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E-UNITARY COVERS FOR INVERSE SEMIGROUPS

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An inverse semigroup is called E-unitary if the equations $ea = e = e^2$ together imply $a^2 = a$. In a previous paper, the first author showed that every inverse semigroup has an E-unitary cover. That is, if S is an inverse semigroup, there is an E-unitary inverse semigroup P and an idempotent separating homomorphism of P onto S. The purpose of this paper is to consider the problem of constructing E-unitary covers for S.

Let S be an inverse semigroup and let F be an inverse semigroup, with group of units G, containing S as an inverse subsemigroup and suppose that, for each $s \in S$, there exists $g \in G$ such that $s \leqq g$. Then $\{(s, g) \in S \times G: s \leqq g\}$ is an E-unitary cover of S. The main result of §1 shows that every E-unitary cover of S can be obtained in this way. It follows from this that the problem of finding E-unitary covers for S can be reduced to an embedding problem. A further corollary to this result is the fact that, if P is an E-unitary cover of S and P has maximal group homomorphic image G, then P is a subdirect product of S and G and so can be described in terms of S and G alone. The remainder of this paper is concerned with giving such a description.

1. E-unitary covers. An inverse semigroup is called E-unitary if the equations $ea = e = e^2$ together imply $a^2 = a$. It was shown in **[4]** that every inverse semigroup S has an E-unitary cover in the sense that there is an E-unitary inverse semigroup P together with an idempotent separating homomorphism θ of P onto S. It was further shown in **[5]** that every E-unitary inverse semigroup is isomorphic to a $P(G, \mathscr{X}, \mathscr{Y})$ where $\mathscr{X}$ is a down directed partially ordered set with $\mathscr{Y}$ an ideal and subsemilattice of $\mathscr{X}$ and where G acts on $\mathscr{X}$ by order automorphisms in such a way that $\mathscr{X} = G\mathscr{Y}$; see **[5]** for details. The group G in $P = P(G, \mathscr{X}, \mathscr{Y})$ is isomorphic to the maximum group homomorphic image P/σ of P where

$$\sigma = \{(a, b) \in P \times P: ea = eb \quad \text{for some} \quad e^2 = e \in P\}.$$

DEFINITION 1.1. Let S be an inverse semigroup and let G be a group. Then an E-unitary inverse semigroup P is an *E-unitary cover of S through G if*

(i) $P/\sigma \approx G$

(ii) there is an idempotent separating homomorphism θ of P onto S.

Thus, if $P = P(G, \mathscr{X}, \mathscr{Y})$ is an E-unitary cover of S then P is an E-unitary cover of S through G. As stated, the problem of finding E-unitary covers of an inverse semigroup S consists of finding homomorphisms onto S. The main result of this section shows that this problem can be replaced by an embedding problem.

DEFINITION 1.2. [2] Let $S = S^1$ be an inverse semigroup, with group of units G. Then S is a *factorizable* inverse semigroup if and only if, for each $a \in S$ there exists $g \in G$ such that $a \leqq g$.

Chen and Hsieh showed in [1] that every inverse semigroup S can be embedded in a factorizable inverse semigroup. Indeed, let θ be a homomorphism of S into the symmetric inverse semigroup $\mathscr{I}_X$ on a set X. Let $Y = X$ if X is finite and $Y = X \cup X'$, with $X \cap X' = \square$, $|X| = |X'|$, otherwise. Then $F = \{\alpha \in \mathscr{I}_Y : \alpha \leqq \gamma$ for some permutation γ of $Y\}$ is a factorizable inverse semigroup which contains $S\theta$.

PROPOSITION 1.3. *Let F be a factorizable inverse semigroup with group of units G and let θ be a one-to-one homomorphism of an inverse semigroup S into F. Suppose that for each $g \in G$ there exists $s \in S$ such that $s\theta \leqq g$. Then*

$$P = \{(s, g) \in S \times G : s\theta \leqq g\}$$

is an E-unitary cover of S through G.

Proof. It is clear that P is an inverse subsemigroup of $S \times G$ and, because F is factorizable, that the first projection $\pi_1 : P \to S$ is an idempotent separating homomorphism of P onto S. Likewise, because of the condition on the homomorphism $\theta : S \to F$, the second projection $\pi_2 : P \to G$ is a homomorphism of P onto G.

Now (s, g), $(t, h) \in P$ with $(s, g)\pi_2 = (t, h)\pi_2$ implies $g = h$ thus $(st^{-1})\theta = s\theta(t\theta)^{-1} \leqq gh^{-1} = 1$. It follows from this that st^{-1} is idempotent so that $es = et$, for some idempotent e in S. But then $(e, 1) \in P$ and $(e, 1)(s, g) = (e, 1)(t, h)$ so that $(s, g)\sigma(t, h)$. This shows that $\pi_2 \circ \pi_2^{-1} \subseteq \sigma$. On the other hand, since π_2 is a homomorphism onto a group, $\sigma \subseteq \pi_2 \circ \pi_2^{-1}$. Hence P has maximum group homomorphic image G.

Finally, since $(s, 1) \in P$ implies $s\theta \leqq 1$, and so $s^2 = s$, if $(s, g)\pi_2 = 1$ then (s, g) is idempotent. Hence P is E-unitary.

It follows from the remarks before Proposition 1.3 that every inverse semigroup has an E-unitary cover. The main result of this section shows that every E-unitary cover of S through G is constructed as in Proposition 1.3 for some factorizable inverse semigroup F, with group of units G, containing S as an inverse subsemigroup. In order to prove this, we need some lemmas.

LEMMA 1.4. *Let $\theta: P(G, \mathscr{X}, \mathscr{Y}) \to S$ be an idempotent separating homomorphism of an E-unitary inverse semigroup $P(G, \mathscr{X}, \mathscr{Y})$ onto S. For each $A \in \mathscr{Y}$ set $N_A = \{g \in G: g^{-1}A \in \mathscr{Y}$ and $(A, g)\theta = (A, 1)\theta\}$; if $X = gA$ with $g \in G$, $A \in \mathscr{Y}$, set $N_X = gN_Ag^{-1}$. Then N_X is well defined. Further, the relation π on $\mathscr{X} \times G$ defined by*

$$(A, g)\pi(B, h) \quad \text{if and only if} \quad A = B \quad \text{and} \quad gh^{-1} \in N_A$$

is an equivalence on $\mathscr{X} \times G$ inducing $\theta \circ \theta^{-1}$ on $P(G, \mathscr{X}, \mathscr{Y})$.

Proof. It was shown in **[5]** that the subgroups N_A, $A \in \mathscr{Y}$ satisfy the following three conditions

(i) $N_A \lhd C_A = \{g \in G: gB = B$ for all $B \leqq A\}$ for $A \in \mathscr{Y}$;

(ii) A, $gA \in \mathscr{Y}$ implies $N_{gA} = gN_Ag^{-1}$;

(iii) $A \leqq B \in \mathscr{Y}$ implies $N_B \subseteq N_A$.

We use (ii), to show that N_X is well defined. Suppose $gA = hB$, A, $B \in \mathscr{Y}$. Then $B = h^{-1}gA$ so that, $N_B = (h^{-1}g)N_A(h^{-1}g)^{-1}$ by (ii). Thus $hN_Bh^{-1} = gN_Ag^{-1}$. When the N_X, $X \in \mathscr{X}$ are defined in this way, it is easy to see that they obey the analogs of (i), (ii), (iii) and the remainder of the lemma follows easily.

For each $C \in \mathscr{X}$, $g \in G$, we shall denote by $[C, g]$ the π-class containing (C, g). Further, we shall denote by $\mathfrak{X}$ the set $(\mathscr{X} \times G)/\pi \cup G$.

LEMMA 1.5. *For each $s \in S$ such that $s = (A, g)\theta$, set*

$$\Delta\rho_s = \{[C, h]: h^{-1}C \leqq A\}$$

$$\text{and} \quad [C, h]\rho_s = [C, hg] \quad \text{for each} \quad [C, h] \in \Delta\rho_s.$$

Then $\rho: s \to \rho_s$ is a faithful representation of S by one-to-one partial transformations of $\mathfrak{X}$.

Proof. This follows straightforwardly, using the fact that $\{N_X: X \in \mathscr{X}\}$ obeys (i), (ii), (iii) of Lemma 1.4.

For each $g \in G$, define $\alpha_g: \mathfrak{X} \to \mathfrak{X}$ by

$$h\alpha_g = hg \quad \text{and} \quad [C, h]\alpha_g = [C, hg]$$

for each $h \in G$, $[C, h] \in (\mathscr{X} \times G)/\pi$. Then, as in Lemma 1.5, it follows that $\alpha: h \to \alpha_h$ is a faithful representation of G by permutations of $\mathfrak{X}$. Let $F = \{\gamma \in \mathscr{I}_{\mathfrak{X}}: \gamma \leqq \alpha_g$ for some $g \in G\}$.

PROPOSITION 1.6. *F is a factorizable inverse semigroup which contains $S\rho$. Further, F has group of units $\{\alpha_g : g \in G\}$ and for each α_g there exists $\rho_s \leqq \alpha_g$; also*

$$P(G, \mathscr{X}, \mathscr{Y}) \approx \{(s, g) \in S \times G : s\rho \leqq \alpha_g\}.$$

Proof. The only part requiring verification is that $P(G, \mathscr{X}, \mathscr{Y}) \approx P$ where $P = \{(s, g) \in S \times G : s\rho \leqq \alpha_g\}$.

Define $\phi : P(G, \mathscr{X}, \mathscr{Y}) \to P$ by $(A, g)\phi = ((A, g)\theta, \alpha_g)$. Then, since θ is idempotent separating and α is faithful, ϕ is one-to-one. Further

$$\begin{aligned}(A, g)\phi(B, h)\phi &= ((A, g)\theta, \alpha_g)((B, h)\theta, \alpha_h) \\ &= ((A \wedge gB, gh)\theta, \alpha_g\alpha_h) \\ &= ((A, g)(B, h))\phi\end{aligned}$$

so ϕ is a homomorphism.

Finally, if $(s, \alpha_g) \in P$, where $s = (B, h)\theta$, then $\rho_s \leqq \alpha_g$ implies $[C, 1]\rho_s = [C, 1]\alpha_g$ for each $C \leqq B$; that is, $[C, h] = [C, g]$ for each $C \leqq B$. In particular, $[B, h] = [B, g]$ so that $s = (B, h)\theta = (B, g)\theta$. Hence $(s, \alpha_g) = (B, g)\phi$ so that ϕ is onto.

Summing up, we have the following theorem.

THEOREM 1.7. *Let G be a group and let S be an inverse semigroup. Let F be a factorizable inverse semigroup with group of units G which contains S as an inverse subsemigroup. Suppose that, for each $g \in G$, there exists $s \in S$ such that $s \leqq g$. Then*

$$\{(s, g) \in S \times G : s \leqq g\}$$

is an E-unitary cover of S through G. Conversely, each E-unitary cover is isomorphic to a cover obtained in this way.

COROLLARY 1.8. *Let P be an E-unitary cover of S through G. Then P is a subdirect product of S and G.*

Theorem 1.7 shows that the problem of finding E-unitary covers of S through G is equivalent to finding an embedding of S into a factorizable inverse semigroup. Such embeddings are hard to classify as they may be much larger than S and G. On the other hand, Corollary 1.8 shows that every E-unitary cover of S through G is a subdirect product of S and G and therefore depends only on S and G. In the next section, we turn to the problem of constructing those subdirect products of S and G which are E-unitary covers of S through G.

2. Subdirect products of inverse semigroups.

DEFINITION 2.1. Let S and T be inverse semigroups. Then a mapping $\phi: S \rightarrow 2^T$ is a *subhomomorphism* of S into T if

(i) $\phi(s) \neq \square$ for each $s \in S$;

(ii) $\phi(s)\phi(t) \subseteq \phi(st)$ for all $s, t \in S$;

(iii) $\phi(s^{-1}) = \phi(s)^{-1}$ for each $s \in S$,

where, for any $A \subseteq T$, $A^{-1} = \{a^{-1}: a \in A\}$.

The set $\phi(S) = \{t \in T: t \in \phi(s) \text{ for some } s \in S\}$ is, from (ii) and (iii), an inverse subsemigroup of T. We say that ϕ is *surjective* if $T = \phi(S)$.

PROPOSITION 2.2. *Let S and T be inverse semigroups and let ϕ be a surjective subhomomorphism of S into T. Then*

$$\Pi(S, T, \phi) = \{(s, t) \in S \times T: t \in \phi(s)\}$$

is an inverse semigroup which is a subdirect product of S and T.

Conversely, suppose that V is an inverse semigroup which is a subdirect product of S and T and let ψ be the induced homomorphism of V into $S \times T$. Then ϕ defined by

$$\phi(s) = \{t \in T: (s, t) \in V\psi\}$$

is a surjective subhomomorphism of S into T. Further

$$V\psi = \Pi(S, T, \phi).$$

Proof. This is straightforward.

Proposition 2.2 shows that every E-unitary cover of S through G is determined by a subhomomorphism of S into G; and, dually, by a subhomomorphism of G into S. We shall consider these two approaches and the relationships between them in the later sections of the paper.

3. Subhomomorphisms into a group. In this section we shall describe the subhomomorphisms of an inverse semigroup S into a group G. Note, however, that not every subhomomorphism ϕ of S into G gives an E-unitary cover of S through G.

DEFINITION 3.1. Let S be an inverse semigroup and let G be a group. Then a subhomomorphism $\phi: S \rightarrow G$ is *unitary* if

$$1 \in \phi(s) \quad \text{implies} \quad s^2 = s.$$

Note that, if s is an idempotent then

$$\phi(s)\phi(s)^{-1} = \phi(s)\phi(s^{-1}) \subseteq \phi(ss^{-1}) = \phi(s)$$

so that $1 \in \phi(s)$.

PROPOSITION 3.2. *Let S be an inverse semigroup and let G be a group. Suppose that ϕ is a surjective unitary subhomomorphism of S into G. Then $\Pi(S, G, \phi)$ is an E-unitary cover of S through G. Conversely, let P be an E-unitary cover of S through G with ψ the induced homomorphism $P \to S \times G$. Then ϕ defined by*

$$\phi(s) = \{g \in G\colon (s, g) \in P\psi\}$$

is a surjective unitary subhomomorphism of S into G.

Proof. Suppose that ϕ is unitary. Then, since the idempotents of $P = \Pi(S, G, \phi)$ are the elements $(e, 1)$ with $e^2 = e$ in S, the projection π_S of P onto S is idempotent separating.

The projection $\pi_G\colon P \to G$ is onto, so, to prove that $G \approx P/\sigma$, we need only show that $\pi_G \circ \pi_G^{-1} = \sigma$. In fact, since π_G is a homomorphism onto a group, so that $\sigma \subseteq \pi_G \circ \pi_G^{-1}$, we need only show that $\pi_G \circ \pi_G^{-1} \subseteq \sigma$.

Suppose that $(s, g)\pi_G = (t, h)\pi_G$ so that $g = h$. Then (s, g), $(t, g) \in P$ implies $(st^{-1}, 1) \in P$. That is, $1 \in \phi(st^{-1})$. Since ϕ is unitary, this implies st^{-1} is idempotent so that $es = et$ for some idempotent e in S. But then $(e, 1)(s, g) = (e, 1)(t, g) = (e, 1)(t, h)$ so that $(s, g)\sigma(t, h)$. Thus $\pi_G \circ \pi_G^{-1} \subseteq \sigma$.

Finally, if $(s, g) \in P$ and $(e, 1)(s, g) = (e, 1)$ then $g = 1$ so that $1 \in \phi(s)$. Since ϕ is unitary, this requires $s^2 = s$; whence (s, g) is idempotent. Hence P is an E-unitary cover of S through G.

Conversely, suppose that P is an E-unitary cover of S through G and let $1 \in \phi(s)$. Then $(s, 1) = p\psi$ for some $p \in P$. But $1 = p\psi\pi_G = p\sigma^{\natural}$ implies $p^2 = p$, since P is E-unitary, so that $s = p\psi\pi_s = p\theta$, where θ is the idempotent separating homomorphism $P \to S$, is also idempotent. Hence ϕ is unitary.

The proof of Proposition 3.2 is strikingly reminiscent of that of Proposition 1.3. This is because Proposition 1.3 is a special case of Proposition 3.2. For, let θ be a homomorphism of S into a factorizable inverse semigroup F, with group of units G, and set

$$\phi(s) = \{g \in G\colon s\theta \leqq g\}.$$

Then $g \in \phi(s)$, $h \in \phi(t)$ implies $s\theta \leqq g$, $t\theta \leqq h$ so that $s\theta t\theta \leqq gh$. Thus

$(st)\theta \leqq gh$; that is, $gh \in \phi(st)$. Hence $\phi(s)\phi(t) \subseteq \phi(st)$. Further $s\theta \leqq g$ if and only if $s\theta^{-1} \leqq g^{-1}$; that is $s^{-1}\theta \leqq g^{-1}$. Hence $\phi(s^{-1}) = \phi(s)^{-1}$ so that ϕ is a subhomomorphism. If θ is one-to-one, then clearly ϕ is unitary so that Proposition 1.3 follows.

In order to obtain a subhomomorphism ϕ, as above, from a mapping $\theta\colon S \to F$ one need not assume that θ is a homomorphism. Only that θ is a v-prehomomorphism in the sense of the following definition.

DEFINITION 3.4. Let S, T be inverse semigroups then a mapping $\theta\colon S \to T$ is a *v-prehomomorphism* if it obeys the following two conditions.

(i) $(st)\theta \leqq s\theta t\theta$ for each $s, t \in S$;

(ii) $(s^{-1})\theta = (s\theta)^{-1}$ for each $s \in S$.

If S and T are semilattices then a v-prehomomorphism is just an isotone mapping of S into T.

The results in the next lemma follow straightforwardly from the definitions.

LEMMA 3.5. *Let S be an inverse semigroup and let F be a factorizable inverse semigroup with group of units G. Suppose that θ is a v-prehomomorphism of S into F. Then ϕ defined by*

$$\phi(s) = \{g \in G\colon s\theta \leqq g\}$$

is a subhomomorphism of S into G. It is surjective if and only if, for each $g \in G$, there exists $s \in S$ such that $s\theta \leqq g$; it is unitary if and only if θ is idempotent determined in the sense that $a\theta$ idempotent implies a idempotent.

Lemma 3.5 shows that v-prehomomorphisms of S into F give rise to subhomomorphisms from S into G. On the other hand, Proposition 1.6 shows that surjective unitary subhomomorphisms of S into G can be obtained from embeddings of S into factorizable inverse semigroups with groups of units G. To end this section, we show that every subhomomorphism of S into G is determined by a v-prehomomorphism θ of S into a factorizable inverse semigroup $\mathcal{K}(G)$ which depends only on G.

It follows from this that every subdirect product of S and G, in particular every E-unitary cover of S through G, is determined by a v-prehomomorphism of S into $\mathcal{K}(G)$. The problem of constructing v-prehomomorphisms between inverse semigroups is considered, in detail, in **[6]**.

Let G be a group. Then we shall denote by $\mathcal{K}(G)$ the set of all cosets $X = Ha$ of G modulo subgroups of G. The following simple lemma characterizes the members of $\mathscr{K}(G)$ among the nonempty subsets of G.

LEMMA 3.6 **[3]**. *Let G be a group and let X be a nonempty subset of G. Then $X \in \mathscr{K}(G)$ if and only if $X = XX^{-1}X$.*

It follows from Lemma 3.6 that any nonempty intersection of cosets is again a coset. We may thus define a binary operation $*$ on $\mathscr{K}(G)$ as follows: for $X, Y \in \mathscr{K}(G)$,

$$X * Y = \text{smallest coset that contains } XY.$$

PROPOSITION 3.7 **[8]**. *Let G be a group. Then $\mathscr{K}(G)$ is a factorizable inverse semigroup with group of units isomorphic to G. The idempotents are the subgroups of G. Further, for $X, Y \in \mathscr{K}(G)$, $X \leqq Y$ if and only if $X \supseteq Y$.*

PROPOSITION 3.8. *Let S be an inverse semigroup and let G be a group. Suppose that ϕ is a subhomomorphism of S into G. Then θ defined by*

$$a\theta = \phi(a) \quad \textit{considered as a member of } \mathscr{K}(G)$$

is a v-prehomomorphism of S into $\mathscr{K}(G)$.

Proof. Let $X = \phi(a)$; then $X \subseteq XX^{-1}X$. On the other hand, if $g_1, g_2, g_3 \in X$ then

$$g_1g_2^{-1}g_3 \in \phi(a)\phi(a)^{-1}\phi(a) = \phi(a)\phi(a^{-1})\phi(a) \subseteq \phi(aa^{-1}a) = \phi(a),$$

since ϕ is a subhomomorphism. Hence $XX^{-1}X \subseteq X$ and so $X = XX^{-1}X$. This shows $X \in \mathscr{K}(G)$, so that θ is a mapping into $\mathscr{K}(G)$.

Next, since ϕ is a subhomomorphism, $a\theta b\theta \subseteq (ab)\theta$ for each $a, b \in S$. But $a\theta * b\theta$ is the smallest coset containing $a\theta b\theta$ so this implies $a\theta * b\theta \subseteq (ab)\theta$. That is, by Lemma 3.7, $a\theta * b\theta \geqq (ab)\theta$. Finally, again since ϕ is a subhomomorphism, $(a^{-1})\theta = \phi(a^{-1}) = \phi(a)^{-1} = (a\theta)^{-1}$ for each $a \in S$. Hence θ is a v-prehomomorphism of S into $\mathscr{K}(G)$.

It follows from Proposition 3.8 that the subdirect products of S and G are determined by v-prehomomorphisms of S into $\mathscr{K}(G)$. More precisely, we have the following theorem, which sums up the results of this section. It should be pointed out however that it may be easier to find subhomomorphisms of S into G directly than to find v-prehomomorphisms of S into $\mathscr{K}(G)$.

THEOREM 3.9. *Let S be an inverse semigroup and let G be a group.*

(A). *Let θ be a $\vee$-prehomomorphism of S into $\mathcal{K}(G)$ and suppose that, for each $g \in G$, there exists $s \in S$ such that $s\theta \leqq \{g\}$; i.e. $g \in s\theta$. Then $\{(s, g): g \in s\theta\}$ is an inverse semigroup which is a subdirect product of S and G. Every subdirect product of S and G is of this form for some $\vee$-prehomomorphism of S into $\mathcal{K}(G)$.*

(B). *With θ as in* (A), *$\{(s, g): g \in s\theta\}$ is an E-unitary cover of S through G if and only if θ is idempotent determined. Every E-unitary cover of S through G is of this form for some idempotent determined $\vee$-prehomomorphism of S into $\mathcal{K}(G)$.*

4. Subhomomorphisms from a group. In §3, we characterized the E-unitary covers of an inverse semigroup S, through a group G, as subdirect products $\Pi(S, G, \phi)$ with ϕ a subhomomorphism of S into G. They can also be described in the form $\Pi(G, S, \phi)$ with ϕ a subhomomorphism of G into S. In this section, we give such a description. As might be expected the results obtained are, in a sense, dual to those in §3.

DEFINITION 4.1. Let S and T be inverse semigroups. Then a mapping $\theta: S \to T$ is a *$\wedge$-prehomomorphism* of S into T if it obeys the following two conditions.

(i) $a\theta b\theta \leqq (ab)\theta$ for each $a, b \in S$;

(ii) $(a^{-1})\theta = (a\theta)^{-1}$ for each $a \in S$.

PROPOSITION 4.2. *Let S be an inverse semigroup and let G be a group. Suppose that T is an inverse semigroup containing S and let θ be a $\wedge$-prehomomorphism of G into T. Then ϕ defined by*

$$\phi(g) = \{s \in S: s \leqq g\theta\}$$

is a subhomomorphism of G into S; ϕ is surjective if and only if, for each $s \in S$ there exists $g \in G$ such that $s \leqq g\theta$.

The semigroup $\Pi(G, S, \phi)$ is E-unitary. It is an E-unitary cover of S through G if ϕ is surjective.

Proof. The fact that ϕ is a subhomomorphism and the statement about the surjectivity of ϕ are readily verified.

Let $a = 1\theta$; then, since θ is a $\wedge$-prehomomorphism

$$a = aa^{-1}a = aaa \leqq (1.1)\theta a = a^2 \leqq a$$

so that a is idempotent. Let $(g, s) \in \Pi(G, S, \phi)$ and suppose that $(g, s)(1, e) = (1, e)$ where e is idempotent. Then $g = 1$ so that $s \leqq 1\theta = a$; thus s is idempotent. Hence $\Pi(G, S, \phi)$ is E-unitary.

Suppose that ϕ is surjective. Then, if we identify $S \times G$ with $G \times S$, $\Pi(G, S, \phi) = \Pi(S, G, \phi^*)$ where $g \in \phi^*(s)$ if and only if $s \in \phi(g)$. Since $1 \in \phi^*(s)$ implies $s \leqq 1\theta = a$, which is idempotent, ϕ^* is unitary. Hence, by Proposition 3.2, $\Pi(G, S, \phi)$ is an E-unitary cover of S through G.

Proposition 4.2 is analogous to Proposition 3.2. The next proposition is similar to Proposition 3.8; it shows that every E-unitary cover of S through G is determined by a $\wedge$-prehomomorphism of G into a semigroup $C(S)$ depending only on S.

DEFINITION 4.3 **[9]**. Let S be an inverse semigroup. Then a nonempty subset H of S is called *permissible* if

(i) $a \in H$, $b \leqq a$ implies $b \in H$;

(ii) $a, b \in H$ implies ab^{-1}, $a^{-1}b$ idempotent.

Schein **[9]** shows that the set $C(S)$ of permissible subsets of S forms an inverse semigroup under subset multiplication. Further S can be embedded in $C(S)$ by means of the homomorphism η given by

$$a\eta = \{x \in S : x \leqq a\}$$

for each $a \in S$.

PROPOSITION 4.4. *Let S be an inverse semigroup and let G be a group. Suppose that ϕ is a surjective subhomomorphism of G into S such that $\Pi(G, S, \phi)$ is an E-unitary cover of S through G. Then $\phi(g)$ is permissible for each $g \in G$ and θ defined by*

$$g\theta = \phi(g) \quad \text{considered as a member of } C(S)$$

is a $\wedge$-prehomomorphism of G into $C(S)$. Further

$$\Pi(G, S, \phi) = \{(g, s) \in G \times S : s \leqq g\theta\};$$

here we identify S with $S\eta$.

Proof. Suppose $a \in \phi(g)$, $b \leqq a$; thus $b = ea$ for some $e^2 = e \in S$. Then $(g, a) \in P = \Pi(G, S, \phi)$ and $(1, e) \in P$ so that $(1, e)(g, a) = (g, b) \in P$. Hence $b \in \phi(g)$. Next, suppose a, $c \in \phi(g)$ then (g, a), $(g, c) \in P$ so that $(1, a^{-1}c) \in P$. Since P is an E-unitary cover of S through G, with $\pi_G \circ \pi_G^{-1} = \sigma$, where π_G denotes the projection of P onto G, this implies that $a^{-1}c$ is an idempotent. Similarly ac^{-1} is an idempotent. Hence $\phi(g)$ is permissible.

It is now easy to show that θ is a $\wedge$-prehomomorphism and, because $X \leqq Y$ in $C(S)$ if and only if $X \subseteq Y$, that $P = \{(g, s) \in G \times S : s \leqq g\theta\}$.

If we combine the results of Propositions 4.2 and 4.4, then we obtain the following dual to Theorem 3.9.

THEOREM 4.5. *Let S be an inverse semigroup and let G be a group. Let θ be a $\wedge$-prehomomorphism of G into $C(S)$ such that, for each $s \in S$ there exists $g \in G$ with $s \leqq g\theta$. Then*

$$\{(g, s) \in G \times S: s \leqq g\theta\}$$

is an E-unitary cover of S through G. Conversely, each E-unitary cover of S through G has this form for some $\wedge$-prehomomorphism θ of G into $C(S)$.

5. Examples

5.1. *Free group covers.* Let S be an inverse semigroup and let X be a set of generators for S as an inverse semigroup. Let X^{-1} be a set in one to one correspondence with, but disjoint from, X. Then there is a homomorphism θ from the free semigroup $F_{X \cup X^{-1}}$, on $X \cup X^{-1}$, onto S such that $x\theta^{-1} = x^{-1}\theta$ for each $x \in X$. Similarly, there is a homomorphism $\psi: F_{X \cup X^{-1}} \to FG_X$, the free group on X such that $x\psi^{-1} = x^{-1}\psi$ for each $x \in X$. Define $\phi: S \to 2^{FG_X}$ by

$$w \in \phi(s) \quad \text{if and only if} \quad w = u\psi \quad \text{for some} \quad u \in F_{X \cup X^{-1}}$$

with $u\theta = s$; that is $\phi(s) = s\theta^{-1}\psi$ for some $s \in S$.

PROPOSITION 5.1. *ϕ is a surjective unitary subhomomorphism of S into FG_X.*

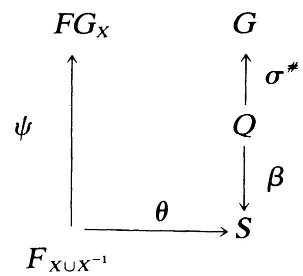
Proof. It is straightforward to show that ϕ is a surjective subhomomorphism of S into FG_X. Suppose that $1 \in \phi(s)$. Then there exists $w \in F_{X \cup X^{-1}}$ such that $w\theta = s$, $w\psi = 1$.

Let η be the canonical homomorphism from $F_{X \cup X^{-1}}$ into the free inverse semigroup FI_X on S. Then both θ and ψ can be factored through η. Since $w\psi = 1$ and FI_X is E-unitary [7] it follows that w, regarded as an element of FI_X, is idempotent. Hence $s = w\theta$ is an idempotent of S. This shows that ϕ is unitary.

As a result of the freeness of FG_X, $\Pi(S, FG_X, \phi)$, with ϕ as above, has a weak universal property.

PROPOSITION 5.2. *Let S be an inverse semigroup and let $P = \Pi(S, FG_X, \phi)$ as above with α the homomorphism $P \to S$. Suppose that Q is an E-unitary cover of S through G with homomorphism $\beta: Q \to S$. Then there is a homomorphism $\gamma: P \to Q$ such that $\alpha = \gamma\beta$.*

Proof. With the notation above, we have the following diagram of maps



For each $x \in X$, choose $y \in Q$ such that $y\beta = x\theta$. Then there is a homomorphism $\nu\colon F_{X\cup X^{-1}} \to Q$ such that $x\nu\beta = x\theta$ and $(x\nu)^{-1} = x^{-1}\nu$ for each $x \in X$. Then $\nu\sigma^{\natural}$ is a homomorphism of $F_{X\cup X^{-1}}$ into G and can be factored through ψ. That is, there is a homomorphism $\delta\colon FG_X \to G$ such that $\nu\sigma^{\natural} = \psi\delta$.

From the definition of P, $P = \{(w\theta, w\psi)\colon w \in F_{X\cup X^{-1}}\}$. Define $\gamma\colon P \to Q$ by $(w\theta, w\psi)\gamma = w\nu$. Then $w\theta = u\theta$, $w\psi = u\psi$ implies $w\psi\delta = u\psi\delta$, that is $w\nu\sigma^{\natural} = u\nu\sigma^{\natural}$ and $w\nu\beta = u\nu\beta$. Since Q is an E-unitary cover of S through G, Corollary 1.8 shows that $u\nu = w\nu$. Hence γ is well defined; it is clearly a homomorphism. Further, from the definition,

$$(w\theta, w\psi)\gamma\beta = w\nu\beta = w\theta = (w\theta, w\psi)\alpha,$$

for each $(w\theta, w\psi) \in P$. Hence $\alpha = \gamma\beta$.

5.2. *The Preston–Vagner cover.* Let $\rho\colon S \to \mathcal{I}_S$ be the Preston–Vagner representation of an inverse semigroup S and let $Y = S$ if S is finite, if not $Y = S \cup S'$ with $S \cap S' = \square$, $|S| = |S'|$. Then $F = \{\alpha \in \mathcal{I}_Y\colon \alpha \leqq \gamma$ for some permutation γ of $Y\}$ is a factorizable inverse semigroup containing $S\rho$. It gives rise to the subhomomorphism ϕ where, for each $s \in S$,

$$\begin{aligned}\phi(s) &= \{\alpha\colon \rho_s \leqq \alpha,\ \alpha \text{ a permutation of } Y\}\\ &= \{\alpha\colon x\alpha = xs \text{ for each } x \in Sss^{-1},\ \alpha \text{ a permutation of } Y\}.\end{aligned}$$

This subhomomorphism gives an E-unitary cover of S through K where

$$K = \{\alpha \in S_Y\colon (xe)\alpha = x(e\alpha) \text{ for all } x \in S \text{ and some } e^2 = e \in S\},$$

where S_Y denotes the symmetric group on Y.

5.3. *E-unitary covers of bisimple inverse semigroups.* A construction is given in **[6]** for the v-prehomomorphisms θ of a bisimple inverse semigroup S into an inverse semigroup T. When applied to the semigroup $\mathscr{K}(G)$ of cosets of a group G, this construction specializes to give the following construction for the E-unitary covers of S through G.

Let H be a subgroup of G and let $S(H)=\{a \in G: aHa^{-1} \subseteq H\}$. Then $S(H)$ is a subsemigroup of G and $(G/H, S(H)/H)$ is a partial semigroup under the multiplication $*$:

$$X * Y = XY \quad \text{for each} \quad X \in S(H)/H, \quad Y \in G/H.$$

Pick an idempotent $e \in S$ and set $R_e = \{a \in S: aa^{-1} = e\}$, $P_e = R_e \cap eSe$ and let $\theta: R_e \to G/H$ be a one-to-one mapping such that the following hold

(i) $a\theta \in S(H)/H$ if $a \in P_e$

(ii) $a\theta b\theta = (ab)\theta$ for $a \in P_e$, $b \in R_e$

(iii) $G = \cup\{a\theta^{-1}b\theta: a, b \in R_e\}$.

Then $\{(s, g) \in S \times G: g \in a\theta^{-1}b\theta$ where $s = a^{-1}b\}$ is an E-unitary cover of S through G. Conversely, each such has this form for some $\theta: R_e \to G/H$ as above.

5.4. *E-unitary covers for semilattices of groups.* A construction is given in **[6]** for the v-prehomomorphisms θ of a semilattice of groups S into an inverse semigroup T. When applied to the semigroup $\mathscr{K}(G)$ of cosets of a group G, this construction specializes to give the following description of the E-unitary covers of S through G.

Let E be a semilattice and let θ be an anti-isotone mapping of E into the lattice of subgroups of G. For each $e \in E$ set $G_e = e\theta$ and $C_e = \{a \in G: aG_f a^{-1} = G_f$ for each $f \leqq e\}$. Then G_e is a normal subgroup of C_e and the groups $K_e = C_e/G_e$ form a semilattice of groups $SL(E, \theta, \mathscr{K}(G))$ with linking homomorphisms $\phi_{e,f}: K_e \to K_f$ given by

$$X\phi_{e,f} = G_f X \quad \text{for each} \quad X \in K_e, \quad e \geqq f.$$

Suppose that S is a semilattice of groups with semilattice of idempotents E. Suppose that θ is an anti-isotone mapping of E into the lattice of subgroups of G and let ϕ be an idempotent determined homomorphism of S into $SL(E, \theta, \mathscr{K}(G))$ such that $G = \cup\{a\phi: a \in S\}$. Then

$$\{(s, g) \in S \times G: g \in s\phi\}$$

is an E-unitary cover of S through G. Conversely, each such has this form for some $\theta: E \to \mathscr{K}(G)$ and $\phi: S \to SL(E, \theta, \mathscr{K}(G))$.

References

1. S. Y. Chen and S. C. Hsieh, *Factorizable inverse semigroups*, Semigroup Forum, **8** (1974), 283–297.

2. A. H. Clifford and G. B. Preston, *Algebraic theory of semigroups*, Vols. 1 and 2. Math. Surveys of the American Math. Soc. (Providence, R. I., Vol. **1** (1961), Vol. **2** (1967)).

3. P. Dubreil, *Contribution à la théorie des demi-groupes*, Mèm. Acad. Sci. Inst. France, **63** (1941), 1–52.

4. D. B. McAlister, *Groups, semilattices and inverse semigroups*, Trans. Amer. Math. Soc., **192** (1974), 227–244.

5. ———, *Groups, semilattices and inverse semigroups, II*, Trans. Amer. Math. Soc., **196** (1974), 351–370.

6. ———, *v-Prehomomorphisms on inverse semigroups*.

7. H. E. Scheiblich, *Free inverse semigroups*, Proc. Amer. Math. Soc., **38** (1973), 1–7.

8. B. M. Schein, *Semigroups of strong subsets* (Russian), Volžskiĭ Matem. Sbornik, Kuĭbyšev, **4** (1966), 180–186.

9. ———, *Completions, translational hulls and ideal extensions of inverse semigroups*, Czechoslovak Math. J., **23** (1973), 575–609.

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