MOMENTS OF MEASURES ON CONVEX BODIES

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In this paper the problem of moments is viewed as one of identifying a class of functions on a semigroup with a class of measures. We present integral representation theorems for linear functionals on algebras (Theorems 2.1 and 2.2) which enable us to solve moment problems for a wide class of compact sets. In particular if K is any compact convex subset of R^3 with nonvoid interior, necessary and sufficient conditions are given for a triple indexed sequence $f(n_1, n_2, n_3)$ to admit an integral representation of the form $f(n_1, n_2, n_3) = \int_K t_1^{n_1} t_2^{n_2} t_3^{n_3} d\mu(t)$ ($t = (t_1, t_2, t_3)$). Here, of course the semigroup S considered is all triples of nonnegative integers under coordinate addition. As in the case of Hausdorff's "little moment problems" the solution depends on certain linear combinations of shift operators.

We consider the three coordinate shift operators E_1, E_2, E_3 defined on the functions f on S, e.g. $(E_2 f)(n_1, n_2, n_3) = f(n_1, n_2 + 1, n_3)$. If K is the octahedron $\{t \in R^3 \mid 1 \pm t_1 \pm t_2 \pm t_3 \geq 0\}$. Then f admits a necessarily unique, nonnegative representing measure if and only if $(\Delta f)(0, 0, 0) \geq 0$, where Δ is any product of difference operators of the form $I \pm E_1 \pm E_2 \pm E_3$. The octahedron example is typical of all bounded nondegenerate convex polyhedra in that the difference operators used to describe those functions which admit representing measures are defined in terms of the facial functionals of the polyhedron K. Necessary and sufficient conditions for the existence of representing signed measures are also given in §3.

In §4 we apply Theorems 2.1 and 2.2 to algebras of shift operators on arbitrary commutative semigroups. This leads to a more general notion of functions of Bounded Variation than has previously been introduced on semigroups cf. [6, 11, 12 and 14]. The classical notion of positive definite function on a group is recast in terms of finite difference. We conclude by giving necessary and sufficient conditions for a linear functional on a commutative B^*-algebra with identity to be in the span of the positive linear functional.

2. Positive and BV-functionals on algebras with involution.

Let $\mathscr{A}$ be a real or complex commutative algebra with identity 1 and involution $*$. If $\mathscr{A}$ is real we assume $x^* = x$ for all $x \in \mathscr{A}$. Let $\mathscr{T}$ be a subset of $\mathscr{A}$ such that:

(i) $\mathcal{T}$ is hermitian ($x^* = x$ for all $x \in \mathcal{T}$)

(ii) $1 - \mathcal{T} \subset \mathrm{Alg\,span}^+ \mathcal{T}$ (For each $x \in \mathcal{T}$ there exist products $x_1, \cdots, x_k$ of members of $\mathcal{T}$ and positive scalars $\alpha_1, \cdots, \alpha_k$ such that $x + \Sigma_{i=1}^k \alpha_i x_i = 1$)

(iii) $\mathrm{Alg\,span}\,\mathcal{T} = \mathcal{A}$ (For each $x \in \mathcal{A}$ there exist scalars $\alpha_1, \cdots, \alpha_k$ and products $x_1, \cdots, x_k$ of members of $\mathcal{T}$ such that $x = \Sigma_{i=1}^k \alpha_i x_i$).

If we set $\mathrm{Alg\,span}^+ \mathcal{T} \equiv P$ then the real span, $\mathcal{A}_\mathbf{R}$, of P is $P - P = \{x - y \,|\, x, y \in P\}$ and the complex span, $\mathcal{A}_\mathbf{C}$, of P is $\mathcal{A}_\mathbf{R} + i\mathcal{A}_\mathbf{R} \equiv \{x + iy \,|\, x, y \in \mathcal{A}_\mathbf{R}\}$. It follows that $\mathcal{A}_\mathbf{R} = \{x \in \mathcal{A}_\mathbf{C} \,|\, x = x^*\}$. Every real, positive homogeneous, additive functional on P admits a unique linear extension to $\mathcal{A}$, this extension necessarily satisfies $f^*(x) = f(x^*)$ for all $x \in \mathcal{A}_\mathbf{C}$, (we are using $*$ to denote both involution and conjugation). If f is multiplicative on P then so is its extension which we will also denote by f. A linear functional f on $\mathcal{A}$ will be called $\mathcal{T}$-positive (or just positive) if $f(x) \geqq 0$ for all $x \in P$. Let P' denote the set of all $\mathcal{T}$-positive linear functionals on $\mathcal{A}$. Our main concern is one of characterizing P'; the real span, $P' - P'$, of P; and the complex span, $(P' - P') + i(P' - P')$. Throughout we will equip $\mathrm{Span}(P')$ with the weak $*$ topology.

PROPOSITION 2.1.1. $1 - \Pi_{j=1}^k x_j \in P$ whenever $x_1, \cdots, x_k \subset \mathcal{T}$.

Proof. The assertion follows from the algebraic identity:

$$\sum_{\sigma \neq 1} \prod_{j=1}^k x_j^{\sigma_j}(1 - x_j)^{1-\sigma_j} = 1 - \prod_{j=1}^k x_j$$

where the left summation is taken over all 0–1 valued functions $\sigma(\cdot)$ defined on the first k natural numbers such that $\sigma \neq 1$. That this identity is valid, follows from induction on k since

$$\sum_{\sigma \neq 1} \prod_{j=1}^{k+1} x_j^{\sigma_j}(1 - x_j)^{1-\sigma_j}$$

$$= x_{k+1} \sum_{\sigma \neq 1} \prod_{j=1}^k x_j^{\sigma_j}(1 - x_j)^{1-\sigma_j}$$

$$+ (1 - x_{k+1}) \sum_{\sigma \neq 1} \prod_{j=1}^k x_j^{\sigma_j}(1 - x_j)^{1-\sigma_j} + \prod_{j=1}^k x_j(1 - x_{k+1})$$

$$= (\text{inductive hypothesis}) \ \left(1 - \prod_{j=1}^k x_j\right) + \prod_{j=1}^k x_j(1 - x_{k+1}) = 1 - \prod_{j=1}^{k+1} x_j.$$

For each $x \in \mathcal{A}$, define the evaluation function $\hat{x}$ on $\mathrm{Span}(P')$ into the scalar field by $\hat{x}(f) = f(x)$. The above proposition implies $\hat{1}$ is strictly positive on the cone P'. For if $f \in P'$ and $f \neq 0$ then there exist

$x_1, \cdots, x_k \in \mathcal{T}$ such that $f(\pi_j x_j) > 0$ so that $f(1 - \pi_j x_j) \geqq 0$ implies $\hat{1}(f) = f(1) \geqq f(\pi_j x_j) > 0$. It follows that $B = \{f \in P' \mid f(1) = 1\}$ is a convex base for P'.

PROPOSITION 2.1.2.   *The base B is compact.*

Proof.   Let F denote the set of all products of members of $\mathcal{T}$ and set $[0, 1]_x = [0, 1]$. Then the restriction $f|_F$ is a distinct member of the product space $\Pi_{x \in F}[0, 1]_x$ for each $f \in B$. Thus Tychonoff's theorem establishes the assertion since $B' = \{f|_F : f \in B\}$ is all functions f on F such that $f(1) = 1$ and $\Sigma\, \alpha_i x_i = \Sigma\, \beta_i x_i$ $(x_i \in F)$ implies $\Sigma\, \alpha_i f(x_i) = \Sigma\, \beta_i f(x_i)$ so that B' is closed relative to the topology of simple convergence.

If we let Γ denote the class of all $\mathcal{T}$-positive multiplicative functionals on $\mathcal{A}$ then Γ is w^*-compact. Note that $\hat{x}\hat{y}|_\Gamma = (xy)^\wedge|_\Gamma$ and $\hat{x}|_\Gamma + \hat{y}|_\Gamma = (x + y)^\wedge|_\Gamma$.

THEOREM 2.1.   *A linear functional f on $\mathcal{A}$ is $\mathcal{T}$-positive if and only if there exists* (a necessarily unique) *regular Borel measure μ_f on Γ such that*

$$f(x) = \int_\Gamma \hat{x}(\chi)d\mu(\chi).$$

Proof.   The existence of μ_f follows from Choquet's theorem c.f. **[4** or **15]**, once we've shown that the extreme points of B are contained in Γ. For this we define the translate f_x of a linear functional f by $f_x(y) = f(xy)$ $(x, y \in \mathcal{A})$ then f_x is linear and $\mathcal{T}$-positive if f is $\mathcal{T}$-positive and $x \in P$. Since $f = f_x + f_{1-x}$ for $x \in \mathcal{T}$ we have $\lambda f = f_x$ whenever f is an extreme point. Evaluation at 1 shows $\lambda = f(x)$ so that $f(x)f(y) = f(xy)$ for all $y \in \mathcal{A}$ and $x \in \mathcal{T}$. It follows that f is multiplicative so that every extreme point of B is multiplicative. See **[1]** for a similar argument. Uniqueness follows from the Stone–Weierstrass theorem since $\{\hat{x}|_\Gamma : x \in \mathcal{A}\}$ is a point separating *-subalgebra of the continuous functions $\mathscr{C}(\Gamma)$ on Γ which contains constants.

The argument used in **[5]** shows that every member of Γ is in fact extreme so that B is a Bauer Simplex, i.e. a simplex whose set of extreme points is closed. Since the map $f \to \mu_f$ of P onto the nonnegative measures is linear it is natural to ask what linear functionals admit a representing measure with respect to real and complex measures? I.e., how can $P' - P'$ and $(P' - P') + i(P' - P')$ be characterized? The answer to this question relies on the concept of BV-function and the proofs which we will use depend heavily on compactness arguments with certain partitions of unity. We begin with some preliminary definitions.

A *cycle* is a finite subset $\{x_1, \cdots, x_k\}$ of P such that $x_1, \cdots, x_{k-1} \in \mathcal{T}$ and $\Sigma_{i=1}^k x_i = 1$. Let $\mathscr{P}$ be a collection of cycles such that each $x \in \mathcal{T}$ is

in at least one cycle of $\mathscr{P}$.   An example of such a $\mathscr{P}$ is $\{x, 1 - x \mid x \in \mathscr{T}\}$.   The cycle $\{x, 1 - x\}$ will be called a *simple cycle*.   The motivation for considering simple cycles comes from considering the algebra $\mathscr{A}$ of shift operators on the additive semigroup S of nonnegative integers in light of Hausdorff's solution to the "little moment problem" [cf. 16].   These cycles correspond to the collection $\{\{t^k (1 - t)^{n-k}\}_{k \leq n}\}_n$ of partitions of unity on $\Gamma \equiv [0, 1]$ and this collection is rich enough to adequately describe the variation of the Borel measures on Γ.   When dealing with more general semigroups we must consider products of these cycles to obtain rich enough partitions.   In this generality $\Gamma = \exp(S)$ [cf. 11].   In trying to describe the variation of measures supported by compact subsets of the semicharacters of S other than $\exp S$, it becomes necessary to consider other than simple cycles.   This becomes particularly important for many of the applications given later and we feel justifies the additional complexity.   With each finite subcollection $\{\{x_{ij}\}_i\}_j$ of $\mathscr{P}$, we associate the partition of unity of all distinct products which contain one factor from each cycle.   When the cycles in the subcollection agree for each $j = 1, 2, \cdots, n$ we get the *multinomial partition*

$$\left\{ \binom{n}{i_1, \cdots, i_m} (x_1^{i_1} \cdots x_m^{i_m}) \,\middle|\, \sum_{l=1}^{m} i_l = n \right\}.$$

In general we will be considering products of multinomial partitions of cycles in $\mathscr{P}$, i.e. partitions of the form

$$\left\{ \binom{n_1}{i_{11}, \cdots} \cdots \binom{n_k}{i_{1k}, \cdots} \prod_{l,j} x_{lj}^{i_{lj}} \,\middle|\, \sum_l i_{lj} = n_j \right\}$$

where $\{x_{lj}\}_l$ is a cycle for each $j = 1, \cdots, k$.   If we set the expression under the bracket equal to p then the functions $\hat{p}$ (restricted to Γ) form a partition of unity in the space of continuous functions $\mathscr{C}(\Gamma)$.   We will consider these products of multinomial partitions of unity in $\mathscr{C}(\Gamma)$ extensively later and without further reference.   For each finite subset $\mathscr{P}_0 = \{\{x_{ij}\}\}_{j=1,2,\cdots,k}$ of $\mathscr{P}$ and each set of integers $\{n_1, \cdots, n_k\}$ define

$$\|f\|_{(\mathscr{P}_0, n_j)} = \sum \binom{n_1}{i_{11}, \cdots} \cdots \binom{n_k}{i_{1k}, \cdots} \left| f\left(\prod_{l,j} x_{lj}^{i_{lj}} \right) \right|,$$

where summation is taken with respect to all choices of i_{lj} such that $\sum_l i_{lj} = n_j$ for each $j = 1, 2, \cdots, k$.
Note that if $\mathscr{P}_0$ is a collection of simple cycles $\{x_j, 1 - x_j\}_{j=1,2,\cdots,k}$ then

$$\|f\|_{\mathscr{P}_0} = \sum \binom{n_1}{i_1} \cdots \binom{n_k}{i_k} |f[\Pi_j \, x_j^{i_j} (1 - x_j)^{n_j - i_j}]|.$$

Finally we define the *total variation* of f by

$$\|f\| = \sup_{(\mathscr{P}_0, n_j)} \|f\|_{(\mathscr{P}_0, n_j)}$$

and $(\mathscr{P}_0, n_j) \leqq (\mathscr{P}'_0, n'_j)$ if $\mathscr{P}_0 \subset \mathscr{P}'_0$ and $n_j \leqq n'_j$ for corresponding j's. The class of all functionals of finite variation are called $\mathscr{T}$-BV (or just BV) and it is this class of functionals which we will show admit integral representations. Our main theorem is as follows.

THEOREM 2.2. *The BV-functionals form the linear span of the positive linear functionals. These are precisely the linear functionals f which admit a* (necessarily unique) *representing measure μ_f. If f is* BV *then $\|\mu_f\| = \|f\|$ and μ_f is real if and only if $f(x^*) = f^*(x)$.*

The proof is broken down into a series of lemmas. Lemmas 2.2.1 through 2.2.3 prove that if f admits a representing measure then f is BV.

LEMMA 2.2.1. *Let $\varepsilon > 0$ be given and K_1 and K_2 be disjoint compact subsets of Γ. There exists a product Λ of multinomial partition of cycles in $\mathscr{P}$ with subpartition Λ_0 such that*

$$\sum_{p \in \Lambda_0} \left| \int \hat{p}\,du \right| \begin{cases} \geqq 1 - \epsilon & \text{on } K_1 \\ \\ < \epsilon & \text{on } K_2. \end{cases}$$

Proof. Let χ_1 and χ_2 be distinct members of Γ. Then there exists $x_1 \in \mathscr{T}$ such that $\chi_1(x_1) \neq \chi_2(x_1)$. These two real numbers can be separated by disjoint open subsets G_1 and G_2 in $[0,1]$ with $\chi_j(x_1) \in G_j$ for $j = 1, 2$. The theory of Bernstein polynomials cf. [8, page 6] implies that for all sufficiently large n the partition of unity $\left\{\binom{n}{i} t^i (1-t)^{n-i}\right\}$ $(0 \leqq t \leqq 1)$ admits a subpartition which is arbitrarily close to zero on G_2 and arbitrarily close to unity on G_1. Inverting, we have the existence of a subpartition of the partition of unity $\left\{\binom{n}{i} \hat{x}_1^i (1 - \hat{x}_1)^{n-i}\right\}$ which is arbitrarily close to zero on the preimage $\hat{x}_1^-(G_2)$ and arbitrarily close to unity on $\hat{x}_1^-(G_1)$. But if $\{x_i\}_{i=1,2,\cdots,k}$ is a cycle in $\mathscr{P}$ which contains x_1, then

$$\sum \binom{n}{i_1, i_2, \cdots, i_k} x_1^{i_1} \cdots x_k^{i_k}$$

$$= \sum_{i_1=0}^{n} \binom{n}{i_1} x_1^{i_1} \sum \binom{n-i_1}{i_2 \cdots i_k} x_2^{i_2} \cdots x_k^{i_k} = \sum_{i=0}^{n} \binom{n}{i} x_1^i (1 - x_1)^{n-i}.$$

Thus the partition of unity

$$\binom{n}{i_1, \cdots, i_k} \hat{x}_1^{i_1} \cdots \hat{x}_k^{i_k}$$

admits a subpartition which separates the disjoint open sets $\hat{x}_1^{-}(G_1)$ and $\hat{x}_1^{-}(G_2)$ which contain the singleton (compact) sets $\{\chi_1\}$ and $\{\chi_2\}$ as in Lemma 3.1 of [10]. The compactness argument used there implies the present lemma.

LEMMA 2.2.2.   *Let μ be a bounded complex valued measure, f and g integrable functions and E a measurable set.   If $1 \geq f \geq 1 - \epsilon$ and $0 \leq g \leq \epsilon$ on E then*

(i) $\left| \int_E f du \right| \geq (1 - 4\epsilon) |\mu(E)|$ *and*   (ii) $\left| \int_E g du \right| \leq 4\epsilon |\mu(E)|$.

Proof.   (i) $|\mu(E)| - \left| \int_E f du \right| \leq \left| \mu(E) - \int_E f du \right|$

$$\leq \int_E (1 - f) d|\mu| \leq \epsilon |\mu|(E) \leq 4\epsilon |\mu(E)|.$$

Similarly (ii) $\left| \int_E g du \right| \leq \int_E g d\mu| \leq \int_E \epsilon d|\mu| \leq 4\epsilon |\mu(E)|.$

LEMMA 2.2.3.   *If μ is a complex regular Borel measure on Γ then the variation is given by*

$$\|\mu\| = \sup_\Lambda \sum_{p \in \Lambda} \left| \int \hat{p} d\mu \right|$$

where Λ is an arbitrary product of multinomial partitions of cycles in $\mathscr{P}$.

Proof.   Choose ϵ such that $0 < \epsilon < 1$.   From regularity, there exists disjoint compact sets $\{K_i \mid i = 1, 2, \cdots, k\}$ such that

(i) $\|\mu\| - \epsilon/3 \leq \sum_{i=1}^{k} |\mu(K_i)|.$

From Lemma 2.2.1, there exists a collection $\{\Lambda^i\}_i$ of products of multinomial partitions with subpartitions $\{\Lambda_0^i\}_i$ such that

(ii) $\displaystyle\sum_{p \in \Lambda_0^i} \hat{p}(x) \begin{cases} \geq \sqrt[k]{1 - \epsilon/12} & \text{on } K_i \\ \\ < 1 - \sqrt[k]{1 - \epsilon/12k} & \text{on } \bigcup_{j \neq i} K_j. \end{cases}$

Let $\Lambda = \Pi_i \Lambda^i = \{\boldsymbol{p}_1, \boldsymbol{p}_2 \cdots \boldsymbol{p}_k \,|\, \boldsymbol{p}_i \in \Lambda^i\}$. Then Λ is a product of multinomial partitions with subpartitions

$$\Lambda^i_{\times} = \Lambda^i_0 \prod_{j \neq i} (\Lambda^j \backslash \Lambda^j_0)$$

Since $\sqrt[k]{1 - \epsilon/12k} \geqq \sqrt[k]{1 - \epsilon/12}$, Lemma 2.2.2 implies

$$\left| \int_{K_i} \sum_{p \in \Lambda^i_{\times}} \hat{p} \, d\mu \right| \geqq \left(1 - \frac{\epsilon}{3} \right) |\mu(K_i)|.$$

Similarly, since $1 - \sqrt[k]{1 - \epsilon/12k} < \epsilon/12k$ we have

$$\left| \int_{\substack{\cup K_j \\ j \neq i}} \sum_{p \in \Lambda^i_{\times}} p \, d\mu \right| \leqq \frac{\epsilon}{3k} \left| \mu\left(\bigcup_{j \neq i} K_j \right) \right| < \frac{\epsilon}{3k} \|\mu\|.$$

Therefore

$$\sum_{p \in \Lambda} \left| \int \hat{p} \, d\mu \right| \geqq \sum_i \sum_{p \in \Lambda^i_{\times}} \left| \int \hat{p} \, d\mu \right|$$

$$\geqq \sum_i \sum_{p \in \Lambda^i_{\times}} \left[\left| \int_{K_i} \hat{p} \, d\mu \right| - \left| \int_{\substack{\cup K_j \\ j \neq i}} p \, d\mu \right| - \left| \int_{\substack{\Gamma \backslash \cup K_j \\ j \neq i}} \hat{p} \, d\mu \right| \right]$$

$$\geqq \sum_i \left(1 - \frac{\epsilon}{3} \right) |\mu(K_i)| - \sum_i \frac{\epsilon}{3k} \|\mu\| - \sum_{p \in \Lambda} \left| \int \hat{p} \, d\mu \right|$$

$$\geqq \left(1 - \frac{\epsilon}{3} \right) \sum |\mu(K_i)| - \frac{\epsilon}{3} \|\mu\| - \sum_{p \in \Lambda} \int \hat{p} \, d|\mu|$$

$$\geqq \text{(by i)} \quad \left(1 - \frac{\epsilon}{3} \right) \left[\|\mu\| - \frac{\epsilon}{3} \right] - \frac{\epsilon}{3} \|\mu\| - |\mu|(\Gamma \backslash \cup K_j).$$

Since $|\mu|(\Gamma) - \epsilon/3 \leqq \Sigma |\mu|(K_i)$, we have $|\mu|(\Gamma \backslash \cup K_j) < \epsilon/3$ so that

$$\sum_{p \in \Lambda} \left| \int p \, d\mu \right| \geqq \left(1 - \frac{2}{3}\epsilon \right) \|\mu\| - \frac{2}{3}\epsilon \quad \text{or} \quad \|\mu\| \leqq \sup_{\Lambda} \sum_{\Lambda} \left| \int \hat{p} \, d\mu \right|.$$

The assertion follows since

$$\sum_{\Lambda} \left| \int \hat{p} \, d\mu \right| \leqq \sum_{\Lambda} \int \hat{p} \, d|\mu| = \|\mu\|.$$

REMARK 2.2.4. If f admits a representing measure μ_f then $\|\mu_f\| = \|f\|$.

Proof. Let $\mathscr{P}_0 = \{\{x_{ij}\}_j\}_i$ be a collection of cycles in $\mathscr{P}$. A typical member of a product Λ of multinomial partitions of $\mathscr{P}$ of the form:

$$p = \binom{n_1}{i_{11}, \ldots} \cdots \binom{n_k}{i_{1k}, \ldots} \Pi_{i,j} x_{ij}^{i_{ij}}$$

so that

$$\|f\|_{(\mathscr{P}_0, n_j)} = \sum_{p \in \Lambda} |f(p)| = \sum_{p \in \Lambda} \left| \int \hat{p} \, d\mu_f \right|$$

Lemma 2.2.3 implies the remark.

In order to prove that every BV-functional admits a representing measure we will define the *variation* of a BV-functional f by $|f|(y) = \|f_y\|$ for all $y \in P$. If $f \in P' - P'$ then $\mu_{f_y} = \hat{y} d\mu_f$ since $f_y(x) = f(yx) = \int yx \, d\mu_f$. Hence $|f|(y) = \|f_y\| = \|\mu_{f_y}\| = \|\hat{y} d\mu_f\|$ and since $\hat{y}$ is nonnegative on Γ, $\|\hat{y} d\mu_f\| = \int \hat{y} d|\mu_f|$. Therefore $|f|(y) = \int \hat{y} d|\mu_f|$ for all $y \in P$. Once we show $|f|(\cdot)$ is finite, additive and positive homogeneous on P we can linearly extend $|f|$ to $\mathscr{A}_\mathbf{R}$ whenever f is real on $\mathscr{A}_\mathbf{R}$. It will be shown that $f = 1/2(|f| \pm f)$ is $\mathscr{T}$-positive so that $f = f^+ - f^- \in P' - P'$ whenever f is real on $\mathscr{A}_\mathbf{R}$ and BV. The complex case easily follows. We will need to verify some preliminary properties of the total variation first. We begin with the following elementary consequence of the triangle inequality and positive homogeneity of the absolute value.

LEMMA 2.2.5.
(i) $\|f + g\|_{\mathscr{P}_0} \leqq \|f\|_{\mathscr{P}_0} + \|g\|_{\mathscr{P}_0}$
(ii) $\|\alpha f\|_{\mathscr{P}_0} = |\alpha| \cdot \|f\|_{\mathscr{P}_0}$

Thus $\|\cdot\|$ is a positive, homogeneous, subadditive function on the BV-linear functionals.

Let $\{\{x_{lj}\}_l\}_j$ be a finite subset of cycles. Then since $(x_{11} + \cdots)^{n_1}(x_{12} + \cdots)^{n_2} \cdots (x_{1l} + \cdots)^{n_l+1} \cdots (x_{1k} + \cdots)^{n_k} = (x_{11} + \cdots)^{n_1}(x_{12} + \cdots)^{n_2}(x_{1l} + \cdots)^{n_l} \cdots (x_{1k} + \cdots)^{n_k}(x_{1l} + \cdots)$,

$$\|f\|_{(\mathscr{P}_0; n_1, \cdots, n_l+1, \cdots, n_k)} = \|f\|_{(\mathscr{P}_1; n_1, \cdots, n_k, 1)}$$

where $\mathscr{P}_1$ is the union of $\mathscr{P}_0$ with the single cycle $\{x_{1l}, x_{2l}, \cdots\}$. By induction we can eliminate all repetitious cycles by increasing the order. In summary we present the following.

LEMMA 2.2.6. *Let $\mathscr{P}_0$ be a finite collection of cycles and $\mathscr{P}_0'$ be the*

collection of all pairwise distinct cycles of $\mathscr{P}_0$. Then $m_1, m_2, \cdots, m_l$ can be chosen so that

$$\|f\|_{(\mathscr{P}_0; n_1, \cdots, n_k)} = \|f\|_{(\mathscr{P}_0; m_1, \cdots, m_l)}.$$

LEMMA 2.2.7. Let $\mathscr{P}_1 = \{x_{1j}, x_{2j}, \cdots\}_{j=1,\cdots,m}$ be a set of m cycles and let $\mathscr{P}_0$ be the first $k - 1$ of them. Then

$$\|f\|_{(\mathscr{P}_1, n_j)} = \sum \binom{n_k}{i_{1k}, \cdots} \cdots \binom{n_m}{i_{1m}, \cdots} \|f_x\|_{(\mathscr{P}_0, n_j)} \text{ where } x = \prod_{j=k}^{m} (x_{1j}^{i_{1j}} \cdots).$$

Proof.

$$\|f\|_{(\mathscr{P}_1, n_j)} = \sum \binom{n_k}{i_{1k}, \cdots} \cdots \binom{n_m}{i_{1m}, \cdots} \sum \binom{n_1}{i_{11} \cdots} \cdots \binom{n_{k-1}}{i_{1k+1} \cdots} \left| f_x \left(\prod_{j=1}^{k} x_{1j}^{i_{1j}} \cdots \right) \right|$$

and the lemma follows.

If we take $m = k$ and $n_m = 1$ then the above lemma along with Lemma 2.2.5 shows $\|f\|_{(\mathscr{P}_1, n_j)} = \Sigma_l \|f_{xkl}\|_{(\mathscr{P}_0, n_j)} \geqq \|\Sigma_l f_{xkl}\| = \|f\|_{(\mathscr{P}_0, n_j)}$. Lemma 2.2.6 now implies the following.

LEMMA 2.2.8. $\|f\|_{(\mathscr{P}_0, n_j)}$ is a nondecreasing function of $\mathscr{P}_0$ and n_j. In particular

$$\|f\| = \lim_{(\mathscr{P}_0, n_j)} \|f\|_{(\mathscr{P}_0, n_j)}.$$

LEMMA 2.2.9. If x is a finite product of member of $\mathscr{T}$ and f is BV then f_x and f_{1-x} are BV and $\|f\| = \|f_x\| + \|f_{1-x}\|$.

Proof. Let $x = \Pi_{j=1}^k x_{1j}$ $(x_{1j} \in \mathscr{T})$ and $\mathscr{P}_0$ be a finite subset of $\mathscr{P}$. For each j, there exists a cycle $C_j = \{x_{1j}, x_{2j}, \cdots, x_{l_j j}\}$ contained in $\mathscr{P}$ which contains x_{1j}. Let $\mathscr{P}_1$ be the collection of cycles obtained by adjoining C_j, with multiplicity 1, to $\mathscr{P}_0$ for each j. Then from Lemma 2.2.7,

$$\|f\|_{\mathscr{P}_1} = \sum_{\eta(\cdot)} \|f_{y_\eta}\|_{\mathscr{P}_0}$$

where $\eta(j) \in \{1, 2, \cdots, l_j\}$ and $y_\eta = \Pi_j x_{\eta(j)j}$. Hence

$$\|f\| \geqq \sum_\eta \|f_{y_\eta}\|_{\mathscr{P}_0} \geqq \|f_x\|_{\mathscr{P}_0} + \sum_{\eta \neq 1} \|f_{y_\eta}\|_{\mathscr{P}_0} \rightarrow \|f_x\| + \|f_{1-x}\| \geqq \|f\|.$$

The assertion follows.

LEMMA 2.2.10. $|f|(\,\cdot\,)$ *is additive and positive homogeneous on P.*

Proof.   Let $x = \sum \alpha_i x_i$ $(\alpha_i \geq 0,\ x_i \in \mathcal{T}^k)$ be an arbitrary member of P.   Then

$$\left\|\sum_i \alpha_i f\right\| = \left\|\sum_i \alpha_i\ (f_{x_i} + f_{1-x_i})\right\| \leq \left\|\sum_i \alpha_i f_{x_i}\right\| + \left\|\sum_i \alpha_i f_{1-x_i}\right\|$$

$$\leq \sum \alpha_i \|f_{x_i}\| + \sum \alpha_i \|f_{1-x_i}\| = \sum_i \alpha_i \|f\| = \left\|\sum_i \alpha_i f\right\|.$$

A second application of the triangle inequality shows $\|\sum \alpha_i f_{x_i}\| = \sum \alpha_i \|f_{x_i}\|$ so that

$$|f|\left(\sum \alpha_i x_i\right) = \|f_{\sum \alpha_i x_i}\| = \left\|\sum \alpha_i f_{x_i}\right\| = \sum \alpha_i \|f_{x_i}\| = \sum \alpha_i |f|(x_i).$$

LEMMA 2.2.11.   *If f is a linear* BV-*functional on* $\mathcal{A}$ *such that* $f^*(x) = f(x^*)$ *for all* $x \in \mathcal{A}$ *then f is the difference of two positive functionals.*

Proof.   Note that f is real on P.   Since $|g(1)| \leq \|g\|$ for all g, we have $|f(x)| = |f_x(1)| \leq \|f_x\| = |f|(x)$ so that $|f|(x) \geq \pm f(x)$ is for all $x \in P$.   Thus $f^\pm = 1/2[|f| \pm f]$ is $\mathcal{T}$-positive whenever f is BV.   The assertion follows since $f = f^+ - f^-$.

Theorem 2.1 now implies that a BV-functional f admits a real representing measure if and only if $f^*(x) = f(x^*)$ for all $x \in \mathcal{A}$.   The proof of Theorem 2.2 will be complete once we've established the following lemma.

LEMMA 2.2.12.   *If* $\mathcal{A}$ *is complex then the set* BV($\mathcal{A}$) *of* BV-*functionals is* $(P' - P') + i(P' - P')$.

Proof.   For each BV-functional f, set $f_*(x) = f^*(x^*)$.   Then f_* is linear.   Also $f = f_1 + if_2$, where $f_1 = (f + f_*)/2$ and $f_2 = (f - f_*)/2i$.   Since $x = x^*$ for all $x \in \mathcal{T}$ we have $|f_*(\Pi x_j)| = |f(\Pi x_j)|$ for $x_j \in \mathcal{T}$.   Thus f_* and hence f_j $(j = 1, 2)$ are BV-functionals.   But since $f_j^*(x) = f_j(x^*)$, we have f in the complex span of P'.   That is BV($\mathcal{A}$) $\subset (P' - P') + i(P' - P')$.   Conversely let $f \in$ C-span(P).   Then $f = f_1 + if_2$ where f_j is BV and $f_j^*(x) = f_j(x^*)$ $(j = 1, 2)$.   Therefore f admits a complex representing measure μ_f whose real and imaginary parts are μ_{f_1} and μ_{f_2} respectively.   It now follows from Remark 2.2.4 that f is BV and the proof is complete.

COROLLARY TO THEOREM 2.2. *A linear functional f is* BV *if and only if*

$$\|f\|_0 = \lim_{(\mathcal{T}, n)} \sum_\sigma \binom{n}{i_1} \cdots \binom{n}{i_k} |f(\Pi x_i^{\sigma_i}(1 - x_i^{1-\sigma_i}))| < \infty,$$

where σ *is a* 0–1 *function on the first k natural numbers and the limit is taken over all finite subsets* $\{x_1, \cdots, x_k\}$ *of* $\mathcal{T}$ *and all integers n. Moreover* $\|f\|_0 = \|f\|$.

Proof. The collection $\mathcal{P} = \{x, 1 - x \mid x \in \mathcal{T}\}$ of simple cycles is an admissible collection of cycles as mentioned above.

3. Application to classical moment problems.

The classical moment problem is one of characterizing functions f which admit integral representations of the form

$$(*) \quad f(n_1, \cdots, n_\nu) = \int_K t_1^{n_1} \cdots t_\nu^{n_\nu} d\mu(t),$$

where K is a subset of ν-dimensional space $\mathbf{R}^\nu$. Restrictions are put on μ. Here we apply the results of §2 to solve this moment problem for compact convex sets K with nonvoid interior and with respect to both nonnegative and signed μ.

To put this in our setting let $\mathcal{A}(x)$ be the algebra of all polynomials in the ν-variables $x = x_1 \cdots x_\nu$. A subset $\mathcal{T}$ of polynomials, nonnegative on K such that

(i) $\mathcal{A}(x) = \text{Alg span } \mathcal{T}$
(ii) $1 - p \in \text{Alg span}^+(\mathcal{T})$ whenever $p \in \mathcal{T}$
(iii) $K = \bigcap_{p \in \mathcal{T}} \{t \in \mathbf{R}^\nu \mid p(t) \geq 0\}$

can be constructed.

3.1. Construction of $\mathcal{T}$.

Let t^0 be an interior point of K and $B(K) = \{p\text{-affine} \mid p(t) \geq 0 \text{ for all } t \in K \text{ and } p(t^0) = 1\}$.

LEMMA 3.1.1. $B(K)$ *is uniformly bounded.*

Proof. There exists $\epsilon > 0$ such that the Euclidean disk $S_\epsilon(t^0) = \{t \in \mathbf{R}^\nu \mid \|t - t^0\| < \epsilon\}$ is contained in K. If $p \in B(K)$ assumes its supremum on K at $t^1 (\sup p(K) = p(t^1))$ then t^1 is an extreme point of K and thus $t^1 \notin S_\epsilon(t^0)$. Also there exists $t^2 \notin S_\epsilon(t^0)$ such that $t^0 = \alpha t^1 + (1 - \alpha)t^2$ $(0 < \alpha < 1)$. Therefore $\epsilon < \|t^2 - t^0\| = \alpha \|t^1 - t^2\| < \alpha \cdot (\text{diam}(K))$ so $\alpha > \epsilon/\text{diam}(K)$ where $\text{diam } K = \sup\{\|s - t\| \mid s, t \in K\}$. Since $1 = \alpha p(t^1) + (1 - \alpha)p(t^2)$, we have

$$p(t^1) = \frac{1 - (1 - \alpha)p(t^2)}{\alpha} \leqq \frac{1}{\alpha} < (\operatorname{diam}(K))/\epsilon.$$

LEMMA 3.1.2. $B(K)$ *is a compact base for* $P_A = \{p\text{-affine} \mid p \geqq 0 \text{ on } K\}$.

Proof.   If we norm the finite dimensional space $A(K)$ of affine functions on K by $\|p\|_K = \sup|p(K)|$.   Lemma 3.1.1 implies $B(K)$ is compact the evaluation functional $\hat{t}^0$ defined by $\hat{t}^0(p) = p(t^0)$ is strictly positive on P_A and the assertion follows since $B(K)$ is contained in the preimage of 1 under $\hat{t}^0$.

PROPOSITION 3.1.3.   *The set* $\mathcal{T} = \{p/\|p\|_K \mid p \in \operatorname{ext} B(K)\} \cup \{0\}$ *of normalized extreme points of* $B(K)$ *along with* 0 *satisfies* 3.(i), 3.(ii) *and* 3.(iii).

Proof.   Since $A(K)$ is finite dimensional, and $B(K)$ is convex and compact, $B(K) = \operatorname{co}[\operatorname{ext}(B(K))]$.   Therefore, $\operatorname{span}^+ \mathcal{T} = P_A$ and since $[p - \inf p(K)]/[\sup p(K) - \inf p(K)] \in P_A$ for every $p \in A(K)$ we have $\operatorname{span} \mathcal{T} = A(K)$ so that (i) follows.   (ii) is clear since $0 \leqq p \leqq 1$ for all $p \in \mathcal{T}$ and $\operatorname{span}^+ \mathcal{T} = P_A$.   The Hahn–Banach Theorem implies $K = \bigcap_{p \in P_A}\{t \mid p(t) \geqq 0\}$ and since $P_A = \operatorname{span}^+ \mathcal{T}$ we have $\bigcap_{p \in \mathcal{T}}\{t \mid p(t) \geqq 0\}$ is also K and (iii) follows.

In the event that K is a convex polytope, the set $\mathcal{T}$ of Proposition 3.1.3 can be more precisely described as the linear functionals which expose the faces of K.   To be explicit there exist a minimal set of linear functionals $\{L_i \mid i = 1, 2, \cdots, k\}$ such that $K = \bigcap_j \{t \mid L_j(t) \geqq \inf L_j(K)\}$.   Let $p_j(K) = [L_j(t) - \inf L_j(K)]/[\sup L_j(K) - \inf L_j(K)]$.

PROPOSITION 3.1.4. $\mathcal{T} = \{p_j \mid j = 1, \cdots, k\} \cup \{1, 0\}$.

Proof.   Let $p \in \operatorname{ext} B(K)$ and $Z(p) = \{t \in K \mid p(t) = 0\}$.   If $Z(p) = \varnothing$ then $\inf p(K) > 0$ so p is no smaller (with respect to the ordering induced by P_A) than the constant function whose value is $\inf p(K)$.   Lemma 3.1.2 implies p is a multiple of this function so that $p \equiv 1$.   If $Z(p) \neq \varnothing$ then $Z(p)$ is a subset of a maximal proper extreme convex subset (face) of K.   But the faces of K are precisely the sets $Z(p_j)$ cf. **[9]**.   Since $Z(p)$ is extremal we have $\operatorname{ext} Z(p) \subset \operatorname{ext} Z(p_j) \subset \operatorname{ext}(K)$.   But $\operatorname{ext}(K)$ is finite.   Hence there exists $\alpha > 0$ such that $p - \alpha p_j$ is nonnegative on $\operatorname{ext}(K)$, whence $p - \alpha p_j \in P_A$ so that $p = \alpha p_j$ or $\mathcal{T} \subset \{p_j\}$.   Conversely if $p \in P_A$ and $p_j - p \in P_A$ then $Z(p_j) \subset Z(p)$.   But $Z(p_j)$ is a face and hence is a maximal proper extremal convex subset of K.   If $p \not\equiv 1$ then $Z(p)$ is a proper extremal convex subset of K so that $Z(p)$ is exactly the face $Z(p_j)$.   But every face of a

convex polytope affinely spans a hyperplane H, c.f. [9]. In as much as $\boldsymbol{p}$ and $\boldsymbol{p}_j$ agree on H we must have $\alpha \boldsymbol{p}_j = \boldsymbol{p}$ or $\boldsymbol{p}_j$ is on an extreme ray of P_A. Thus $\mathscr{T} \subset \{\boldsymbol{p}_j\}$.

3.2. Solution of the moment problem.

Let $\mathscr{T}$ be any subset of $\mathscr{A}(x)$ which satisfies 3.(i), 3.(ii) and 3.(iii).

PROPOSITION 3.2.1. *A multiplicative linear functional χ is $\mathscr{T}$-positive if and only if there exist $t \in K$ such that $\chi(\boldsymbol{p}) = \boldsymbol{p}(t)$. Moreover the set of positive multiplicative linear functionals Γ on $\mathscr{A}(x)$ is homeomorphic to K under the map $\chi \to t$.*

Proof. Let $\mathbf{p}_i$ be the polynomial defined by $\boldsymbol{p}_i(x) = x_i$ (evaluation at the ith coordinate) and let $t_i = \chi(\boldsymbol{p}_i)$. If $\boldsymbol{p} \in \mathscr{T}$ such that $\boldsymbol{p}(x) = \sum \alpha_{i_1,\cdots,i_\nu} x_1^{i_1} \cdots x_\nu^{i_\nu}$ then $\chi(\boldsymbol{p}) = \sum \alpha_{i_1,\cdots,i_\nu} t_1^{i_1} \cdots t_\nu^{i_\nu}$, so $\chi(\boldsymbol{p}) = \boldsymbol{p}(t_1 \cdots t_\nu)$ for all $\boldsymbol{p} \in \tau$ and hence all $\boldsymbol{p}$. The rest of the assertion follows easily.

A linear functional L in $\mathscr{A}(x)$ is uniquely determined by its moments $f(n_1, \cdots, n_\nu) = L(t_1^{n_1} \cdots t_\nu^{n_\nu})$.

COROLLARY 3.2.2. *A real valued function f on the ν-tuples of nonnegative integers admits a real integral representation of the form*

$$f(n_1, \cdots, n_\nu) = \int_K t_1^{n_1} \cdots t_\nu^{n_\nu} d\mu_f$$

if and only if the linear functional L_f determined by the moments $f(n_1, \cdots, n_\nu)$ is $\mathscr{T}$–BV; μ_f being nonnegative whenever L_f is $\mathscr{T}$-positive.

In order to more precisely describe those functions which are solutions to the moment problem let E_i denote the shift operator as defined by $(E_i f)(n_1, \cdots, n_\nu) = f(n_1, \cdots, n_i + 1, \cdots, n_\nu)$ for each $i = 1, 2, \cdots, \nu$. We define the product of two shift operators by composition and let $\mathscr{A}(E)$ denote the algebraic span of the shift operators. Then $\mathscr{A}(E)$ is isomorphic to $\mathscr{A}(x)$; $\boldsymbol{p}(E_1, \cdots, E_\nu)$ being mapped onto $\boldsymbol{p}(x_1, \cdots, x_\nu)$.

THEOREM 3.2.2. *A function f is a solution to the classical moment problem $(*)$ with respect to a positive measure μ if and only if*

$$[\boldsymbol{p}_1 \cdots \boldsymbol{p}_k(E)]f(0) \geqq 0 \text{ for all } \boldsymbol{p}_1, \cdots, \boldsymbol{p}_k \in \mathscr{T} \quad (0 = (0, \cdots, 0));$$

and f is a solution with respect to a signed measure if and only if

$$\lim \sum \binom{n}{i_1, \cdots} \cdots \binom{n}{i_{1k}, \cdots} \left| [\Pi_{i,j} \boldsymbol{p}_{ij}^{i_{ij}}(E)]f(0) \right| < \infty$$

where the limit is taken over all finite subsets of a set of cycles which include each $p \in \mathcal{T}$.

Proof. Let $p(x) = \sum \alpha_{n_1, \cdots, n_\nu} x_1^{n_1} \cdots x_\nu^{n_\nu}$. Then

$$
\begin{aligned}
L_f(p) &= \sum \alpha_{n_1, \cdots, n_\nu} f(n_1, \cdots, n_\nu) \\
&= \sum \alpha_{n_1, \cdots, n_\nu} E_1^{n_1} \cdots E_\nu^{n_\nu} f(0) \\
&= [p(E)f](0).
\end{aligned}
$$

Thus $L_f(p_1 \cdots p_k) = ([(p_1 \cdots p_k)(E)]f)(0)$ and the assertion follows.

COROLLARY 3.2.3.   *The set $P = \text{Alg span}^+(\mathcal{T})$ is uniformly dense in the nonnegative continuous functions on K.*

Proof. Let f be a continuous function which is nonnegative on K but not in the uniform closure of P.   The Hahn–Banach theorem implies the existence of a linear functional L on $C(K)$ such that $L(f) < 0 \leqq \inf L(P)$. But then L admits a nonnegative representing measure μ_L on K such that $L(p) = \int_K p \, d\mu_L$ for all $p \in \mathcal{A}$.   Since f is nonnegative on K we reach the contradiction that $L(f) = \int_K f \, d\mu_L \geqq 0$.

3.3.   Specific examples

3.3.1.   The Hausdorff (1-dimensional) moment problem.   Take $K = [a, b]$, then from Proposition 3.1.4,

$$
\mathcal{T} = \left\{ 1, 0, \left(\frac{1}{b-a} x - \frac{a}{b-a} \right) \right\}
$$

so that if $(Ef)(n) = f(n+1)$, then Theorem 3.2.2 implies:
f is a solution of the moment problem $\int_0^1 t^n \, d\mu = f(n)$ if and only if

(i) $\lim_n \dfrac{1}{(b-a)^n} \displaystyle\sum_{i=0}^n \binom{n}{i} |(E - aI)^i(-E + bI)^{n-i} F(a)| < \infty$.

Moreover μ is nonnegative if and only if

(ii) $(E - aI)^i(-E + bI)^j f(0) \geqq 0$

for all pairs on nonnegative integers i and j.

Note that if $a = 0$ and $b = 1$ then $-E + b$ is the difference operator $\Delta \equiv I - E$ and (i) reduces to $\lim_n \sum \binom{n}{i} |\Delta^{n-i} f(i)| < \infty$ while (ii) becomes $\Delta^i f(i) \geq 0$; Hausdorff's well known solution to the little moment problem, c.f. [6 or 14]. In this case Corollary 3.2.3 implies that every continuous function on $[0, 1]$ can be uniformly approximate by polynomials of the form $\sum \alpha_{ij} t^i (1 - t)^j$, $(\alpha_{ij} > 0)$, an obvious corollary to the theory of Bernstein polynomials.

3.3.2. The 3-dimensional cube. $K = \{t \in R^3 | 0 \leq t_i \leq 1, i = 1, 2, 3\}$. Then $\mathcal{T} = \{1, 0, 1 - x_1, 1 - x_2, 1 - x_3, x_1, x_2, x_3\}$ so that f is a solution of (*) if and only if

$$\lim_n \sum \binom{n}{i_1}\binom{n}{i_2}\binom{n}{i_3} |(I - E_1)^{n-i_1}(I - E_2)^{n-i_2}(I - E_3)^{n-i_3} f(i_1, i_2, i_3)| < \infty$$

and μ is nonnegative if and only if $(I - E_1)^{i_1}(I - E_2)^{i_2}(I - E_3)^{i_3} f(j_1, j_2, j_3) \geq 0$ for all choices of nonnegative integers, $i_1, i_2, i_3, j_1, j_2, j_3$.

3.3.3. The 2-dimensional simplex. If $K = \{t | 1 - t_1 - t_2 \geq 0, t_1 \geq 0, t_2 \geq 0\}$, then $\mathcal{T} = \{1, 0, 1 - x_1 - x_2, x_1, x_2\}$. Thus f is a solution of (*) if and only if

$$\lim_n \sum \binom{n}{i_1 \ i_2 \ i_3} |(I - E_1 - E_2)^{i_3} f(i_1, i_2)| < \infty$$

and the measure μ of determinancy is nonnegative if and only if $(I - E_1 - E_2)^{i_3} f(i_1, i_2) \geq 0$ for all choices of nonnegative integers i_1, i_2 and i_3. Corollary 3.2.3 implies that every continuous function on K can be uniformly approximated by polynomials of the form

$$\sum \alpha_{i_1 i_2 i_3} t_1^{i_1} t_2^{i_2} (1 - t_1 - t_2)^{i_3} \quad (\alpha_{i_1 i_2 i_3} > 0);$$

a fact which also follows from [8, page 51.]

4. Applications to semigroups. Let S be a commutative semigroup with identity 1 and involution * c.f.[8]. For each $x \in X$ we define the shift operator E_x on the functions on S by $E_x f(y) = f(xy)$. The product $E_x E_y$ of two shift operators is defined by composition and the linear sum is defined by $(\alpha E_x + \beta E_y) f(z) = \alpha f(xz) + \beta f(yz)$. Note that $E_x E_y = E_{xy}$ for all $x, y \in S$. The algebra $\mathcal{A}(E)$ spanned by the shift operators is then a commutative algebra with identity E_1 which can be made into an involution algebra by defining $(\sum \alpha_i E_{x_i})^* = \sum \alpha_i^* E_{x_i^*}$. The functions on S can be biuniquely identified

with the linear functionals L_f on $\mathcal{A}(E)$ by $L_f(\Sigma \, \alpha_i E_{x_i}) = \Sigma \, \alpha_i f(x_i)$. Then the multiplicative functionals are homeomorphic to the semigroup semicharacters relative to the topology of simple convergence. Thus the $\mathcal{T}$-positive and BV-functionals on $\mathcal{A}(E)$ correspond to certain classes of functions on S. Various possibilities exist for $\mathcal{T}$. In defining $\mathcal{T}$ more generality is obtained by considering a *generator* set for S, i.e. a subset X of S such that every element of $\mathcal{T}$ is a product of finitely many elements of $X \cup X^*$.

4.1.   Completely monotonic functions.   We assume $x^* = x$ for all $x \in S$ and set $\mathcal{T} = \{E_x, 1 - E_x \ x \in X\}$. Then L_f is positive (or BV) if and only if f is completely monotonic cf. [5] (or BV in the sense of [11]). If $S = (\{0, 1, \cdots\}, +)$, then these are the solutions to the little moment problem described in 3.3.1. If S is the set of all triples of nonnegative integers under coordinate addition, the situation is as described in §3.3.2.

4.2.   Moment problems in $\mathbf{R}^\nu$.   For a given convex body in $\mathbf{R}^\nu$ we let S be the semigroups of all ν-tuples of nonnegative integers under coordinate addition and the identity involution. We define $\mathcal{T}$ as in Theorem 3.2.2. Then the $\mathcal{T}$-BV functionals can be viewed as those functions which are solutions of the moment problem of §3.(*). The situation is clarified by the examples given in §3.

4.3.   Positive definite functions on groups and involution semigroups.   If S is a group or inverse semigroup (c.f. [2]), then $x^* = x^{-1}$ provides a natural involution on S. A natural involution on the semigroup of all order pairs of nonnegative integers which is considered in [10] is given by $(i, j)^* = (j, i)$. For arbitrary involution semigroup with generator set X we define a difference operator $T_{\sigma_x} = 1/2 I + (\sigma/4)E_x + (\sigma^*/4)E_{x^*}$, for each $x \in S$ and fourth root of unit σ. $\mathcal{T}$ is then selected as $\{T_{\sigma_x} \mid x \in X\}$. A linear functional L_f on $\mathcal{A}(E)$ is $\mathcal{T}$-positive if and only if finite products of the form $(\Pi_j T_{\sigma_{x_j}} f)(1)$ are nonnegative for all fourth roots of unity σ_j and all choices of $x_j \in X$. Such functions f on S are called (X)-*positive definite*. Moreover L_f is $\mathcal{T}$-BV if and only if

$$\lim_{(X,n)} \sum_\sigma \prod_j \left[\binom{n}{i_j} (T^i_{\sigma_{x_j}} T^{n-i}_{-\sigma_{x_j}} f)(1) \right] < \infty.$$

The functions f on S for which L_f is $\mathcal{T}$-BV are called X-BV functions on S. These functions are completely discussed in [12] and the results of §2 can be used to recover the theorems there. When S is a group, Γ

corresponds to the group characters $\text{Char}(S)$ and Theorem 2.1 takes the form of Bochner theorem for discrete groups.

COROLLARY 4.3.1. *If S is a commutative group, then a complex valued function f is positive definite if and only if f admits an integral representation*

$$f(x) = \int_{\text{Char}S} \chi(x)\,d\mu(\chi).$$

In particular the classically defined positive definite functions on groups agree with those defined here. Theorem 2.2 describes those functions on S which are in the span of the positive definite functions.

When S is the semigroup of pairs of nonnegative integers as described above and $X = \{(1,0),(1,1)\}$ then the multiplicative linear functional $L_x \in \Gamma$ can be identified with the points of the unit disk D and Theorems 2.1 and 2.2 characterize those functions f on S which are

solutions of the moment problem $f(i,j) = \int_D z^i (z^*)^j d\mu(z)$ for nonnegative and complex μ respectively. The solution to this moment problem for nonnegative μ leads to a derivation of the spectral theorem for normal operators [10]. For general involution semigroups, if we select $X = S$ then Γ corresponds to the bounded semicharacters χ such that $\chi(x^*) = \overline{\chi(x)}$ and Theorem 2.1 shows that f is $\mathcal{T}$-positive if and only if f is *-definite as defined in [7].

5. Applications to B*-algebras. Let $\mathcal{A}$ be a commutative B^*-algebra with identity 1 such that $\|x\| = \|x^*\|$ and let $\mathcal{T} = \{xx^* \mid \|x\| < 1\}$. Then each $x \in \mathcal{A}$ can be expressed as $x_1 + ix_2$ where $x_j^* = x_j$ $(j = 1,2)$. Thus

$$\alpha x = [(\alpha - 1) + (\alpha - 1)i]1 + [1 - \alpha(1 - x_1)] + [1 - \alpha(1 - x_2)]i.$$

If we choose $\alpha > 0$ such that $\|\alpha(1 - x_j)\| < 1$ $(j = 1,2)$ then the square root lemma implies the existence of $y_j \in \mathcal{T}$ such that $y_j = 1 - \alpha(1 - x_j)$ or

$$\alpha x = [(\alpha - 1) + (\alpha - 1)i]1 + y_1 + y_2.$$

Thus $\mathcal{T}$ generates $\mathcal{A}$. The square root lemma also implies $1 - x \in \mathcal{T}$ whenever $x \in \mathcal{T}$. If we let Γ be the set of multiplicative linear functionals χ such that $\chi^*(x) = \chi(x^*)$, then Theorems 2.1 and 2.2 establish the following.

THEOREM 5.1. *A linear functional admits an integral representation*

$f(x) = \int_\Gamma \chi(x)d\mu(x)$ with respect to a necessarily unique regular Borel measure μ if and only if

$$\lim_{(\mathcal{F},n)} \sum \binom{n}{i_1}\cdots\binom{n}{i_k}\left| f\left[\prod_j x_j^{i_j}(1-x_j)^{n-i_j}\right]\right| < \infty.$$

Also f admits a nonnegative representing measure if and only if $f(xx^*) \geqq 0$ for all $x \in \mathcal{A}$.

The second assertion is well known, c.f. [15].

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