersen has proved that if $\mathscr{A}$ is a C^*-algebra and $\mathscr{I}$ is a closed ideal in $\mathscr{A}$ then for any $A \in \mathscr{A}$ and for any n,

$$\|A^n + \mathscr{I}\| = \inf \|(A+B)^n\|, \qquad B \in \mathscr{I}.$$

Let $\mathscr{K}$ denote the closed two-sided ideal in $\mathscr{B}(\mathscr{H})$ of compact operators, and let $\pi\colon \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})/\mathscr{K}$ be the natural homomorphism onto the quotient C^*-algebra, the *Calkin algebra*. Then the essential norm of T in $\mathscr{B}(\mathscr{H})$ defined above is actually the C^*-norm of $\pi(T)$ in the Calkin algebra. We say that $D \in \mathscr{B}(\mathscr{H})$ is *diagonal* if there is an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors for D. A *finite rank* operator is one with finite-dimensional range. The *range projection* of $T \in \mathcal{B}(\mathcal{H})$ is the smallest projection Q such that $QT = T$, and the *support projection* P is the smallest projection such that $TP = T$. Throughout the paper we use $|T|$ to denote $(T^*T)^{1/2}$ and $\sigma(T)$ to denote the spectrum of $T \in \mathcal{B}(\mathcal{H})$. The reader is referred to **[5]** for general facts about Hilbert space operators.

1. Reducing the norm of a product by perturbing its factors. This first theorem is the heart of the paper.

THEOREM 1. *Let A, B in $\mathcal{B}(\mathcal{H})$ be such that the product AB is not compact. Then there is a compact operator K such that*

$$\|A(I-K)B\| = \|AB\|_e.$$

Furthermore, if $\{e_n\}_n$ is any orthonormal basis for $\mathcal{H}$, then K can be constructed to be diagonal relative to that basis, with $0 \leqq K \leqq I$.

Before beginning the proof we make some relevant observations. If D is any diagonal operator with $\|D\| \leqq 1$, then it is trivial that $\|DABx\| \leqq \|ABx\|$, for any $x \in \mathcal{H}$ and any A, B in $\mathcal{B}(\mathcal{H})$. It is also obvious that $\|ABD\| \leqq \|AB\|$, although $\|ABDx\| \leqq \|ABx\|$ may not hold for every x. On the other hand, there is no general relationship between $\|ADB\|$ and $\|AB\|$.

We remark also that this theorem is false if the product AB is compact. To see this let A be any injective compact operator and let $B = I$. Then $\|AB\|_e = 0$, but $A(I-K)B$ cannot be zero if K is any compact operator.

Proof. We may assume that $\|A\| \leqq 1$ and $\|B\| \leqq 1$. Let $\{P_k\}_k$ be the increasing sequence of finite rank projections with range $(P_k) = \operatorname{span}\{e_1, \cdots, e_k\}$.

Let μ be any number with $\|AB\|_e < \mu < \|AB\| = \mu_0$. We will first construct a finite rank perturbation D of I so that $0 \leqq D \leqq I$; D will be $\{e_n\}_n$-diagonal; and with $\|ADB\| \leqq \mu$. Then we will show how this construction is repeated, to define by induction the desired operator $I - K$. In order to be able later to set up the induction, we will write in the factor I, which is being perturbed.

Let $E(\lambda)$ be the spectral resolution for $|AIB|$. Set $E = E((\mu - 2\delta, \mu_0])$, where $\delta > 0$ is a small number with $\mu - 2\delta > \|AB\|_e$. Then E must be a finite rank projection: otherwise, we could find an infinite orthonormal set $\{x_n\}_n$ such that $x_n \in \operatorname{ran}(E)$, and hence for which

$$\| AIBx_n \| = \| \, | AIB | \, x_n \| > \mu - 2\delta.$$

But this would imply

$$\| AIB \|_e \geqq \mu - 2\delta > \| AIB \|_e,$$

a contradiction.

Let G be the projection onto $\operatorname{ran}(IBE)$, so G is finite rank. Choose P_{k_1} from the sequence $\{P_k\}$ large enough so that

$$\| (I - P_{k_1})G \| < \nu,$$

where $\nu > 0$ is a very small number to be determined. Let Q_1 be the finite rank projection onto $\operatorname{ran}(AP_{k_1})$. Let H_1 be the support projection of the finite rank operator $Q_1 A (I - P_{k_1})$, so $H_1 \leqq I - P_{k_1}$.

Choose $k_2 > k_1$ sufficiently large so that

$$\| Q_1 A (I - P_{k_2}) \| = \| Q_1 A (I - P_{k_1}) H_1 (I - P_{k_2}) \| \leqq \| H_1 (I - P_{k_2}) \| < \nu.$$

Let Q_2 be the finite rank projection onto $\operatorname{ran}(AP_{k_2})$. Let H_2 be the finite rank support projection of $Q_2 A (I - P_{k_2})$, so $H_2 \leqq I - P_{k_2}$.

Choose $k_3 > k_2$ sufficiently large so that

$$\| Q_2 A (I - P_{k_3}) \| = \| Q_2 A (I - P_{k_2}) H_2 (I - P_{k_3}) \| \leqq \| H_2 (I - P_{k_3}) \| < \nu.$$

Repeat this process m times, where m is to be determined, to get two increasing sets of finite rank projections $\{Q_n\}_{n=1}^m$, $\{P_{k_n}\}_{n=1}^m$. Set $E_1 = P_{k_1}$, $E_2 = P_{k_2} - P_{k_1}, \cdots, E_m = P_{k_m} - P_{k_{m-1}}$, $E_{m+1} = I - P_{k_m}$. Set $F_1 = Q_1$, $F_2 = Q_2 - Q_1, \cdots, F_m = Q_m - Q_{m-1}$, $F_{m+1} = I - Q_m$.

Observe now that $F_j A E_n = 0$, if $n < j$: for,

$$F_j A E_n = F_j A P_{k_n} E_n = F_j Q_n A P_{k_n} E_n = (Q_j - Q_{j-1}) Q_n A P_{k_n} E_n = 0,$$

whenever $n < j$.

Observe also that $\| F_j A E_n \| < \nu$ if $n > j + 1$ for then $E_n = (I - P_{k_{j+1}}) E_n$, so that

$$\| F_j A E_n \| = \| F_j Q_j A (I - P_{k_{j+1}}) E_n \| \leqq \| Q_j A (I - P_{k_{j+1}}) \| < \nu.$$

Now, set $\gamma = (\mu - 2\delta)/\| AIB \|$, so $0 < \gamma < 1$.

Define $D = I \sum_{j=1}^{m+1} \eta_j E_j$, where $\gamma = \eta_1 < \eta_2 < \cdots < \eta_{m+1} = 1$, is an even partition of the interval $[\gamma, 1]$. We choose a small $\epsilon > 0$ to be determined, and we now determine m: so that $m\epsilon > 1 - \gamma$. In other words, $\eta_j - \eta_{j-1} < \epsilon$. Thus D is a finite rank perturbation of I, and is $\{e_n\}_n$-diagonal.

We will now show that $\|ADB\| \leqq \mu$. (Note that so far we have that $\gamma = \gamma(\delta, \mu)$, and $m = m(\gamma, \epsilon)$; but we are free to choose ϵ and ν as small as we wish.)

Let z be a unit vector of $\mathscr{H}$, and write $z = \alpha x \oplus \beta y$, where $|\alpha|^2 + |\beta|^2 = 1$, $\|x\| = 1 = \|y\|$, and $x \in \operatorname{ran} E((\mu - 2\delta, \mu_0])$, $y \in \operatorname{ran} E([0, \mu - 2\delta])$. Since these are orthogonal spectral projections for $|AIB|$, this means $AIBx$ is orthogonal to $AIBy$. Now,

$$\|ADBz\|^2 \leqq |\alpha|^2 \|ADBx\|^2 + |\beta|^2 \|ADBy\|^2 + 2|\alpha\beta| |\langle ADBx, ADBy\rangle|,$$

and we consider the three summands separately.

First,

$$\begin{aligned}
\|ADBx\| &= \left\| A \sum_{j=1}^{m+1} \eta_j E_j IBx \right\| \\
&\leqq \left\| A \sum_{j=1}^{m+1} \eta_j E_j GIBx \right\| \\
&\leqq \|A\eta_1 E_1 GIBx\| + \nu \qquad (\|(I - E_1)G\| < \nu) \\
&\leqq \eta_1 \|AGIBx\| + 2\nu \\
&= \gamma \|AIBx\| + 2\nu \qquad (\eta_1 = \gamma) \\
&\leqq \frac{\mu - 2\delta}{\|AIB\|} \|AIBx\| + 2\nu \\
&\leqq \mu - 2\delta + 2\nu.
\end{aligned}$$

Now consider

$$\begin{aligned}
\|ADBy\| = \left\| A \sum_{n=1}^{m+1} \eta_n E_n IBy \right\| &= \left\| \sum_{j=1}^{m+1} F_j A \sum_{n=1}^{m+1} \eta_n E_n IBy \right\| \\
= \Bigg\| & F_1 A\eta_1 E_1 IBy + F_1 A\eta_2 E_2 IBy + F_1 A \sum_{n=3}^{m+1} \eta_n E_n IBy \\
& + F_2 A\eta_2 E_2 IBy + F_2 A\eta_3 E_3 IBy + F_2 A \sum_{n=4}^{m+1} \eta_n E_n IBy \\
& \vdots \\
& + F_{m-1} A\eta_{m-1} E_{m-1} IBy + F_{m-1} A\eta_m E_m IBy \\
& + F_{m-1} A\eta_{m+1} E_{m+1} IBy \\
& + F_m A\eta_m E_m IBy + F_m A\eta_{m+1} E_{m+1} IBy \\
& + F_{m+1} A\eta_{m+1} E_{m+1} IBy \Bigg\|,
\end{aligned}$$

since $F_j A E_n = 0$ if $n < j$;

$$\leqq \left\| \sum_{j=1}^{m} F_j A \eta_j (E_j + E_{j+1}) IBy + \sum_{j=1}^{m} (\eta_{j+1} - \eta_j) F_j A E_{j+1} IBy \right.$$

$$\left. + F_{m+1} A \eta_{m+1} E_{m+1} IBy \right\| + \frac{m(m-1)}{2} \nu,$$

since $\|F_j A E_n\| < \nu$ if $n > j+1$;

$$\leqq \left\| \sum_{j=1}^{m+1} \eta_j F_j AIBy - \sum_{j=1}^{m-1} \eta_j F_j A \sum_{n=j+2}^{m+1} E_n IBy \right\| + \epsilon \left\| \sum_{j=1}^{m} F_j A E_{j+1} IBy \right\|$$

$$+ \frac{m(m-1)}{2} \nu,$$

since $\eta_{j+1} - \eta_j < \epsilon$;

$$\leqq \left\| \sum_{j=1}^{m+1} \eta_j F_j AIBy \right\| + m(m-1)\nu + \epsilon$$

$$\leqq \|AIBy\| + m(m-1)\nu + \epsilon$$

$$\leqq \mu - 2\delta + m(m-1)\nu + \epsilon.$$

Finally, consider

$$|\langle ADBx, ADBy \rangle| = \left| \left\langle A \sum_{j=1}^{m+1} \eta_j E_j IBx, ADBy \right\rangle \right|$$

$$\leqq |\langle A\eta_1 E_1 GIBx, ADBy \rangle|$$

$$+ \left| \left\langle A \sum_{j=2}^{m+1} \eta_j E_j (I - P_{k_1}) GIBx, ADBy \right\rangle \right|$$

$$\leqq \left| \left\langle F_1 A \eta_1 E_1 GIBx, F_1 A \sum_{n=1}^{m+1} \eta_n E_n IBy \right\rangle \right| + \nu,$$

since $F_1 A E_1 = A E_1$, and $\|(1 - P_{k_1})G\| < \nu$;

$$= |\langle F_1 A \eta_1 E_1 GIBx, F_1 A (\eta_1 E_1 + \eta_2 E_2) IBy \rangle| + (m-1)\nu,$$

since $\|F_j A E_n\| < \nu$ if $n \geqq j+2$;

$$\leqq |\langle F_1 A \eta_1 E_1 GIBx, F_1 A \eta_1 (E_1 + E_2) IBy \rangle| + \epsilon + (m-1)\nu$$

$$\leqq |\langle F_1 A E_1 GIBx, F_1 A (E_1 + E_2) IBy \rangle| + \epsilon + (m-1)\nu$$

$$\leqq \left| \left\langle F_1 A E_1 GIBx, F_1 A \sum_{n=1}^{m+1} E_n IBy \right\rangle \right| + \epsilon + (2m-2)\nu$$

$$= |\langle AE_1 GIBx, AIBy \rangle| + \epsilon + (2m-2)\nu$$

$$\leqq |\langle AGIBx, AIBy \rangle| + \epsilon + 2m\nu$$

$$= \epsilon + 2m\nu.$$

Now determine $\epsilon = \epsilon(\delta, \mu)$ sufficiently small ($2\epsilon < \delta$ and $(\mu - \delta)^2 + 2\epsilon < \mu^2$), and $\nu = \nu(m, \epsilon)$ sufficiently small ($2m\nu < \epsilon$, $\nu < \epsilon$, $m^2\nu < \epsilon$), so that

$$\begin{aligned}\|ADBz\|^2 &\leqq |\alpha|^2\|ADBx\|^2 + |\beta|\,\|ADBy\|^2 + 2|\alpha\beta|\,|\langle ADBx, ADBy\rangle| \\ &< |\alpha|^2(\mu - \delta)^2 + |\beta|^2(\mu - \delta)^2 + 2|\alpha\beta|2\epsilon \\ &< (\mu - \delta)^2 + 2\epsilon \\ &< \mu^2.\end{aligned}$$

Thus we have D with the desired properties.

This construction is the first step in an induction. To view it as such, rename $D = D_1$, $\mu = \mu_1$, $\delta = \delta_1$, $\gamma = \gamma_1$, $m = m_1$, $\epsilon = \epsilon_1$, $\nu = \nu_1$, $\{E_j\}_{j=1}^{m+1}$ as $\{E_{1j}\}_{j=1}^{m_1+1}$, and $\{\eta_j\}_{j=1}^{m+1}$ as $\{\eta_{1j}\}_{j=1}^{m_1+1}$. A decreasing sequence $\{D_n\}_n$ of $\{e_n\}_n$-diagonal operators will be constructed by induction; each a finite rank perturbation of I. Then the operator $D_0 = \inf D_n$, will be the desired compact perturbation, $D_0 = I - K$.

We specify the sequences of constants to be used (the first terms as above):

(1) Choose a strictly decreasing sequence of positive numbers $\{u_n\}_n$ with μ_1 (as above) $< \mu_0 = \|AB\|$ and $\lim \mu_n = \|AB\|_e \neq 0$. The sequence $\{D_n\}_n$ will satisfy $\|AD_nB\| \leqq \mu_n$.

(2) Choose $\{\delta_n\}_n$ positive numbers decreasing to zero, so that $2\delta_n < \mu_n - \mu_{n+1}$.

(3) Let $\{\gamma_n\}_n$ be the positive sequence converging to 1 given by $\gamma_n = (\mu_n - 2\delta_n)/\mu_{n-1}$.

Then from (2) we have

$$\frac{\mu_{n+1}}{\mu_{n-1}} < \gamma_n < \frac{\mu_n}{\mu_{n-1}},$$

so that the infinite product $\Pi\gamma_n$ converges to a nonzero limit precisely when the operator AB is not compact; i.e., when the $\lim \mu_n = \|AB\|_e \neq 0$.

(4) Choose $\{\epsilon_n\}_n$ decreasing to zero, such that $2\epsilon_n < \delta_n$, and $(\mu_n - \delta_n)^2 + 2\epsilon_n < \mu_n^2$.

(5) Choose integers $\{m_n\}_n$ such that $1 - \gamma_n < m_n\epsilon_n$.

(6) Finally, choose positive $\{\nu_n\}_n$ converging to zero, so that $\nu_n < \epsilon_n$, $m_n^2\nu_n < \epsilon_n$, and $2m_n\nu_n < \epsilon_n$.

Now repeat the above construction, line for line, with D_1 in place of I, using μ_2, δ_2, ϵ_2, m_2, ν_2, and specifying $\{E_{2i}\}_{i=1}^{m_2+1}$ and $\{\eta_{2i}\}_{i=1}^{m_2+1}$; the only additional stipulation being that we choose $E_{21} > \Sigma_{i=1}^{m_1} E_{1i}$. Thus we obtain a $D_2 \leqq D_1$, D_2 a finite rank perturbation of D_1, and hence of I, with $\|AD_2B\| < \mu_2$:

$$D_2 = D_1 \sum_{j=1}^{m_2+1} \eta_{2j} E_{2j} = \left[\sum_{i=1}^{m_1+1} \eta_{1i} E_{1i} \right] \left[\sum_{j=1}^{m_2+1} \eta_{2j} E_{2j} \right]$$
$$= \left[\sum_{i=1}^{m_1} \eta_{1i} E_{1i} \right] \eta_{21} E_{21} + E_{1,m_1+1} \left[\sum_{j=1}^{m_2+1} \eta_{2j} E_{2j} \right]$$
$$= \gamma_2 \sum_{i=1}^{m_1} \eta_{1i} E_{1i} + \eta_{21} \left(E_{21} - \sum_{i=1}^{m_1} E_{1i} \right) + \sum_{j=2}^{m_2+1} \eta_{2j} E_{2j}$$

recalling that γ_2 is η_{21}. The point of this equation is to exhibit the diagonal operator D_2 as a linear combination of orthogonal projections.

Assume for induction we have recursively constructed $D_1 \geqq D_2 \geqq \cdots \geqq D_{k-1}$ as above using in turn the specified constants and such that

$$E_{j1} > \sum_{i=1}^{m_{j-1}} E_{j-1,i}, \qquad j = 1, \cdots, k-1.$$

Then repeat the above construction with the kth constants, choosing

$$E_{k1} > \sum_{i=1}^{m_{k-1}} E_{k-1,i},$$

to obtain $\|AD_kB\| \leqq \mu_k$, and $D_k \leqq D_{k-1}$, where, as an orthogonal sum, we have

$$D_k = \prod_{j=2}^{k} \gamma_j \left[\eta_{11} \left(E_{11} - 0 \right) + \sum_{i=2}^{m_1} \eta_{1i} E_{1i} \right]$$
$$+ \prod_{j=3}^{k} \gamma_j \left[\eta_{21} \left(E_{21} - \sum_{i=1}^{m_1} E_{1i} \right) + \sum_{i=2}^{m_2} \eta_{2i} E_{2i} \right]$$
$$\vdots$$
$$+ 1 \left[\eta_{k1} \left(E_{k1} - \sum_{i=1}^{m_{k-1}} E_{k-1,i} \right) + \sum_{i=2}^{m_k} \eta_{ki} E_{ki} \right]$$
$$+ \left[I - \sum_{i=1}^{m_k} E_{ki} \right],$$

noting that the last summand equals $\eta_{k,m_k+1} E_{k,m_k+1}$.

By induction we now have the desired sequence $\{D_n\}_n$ defined. We show that $\{D_n\}_n$ converges uniformly to $\inf D_n = D_0$, with

$$D_0 = \sum_{n=1}^{\infty} \left(\prod_{j=n+1}^{\infty} \gamma_j \right) \left[\eta_{n1} \left(E_{n1} - \sum_{i=1}^{m_{n-1}} E_{n-1,i} \right) + \sum_{i=2}^{m_n} \eta_{ni} E_{ni} \right]$$

(where $E_{0i} = 0$, all i). This will complete the proof: for then, AD_nB converges to AD_0B, so that $\|AD_nB\| \leqq \mu_n$ each n, implying that

$\|AD_0B\| \leqq \lim \mu_n = \mu$. And since $I - D_n$ is finite rank for each n, therefore $I - D_0$ must be compact. Then $K = I - D_0$ will satisfy the conclusion of the theorem.

The convergence of $\{D_n\}$ follows simply because the product $\Pi\gamma_j$ converges. That is,

$$\begin{aligned} D_k - D_0 = \Bigg\{ &\left(1 - \prod_{j=k+1}^{\infty} \gamma_j\right) \prod_{j=2}^{k} \gamma_j \left[\eta_{11}E_{11} + \sum_{i=2}^{m_1} \eta_{1i}E_{1i}\right] \\ &+ \left(1 - \prod_{j=k+1}^{\infty} \gamma_j\right) \prod_{j=3}^{k} \gamma_j \left[\eta_{21}\left(E_{21} - \sum_{i=1}^{m_1} E_{1i}\right) + \sum_{i=2}^{m_2} \eta_{2i}E_{2i}\right] \\ &\vdots \\ &+ \left(1 - \prod_{j=k+1}^{\infty} \gamma_j\right) (1) \left[\eta_{k1}\left(E_{k1} - \sum_{i=1}^{m_{k-1}} E_{k-1,i}\right) + \sum_{i=2}^{m_k} \eta_{ki}E_{ki}\right]\Bigg\} \\ &+ \sum_{n=k+1}^{\infty} \Bigg\{\left(1 - \prod_{j=n}^{\infty} \gamma_j\right)\left(E_{n1} - \sum_{i=1}^{m_{n-1}} E_{n-1,i}\right) \\ &\qquad + \sum_{i=2}^{m_n} \left[1 - \left(\prod_{j=n+1}^{\infty} \gamma_j\right) \eta_{ni}\right] E_{ni}\Bigg\}, \end{aligned}$$

(recall $\gamma_n = \eta_{n1}$). Note that for each n,

$$1 - \left(\prod_{j=n+1}^{\infty} \gamma_j\right) \eta_{ni} < 1 - \prod_{j=n}^{\infty} \gamma_j.$$

Thus,

$$\|D_k - D_0\| \leqq \sup_{n \geqq k+1} \left(1 - \prod_{j=n}^{\infty} \gamma_j\right) \|R\|,$$

where R is a sum of orthogonal projections multiplied by constants that are between zero and one. Thus $\lim_k \|D_k - D_0\| = 0$, and the theorem is proved.

As immediate corollaries, we get the following:

THEOREM 2. *For any A, B in $\mathcal{B}(\mathcal{H})$, and any $\epsilon > 0$, there is a finite rank operator F with $0 \leqq F \leqq I$ such that*

$$\|A(I - F)B\| < \|AB\|_e + \epsilon.$$

Furthermore, given any orthonormal basis, F can be constructed to be diagonal relative to that basis.

Proof. This is simply the first construction in the preceding proof, and it does not require noncompactness of the product AB.

THEOREM 3. *Let $T_1, \cdots, T_n$ be in $\mathcal{B}(\mathcal{H})$ such that ΠT_j is not compact. Then for any j there is a compact perturbation S_j of T_j such that*

$$\| T_1 \cdots T_{j-1} S_j T_{j+1} \cdots T_n \| = \| \Pi T_j \|_e.$$

If T_j is diagonal, S_j may be obtained by reducing the moduli of some eigenvalues of T_j.

Proof. For $j = 1$, set $A = I$, $B = \Pi T_j$ and apply Theorem 1 to get a compact K with $\|(I-K)\Pi T_j\| = \|\Pi T_j\|_e$. Then set $S_1 = (I-K)T_1$. If T_1 is diagonal, construct K to be diagonal relative to the same basis as T_1. If $j = 2$, set $A = T_1$ and $B = \Pi_{j>1} T_j$, and proceed similarly; the other cases are the same.

In order to obtain a corresponding theorem for compact products of operators we require some preliminary results.

PROPOSITION 4. *Any T in $\mathcal{B}(\mathcal{H})$ has a compact perturbation S where $|S|$ is diagonal.*

Proof. Let $T = U|T|$ be the polar decomposition for T. Let $E = U^*U$ and regard $|T|$ as a positive operator in $\mathcal{B}(E\mathcal{H})$. By a theorem of H. Weyl **[10]**, there is a compact operator K in $\mathcal{B}(E\mathcal{H})$ with $|T| + K$ diagonal relative to some orthonormal basis for $E\mathcal{H}$. Consider this as a diagonal operator on $\mathcal{H}$: $\sigma(|T|+K)$ is the closure of the set $\{d_n\}_n$ of diagonal entries. The Weyl spectrum of $|T|+K$ is

$$\sigma_w(|T|+K) = \bigcap_{C \text{ compact}} \sigma(|T|+K+C).$$

Since $|T|+K$ is normal, by Weyl's Theorem, $\sigma_w(|T|+K)$ consists of the cluster points of $\sigma(|T|+K)$ union the eigenvalues that are repeated infinitely often **[1]**. Now,

$$\sigma_w(|T|+K) = \sigma_w(|T|) \subset \sigma(|T|),$$

so $\sigma_w(|T|+K)$ consists of nonnegative real numbers. Thus the subset of $\{d_n\}_n$ consisting of nonzero, nonpositive numbers has no accumulation points and no infinitely repeated numbers. If we replace such d_n by the element of $\sigma_w(|T|+K)$ nearest d_n, the result is a positive diagonal operator D which is a compact perturbation of $|T|+K$, and such that $ED = D$. Then $S = UD$ is the desired compact perturbation of T.

PROPOSITION 5. *Let A, B be in $\mathscr{B}(\mathscr{H})$ and let K be any compact operator. There are compact perturbations A' and B' of A and B and a projection E such that*

$$A'B' = (AB + K)E.$$

Proof. Let $U|B|$ be the polar decomposition for B. Using the previous result, assume that $|B|$ is diagonal relative to an orthonormal basis $\{e_n\}_n$, with diagonal sequence $\{b_n\}_n$.

To motivate the proof, we remark that, since B may not be invertible, we cannot simply set $A' = A + KB^{-1}$, to get $A'B = AB + K$. However, if we first erase a subsequence of $\{b_n\}_n$ which converges to zero "too fast", then this approach will work.

Let P_n be the finite rank projection onto span $\{e_1, \cdots, e_n\}$. Then $\{P_nKP_n\}_n$ converges uniformly to K, so choose a subsequence $\{P_{n_k}\}_k$ with

$$\|K - P_{n_k}KP_{n_k}\| < \frac{1}{2^{2k}}.$$

Define a sequence of nonnegative real numbers $\{c_m\}_m$ by

$$c_m = \begin{cases} 0 & \text{if} \quad b_m < \dfrac{1}{2^k} \\ b_m & \text{if} \quad b_m \geqq \dfrac{1}{2^k} \end{cases}$$

whenever $n_{k-1} < m \leqq n_k$, for $k = 1, 2, \cdots$, and $n_0 = 0$. Define another sequence $\{d_m\}_m$ by

$$d_m = \begin{cases} 1 & \text{if} \quad c_m = 0 \\ \dfrac{1}{c_m} & \text{if} \quad c_m \neq 0. \end{cases}$$

Note that for $m \leqq n_k$, $d_m \leqq 2^k$. Let $C \in \mathscr{B}(\mathscr{H})$ be the diagonal operator with diagonal $\{c_m\}_m$ relative to $\{e_m\}_m$, and let D be the unbounded densely defined diagonal operator with diagonal $\{d_m\}_m$ relative to $\{e_m\}_m$. Clearly $|B| - C$ is a compact operator.

Furthermore KD is a compact operator: in particular, the sequence $\{P_{n_k}KDP_{n_k}\}$ is uniformly Cauchy. For, assuming $k > i$,

$$\begin{aligned}\|P_{n_k}KDP_{n_k} - P_{n_i}KDP_{n_i}\| &\leqq \sum_{j=i+1}^{k} \|P_{n_j}KDP_{n_j} - P_{n_{j-1}}KDP_{n_{j-1}}\| \\ &\leqq \sum_{j=i+1}^{k} \|P_{n_j}(K - P_{n_{j-1}}KP_{n_{j-1}})DP_{n_j}\| \\ &\leqq \sum_{j=i+1}^{k} \|K - P_{n_{j-1}}KP_{n_{j-1}}\| \|DP_{n_j}\| \\ &\leqq \sum_{j=i+1}^{k} \frac{1}{2^{2(j-1)}} 2^j = \sum_{j=i+1}^{k} \frac{1}{2^{j-2}} < \frac{1}{2^{i-2}}.\end{aligned}$$

Since $\{P_{n_k}\}$ converges strongly to I, then $\{P_{n_k}KDP_{n_k}\}$ converges uniformly to KD.

Let E be the projection whose range is $\overline{\text{span}}\{e_n: c_n \neq 0\}$. Thus $C = |B|E$ and $DC = E$.

To finish the proof, set $A' = A + KDU^*$, $B' = UC$. Then

$$\begin{aligned}A'B' = (A + KDU^*)(UC) = AUC + KDC = AU|B|E + KE \\ = (AB + K)E\end{aligned}$$

and we are done.

Using Theorem 2, it is possible to reduce the norm of a compact product by perturbing any one factor. However it may be necessary to perturb every factor to get a zero product. For example, let C be any one-to-one compact operator, and let $A = \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix}$, $B = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}$, $AB = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$, and let $\begin{pmatrix} K & L \\ M & N \end{pmatrix}$ be any compact operator. Then

$$\left[A + \begin{pmatrix} K & L \\ M & N \end{pmatrix}\right] B = \begin{pmatrix} C + KC & L \\ MC & N + C \end{pmatrix}$$

which equals zero only if $C(I + K) = 0$, an impossibility. Thus the next theorem is the best possible general result.

THEOREM 6. *Let $T_1, \cdots, T_n$ be in $\mathcal{B}(\mathcal{H})$ such that ΠT_j is compact. Then there are compact perturbations $S_1, \cdots, S_n$ of $T_1, \cdots, T_n$ with $\Pi S_j = 0$.*

Proof. The product $T_1(\Pi_{j=2}^n T_j) = C$, a compact operator.

By the previous proposition, there are compact K_1, L_1 and a projection E_1 with

$$(T_1 + K_1)\left(\prod_{j=2}^{n} T_j + L_1\right) = \left[T_1\left(\prod_{j=2}^{n} T_j\right) - C\right] E_1 = 0.$$

Now apply the proposition to $T_2(\Pi_{j=3}^n T_j) + L_1$ to get compact K_2 and L_2 and a projection E_2 with

$$(T_2 + K_2)\left(\prod_{j=3}^{n} T_j + L_2\right) = \left[T_2\left(\prod_{j=3}^{n} T_j\right) + L_1\right] E_2.$$

Thus

$$(T_1 + K_1)(T_2 + K_2)\left(\prod_{j=3}^{n} T_j + L_2\right) = \left(\prod_{j=1}^{n} T_j - C\right) E_1E_2 = 0.$$

Repeated applications of the proposition yield:

$$\prod_{j=1}^{n-2} (T_j + K_j)(T_{n-1}T_n + L_{n-2}) = \left(\prod_{j=1}^{n} T_j - C\right) \prod_{i=1}^{n-2} E_i = 0.$$

And, a final application gives compact K_{n-1} and L_{n-1}, and a projection E_{n-1} with

$$(T_{n-1} + K_{n-1})(T_n + L_{n-1}) = (T_{n-1}T_n + L_{n-2})E_{n-1}$$

so that for $K_n = L_{n-1}$, we have

$$\prod_{j=1}^{n} (T_j + K_j) = \prod_{j=1}^{n-2} (T_j + K_j)(T_{n-1}T_n + L_{n-2})E_{n-1} = 0,$$

and the theorem is proved.

2. Attaining the essential norm for polynomials in an operator.

In this section we first show that any bounded operator can be perturbed to attain $\|T\|_e$, $\|T^2\|_e$, or $\|T^3\|_e$; in most cases all three norms are achieved by a single compact perturbation of T. We then consider special classes of operators, weighted shifts and n-normal operators, for which stronger results are obtained. The first theorem follows by repeated applications of Theorem 3.

THEOREM 7. *Any T in $\mathcal{B}(\mathcal{H})$ with T^3 not compact has a compact perturbation S with $\|S\| = \|T\|_e$, $\|S^2\| = \|T^2\|_e$ and $\|S^3\| = \|T^3\|_e$.*

Proof. Using Proposition 4, we may assume that $|T|$ is diagonal, where $U|T|$ is the polar decomposition for T. Assume also $\|T\| \leqq 1$.

Let $\{\lambda_n\}_n$ be the sequence of diagonal entries of $|T|$ such that $\lambda_n > \||T|\|_e = \|T\|_e$: then, $\lim \lambda_n = \|T\|_e$. Obtain a compact perturbation T_1 of T by replacing each λ_n with $\|T\|_e$, to get $|T_1|$ from $|T|$, and then setting $T_1 = U|T_1|$. Clearly $\|T_1\| = \|T\|_e$.

Now apply Theorem 3 to the product $T_1^2 = U|T_1|T_1$, to get a compact perturbation $|T_1|'$ of $|T_1|$ by reducing some of the eigenvalues of $|T_1|$, such that

$$\|U|T_1|'T_1\| = \|U|T_1|T_1\|_e = \|T^2\|_e.$$

Since reducing the eigenvalues in a diagonal first or last factor does not raise the norm of a product, we have

$$\|U|T_1|'U|T_1|'\| \leqq \|U|T_1|'U|T_1|\| = \|T^2\|_e.$$

So let $T_2 = U|T_1|'$ (then $|T_2| = |T_1|'$).

Finally, apply Theorem 3 to the product $|T_2|U|T_2|T_2$ to get a compact perturbation $|T_2|'$ of $|T_2|$ by reducing some of the eigenvalues of $|T_2|$, such that

$$\||T_2|U|T_2|'T_2\| = \||T_2|U|T_2|T_2\|_e = \|U|T_2|U|T_2|T_2\|_e = \|T^3\|_e,$$

since $U^*U|T_2| = |T_2|$. Thus

$$\|(U|T_2|')^3\| = \||T_2|'U|T_2|'U|T_2|'\| \leqq \||T_2|U|T_2|'T_2\| = \|T^3\|_e.$$

Now let $S = U|T_2|'$ (so $|S| = |T_2|'$). Then $\|S^3\| = \|T^3\|_e$, but also $\|S^2\| = \|T^2\|_e$, for,

$$\|S^2\| = \||T_2|'U|T_2|'\| \leqq \||T_2|U|T_2|\| = \|T_2^2\| = \|T^2\|_e.$$

Similarly $\|S\| = \|T\|_e$, so the proof is complete.

REMARK 8. One can see from this proof, that this approach does not extend to higher powers of T. The difficulty in simultaneously getting identical perturbations of two inside factors of T^4, in order to reduce the norm of T^4, seems to be beyond these techniques.

We have been unable to get the result in Theorem 7 only for the case where T^3 is compact and T^2 is not compact. The complication lies in finding a compact perturbation S with both $S^3 = 0$ and $\|S\| = \|T\|_e$. On the one hand, this is a fairly special case, reducing to a 3×3 upper triangular operator matrix. On the other hand, it points up a general limitation involved in trying to combine the totally unrelated methods for perturbing compact and noncompact products. Our results are summarized in the following:

THEOREM 9. *Let T be any operator in $\mathcal{B}(\mathcal{H})$. Then*

(i) *there is a compact perturbation S with $\|S\| = \|T\|_e$ and $\|S^2\| = \|T^2\|_e$,*

(ii) *there is a compact perturbation* R *with* $\|R^2\| = \|T^2\|_e$ *and* $\|R^3\| = \|T^3\|_e$,

(iii) *if* T^2 *is compact or* T^3 *is not compact we can choose* $S = R$.

Proof of (i). If T^2 is not compact we can argue as in the beginning of the previous proof. If T^2 is compact, then using Theorem 2.4 of **[6]**, we get a compact perturbation T_1 of T with $T_1^2 = 0$. Then T_1 is equivalent to an operator matrix $T_1 = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$, on a Hilbert space $\mathscr{E} \oplus \mathscr{E}$, with $\|T_1\| = \|A\|$. Let A' be a compact perturbation of A with $\|A'\| = \|A\|_e$, then $S = \begin{pmatrix} 0 & A' \\ 0 & 0 \end{pmatrix}$ satisfies (i).

Proof of (ii). If T^2 is compact, (i) applies. If T^3 is not compact, use the preceding theorem. Otherwise, let T_1 be a compact perturbation of T with $T_1^3 = 0$ **[6]**, so T_1 is equivalent to

$$T_1 = \begin{pmatrix} 0 & A & B \\ 0 & 0 & C \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{with} \quad T_1^2 = \begin{pmatrix} 0 & 0 & AC \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\|T_1^2\| = \|AC\|$. Apply Theorem 3 to AC to get $\|A'C\| = \|AC\|_e$, and set

$$S = \begin{pmatrix} 0 & A' & B \\ 0 & 0 & C \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof of (iii). By (i) and the previous theorem.

We remark that the full strength of Theorem 3 (and hence of Theorem 1) is not required for part (i) of Theorem 9. In particular, Theorem 1 can be proved much more easily if the factor $A = I$; and (i) follows readily from this case.

An operator $T \in \mathscr{B}(\mathscr{H})$ is a *weighted shift* of multiplicity k if there is an orthonormal basis $\{e_n\}_n$ for $\mathscr{H}$ on which T is defined by $Te_n = a_n e_{n+k}$, $n = 1, 2, \cdots$, where $\{a_n\}_n$ is a sequence of complex numbers. In order to prove our theorem for weighted shifts we need the following elementary lemma.

LEMMA 10. *Assume* $\alpha > \beta > 0$, *and let* $a_1, \cdots, a_n$ *be an* n*-tuple of positive numbers with* $\Pi a_i = \alpha$. *Then there is an* n*-tuple of positive numbers* $b_1, \cdots, b_n$ *with* $\Pi b_i = \beta$; $b_i \leqq a_i$ *all* i; *and* $\max(a_i - b_i) \leqq \alpha^{1/n} - \beta^{1/n}$.

Proof. If there is some i with $a_i < \alpha^{1/n} - \beta^{1/n}$ the result is trivial. So assume $a_i \geqq \alpha^{1/n} - \beta^{1/n}$, all i; then $a_i \leqq \alpha/(\alpha^{1/n} - \beta^{1/n})^{n-1}$.

Now, define $(c_1, \cdots, c_n)$ by $c_i = a_i - (\alpha^{1/n} - \beta^{1/n})$. To finish the proof, it suffices to show that $\Pi c_i \leqq \beta$. For, by the continuity of the product, we can then find $(b_1, \cdots, b_n)$ with $c_i \leqq b_i \leqq a_i$ all i, and $\Pi b_i = \beta$.

Set $\gamma = \alpha^{1/n} - \beta^{1/n}$ and consider the function $f(a_1, \cdots, a_n) = \Pi c_i = \Pi(a_i - \gamma)$ defined on the compact set X of $\mathbf{R}^n$ where $\gamma \leqq a_i \leqq \alpha/\gamma^{n-1}$ and where $\Pi a_i = \alpha$. Then f has a maximum value M on X: suppose it occurs at $(a_1, \cdots, a_n)$ with $a_1 > a_2$. Consider $(\sqrt{a_1 a_2}, \sqrt{a_1 a_2}, a_3, \cdots, a_n) \in X$. Note that $\sqrt{a_1 a_2} < \frac{1}{2}(a_1 + a_2)$. Thus

$$\begin{aligned} f(\sqrt{a_1 a_2}, \sqrt{a_1 a_2}, a_3, \cdots, a_n) &= (\sqrt{a_1 a_2} - \gamma)^2 \prod_3^n (a_i - \gamma) \\ &> (a_1 a_2 - \gamma(a_1 + a_2) + \gamma^2) \prod_3^n (a_i - \gamma) \\ &= \Pi(a_i - \gamma) = M, \end{aligned}$$

a contradiction. Hence $a_1 = a_2$, and by symmetry, f takes its maximum at $(\alpha^{1/n}, \cdots, \alpha^{1/n})$, so $M = \beta$. The lemma is proved.

THEOREM 11. *Let T be a weighted shift. Then there is a compact perturbation S of T with $\| S^n \| = \| T^n \|_e$, for all $n = 1, 2, \cdots$.*

Proof. Let T be a shift with weight sequence $(a_j)_j$. We give the proof for a shift of multiplicity 1: in this case,

$$\| T^n \| = \sup_j | a_j a_{j+1} \cdots a_{j+n-1} |.$$

The proof for T of multiplicity k is similar, where

$$\| T^n \| = \sup_j | a_j a_{j+k} \cdots a_{j+(n-1)k} |.$$

A straightforward computation allows us to assume $\| T \| \leqq 1$. Let $\mu_n = \| T^n \|$, $\nu_n = \| T^n \|_e$, $n = 1, 2, \cdots$. We will define by induction a sequence $\{S_n\}_n$ of weighted shifts, each obtained by reducing the moduli of the weights of the preceding, and which converges to the desired perturbation S of T.

Let S_1 be the shift with weights $\{a_{1j}\}_j$, where

$$a_{1j} = \begin{cases} a_j & \text{if } |a_j| \leqq \nu_1 \\ a_j \dfrac{\nu_1}{|a_j|} & \text{if } |a_j| > \nu_1. \end{cases}$$

Then $\|S_1\| = \nu_1 = \|T\|_e$, and $|a_{1j} - a_j|$ is a sequence converging to zero and hence $T - S_1$ is compact.

Assume for induction, that $S_1, S_2, \cdots, S_{n-1}$ have been constructed so that for each $k = 1, 2, \cdots, n-1$, and for each $j \leqq k$:

(i) $T - S_k$ is compact;

(ii) $\|S_k^k\| = \nu_k$;

(iii) $\|S_j - S_k\| \leqq \max\{\mu_j^{1/j} - \nu_j^{1/j} \cdots \mu_k^{1/k} - \nu_k^{1/k}\}$;

(iv) if S_j and S_k have weights $\{a_{ji}\}_i$ and $\{a_{ki}\}_i$ resp., then $|a_{ji}| \geqq |a_{ki}|$, each i.

Construct S_n as follows: note that $\|T^n\| = \sup_j |a_j a_{j+1} \cdots a_{j+n-1}| = \mu_n$. Let Λ be the set of j with $|a_j \cdots a_{j+n-1}| > \nu_n$. Define $\gamma_j = |a_j \cdots a_{j+n-1}|$, for $j \in \Lambda$. Then

$$\lim_j \gamma_j = \|T^n\|_e = \nu_n.$$

Applying the preceding Lemma, we see that for each $j \in \Lambda$, there is an n-tuple $(b_j, b_{j+1}^{(1)}, b_{j+2}^{(2)}, \cdots, b_{j+n-1}^{(n-1)})$ satisfying:

(1) $|b_j b_{j+1}^{(1)} \cdots b_{j+n-1}^{(n-1)}| = \nu_n$;

(2) $|b_j| \leqq |a_j|$, $|b_{j+i}^{(i)}| \leqq |a_{j+i}|$, $i = 1, 2, \cdots, n-1$;

(3) $\max\{|a_j - b_j|, \cdots, |a_{j+i} - b_{j+1}^{(i)}|\} \leqq \gamma_j^{1/n} - \nu_n^{1/n}$.

Choose c_j to be one among a_j and those of $b_j, b_j^{(1)}, \cdots, b_j^{(n-1)}$ which are defined, having a minimum modulus (note that since γ_j is only defined for $j \in \Lambda$, some of the b_j, $b_j^{(i)}$ may not be defined). Let T_n be the shift with weights $\{c_j\}_j$.

Note that $T - T_n$ is compact, since either $a_j = c_j$, or

$$|a_j - c_j| \leqq \max_k \{\gamma_k^{1/n} - \nu_n^{1/n}: k \in \Lambda \quad \text{with} \quad k = j - n + 1, \cdots, j\},$$

where $\lim_j \gamma_j = \nu_n$. This inequality also shows that $\|T - T_n\| \leqq \mu_n^{1/n} - \nu_n^{1/n}$, since $\mu_n = \sup \gamma_k$. Also, $\|T_n^n\| = \nu_n$.

Define S_n to be the shift with weights $\{a_{nj}\}_j$ where a_{nj} is the one of $a_{n-1,j}$ and c_j having minimum modulus.

Clearly $T - S_n$ is compact; $|a_{ni}| \leqq |a_{n-1,i}|$ all $i = 1, 2, \cdots$; and $\|S_n^n\| = \nu_n$. Also, we see that

$$\|S_j - S_n\| \leqq \max\{\mu_j^{1/j} - \nu_j^{1/j}, \cdots, \mu_n^{1/n} - \nu_n^{1/n}\},$$

by comparing the ith weights of these operators: since $\|T - S_n\| \leqq \mu_n^{1/n} - \nu_n^{1/n}$, since $|a_{ni}| \leqq |a_{n-1,i}|$ all i, and by induction hypothesis (iii).

So, the sequence $\{S_n\}_n$ is constructed, and we will now see that it converges uniformly to some bounded operator S. The spectrum of any shift is circularly symmetric about the origin [**5**, p. 43]. Thus $\partial\sigma(T)$ consists of one or more circles. Now $\sigma(T)$ contains the spectrum of $\pi(T)$ in the Calkin algebra. By a theorem of C. Putnam [**9**], $\partial\sigma(T) \subset \sigma(\pi(T)) \cup \{$isolated eigenvalues of T of finite multiplicity$\}$. Thus we conclude that $\sigma(T)$ and $\sigma(\pi(T))$ have the same radius. Thus

$$\lim \| T^n \|^{1/n} = \lim \| \pi(T)^n \|^{1/n} = \lim \| T^n \|_e^{1/n},$$

so $\lim \mu_n^{1/n} - \nu_n^{1/n} = 0$. Hence property (iii) implies that $\{S_n\}_n$ is uniformly Cauchy; so set $\lim S_n = S$. Then $T - S_n$ converges to a compact operator, $T - S$. From the construction of $\{S_n\}_n$, in particular property (iv), it is clear that S is a shift whose jth weight has modulus $\leqq$ the modulus of the jth weight of each S_n. Thus, $\|S^n\| \leqq \|S_n^n\| = \nu_n$, each $n = 1, 2, \cdots$, and the result is proved.

The best possible result is attainable for operators which are direct sums of matrices of bounded degree.

THEOREM 12. *Let* $T = \sum_{k=1}^{\infty} \oplus T_k$, *a direct sum of* $m \times m$ *matrices. Then there is a compact perturbation* S *of* T *such that* $\|p(S)\| = \|p(T)\|_e$, *for every complex polynomial* p.

Proof. Consider each T_k as an element of $\mathbf{C}^{m^2}$. Since T is a bounded operator, the set $\{T_k\}_k$ is a bounded set in $\mathbf{C}^{m^2}$, so that the set $X \subset \mathbf{C}^{m^2}$ of accumulation points of $\{T_k\}_k$ is a compact set. We include in X any T_k which are repeated infinitely many times. Then $\{T_k\} \backslash X$ has no accumulation points, so that if

$$d_k = \text{distance}(T_k, X),$$

then $\lim d_k = 0$. For each T_k choose some $S_k \in X$ with $\| T_k - S_k \| = d_k$ (since all topologies on $\mathbf{C}^{n^2}$ are equivalent, we simply use the operator norm).

Let $S = \sum_{k=1}^{\infty} \oplus S_k$. Clearly S is a compact perturbation of T. Furthermore, every element of the set $\{S_k\}_k \subset \mathbf{C}^{n^2}$ is an accumulation point of that set or occurs infinitely often, and thus for any complex polynomial p, the same is true for the set $\{p(S_k)\}_k$. Therefore

$$\|p(S)\| = \left\| \sum_k \oplus p(S_k) \right\| = \sup_k \|p(S_k)\|$$

$$= \limsup_k \|p(S_k)\| = \limsup_k \|p(T_k)\|.$$

Let $E_n = \Sigma_{k=1}^n \oplus I_k$. Now, any compact operator K satisfies

$$\lim_n \|(I - E_n)K(I - E_n)\| = 0.$$

Thus

$$\begin{aligned}\|p(T)+K\| &\geqq \limsup_n \|(I-E_n)(p(T)+K)(I-E_n)\| \\ &= \limsup_n \|(I-E_n)p(T)(I-E_n)\| \\ &= \limsup_k \|p(T_k)\| \\ &= \|p(S)\|,\end{aligned}$$

so $\|p(S)\| = \|p(T)\|_e$.

COROLLARY 13. *If $T \in \mathcal{B}(\mathcal{H})$ is a nilpotent weighted shift, there is a compact perturbation S with $\|p(S)\| = \|p(T)\|_e$, for every complex polynomial p.*

Proof. Any such T satisfies the hypotheses of Theorem 12.

COROLLARY 14. *Let T be an n-normal operator. Then there is a compact perturbation S such that $\|p(S)\| = \|p(T)\|_e$ for every complex polynomial p.*

Proof. The operator T may be regarded as an $n \times n$ operator matrix whose entries are commuting normal operators $\{T_j\}$ on a Hilbert space $\mathcal{E}$. It follows by a theorem of L. G. Brown, R. G. Douglas, and P. A. Fillmore [**2**, Corollary 5.4, p. 83] that there is an orthonormal basis of $\mathcal{E}$ with each $T_j = D_j + K_j$, where D_j is diagonal relative to this basis, for every $j = 1, 2, \cdots, n^2$, and where K_j is compact. Let K be the $n \times n$ operator matrix whose entries are the K_j, $j = 1, \cdots, n^2$. Then $S = T - K$ is an $n \times n$ operator matrix with simultaneously diagonal entries D_j, so that S is unitarily equivalent to an infinite direct sum of $n \times n$ matrices, and the previous theorem applies.

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