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SPACES OF SIMILARITIES IV: (s, t)-FAMILIES

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The determination of spaces of similarities is a generalization of the Hurwitz problem of compasition of quadratic forms. For forms σ, q over the field F, we write $\sigma < \mathrm{Sim}(q)$ if q admits composition with σ. When F is the real or complex field, the possible dimensions of σ and q were determined long ago by Radon and Hurwitz. We show that these classical bounds are still correct over any field F of characteristic not 2.

This paper deals with the more delicate question of which quadratic forms σ, q over F can admit composition. The motivation of much of this work is the Pfister factor conjecture: if q is a form of dimension 2^m, and $\sigma < \mathrm{Sim}(q)$ for some form σ of large dimension, then q must be a Pfister form. We prove this true in general when $m \leqq 5$, and we also prove it true for all m for a certain class of fields which includes global fields.

Introduction. This paper continues the work on similarities initiated in [11], [12]. The objects studied are nonsingular λ-forms ($\lambda = \pm 1$) over a field F of characteristic not 2. We follow the notation of [8]. The first two sections concern the following question: Given $\lambda = \pm 1$ and n, what quadratic forms σ can be realized as a subspace of $\mathrm{Sim}(V, B)$, for some n-dimensional λ-space (V, B)? When $n = 2^m \cdot n_0$, n_0 odd, and $\dim \sigma \geqq 2m - 1$, a complete solution is found, characterizing such forms σ in terms of the signed determinant $d_{\pm}\sigma$ and the Witt invariant $c(\sigma)$. In fact, a more general characterization of (s, t)-families on $\mathrm{Sim}(V, B)$ is found when $s + t \geqq 2m - 1$. In working with (s, t)-families consistently, the results are more symmetrical and easier to prove. For example, we obtain a new computation of the values of the Hurwitz functions $\rho_t^{\lambda}(n)$.

In the third section, the Pfister factor conjecture of [12; (7.1)] is restated in terms of (s, t)-families. An inductive method is then used to give new proofs of this conjecture in the cases $m = 4, 5$. This method is also used to prove the conjecture for all values of m when F is a global field.

The last two sections of this paper deal with the odd factor conjecture [12; (7.4)]. This question is settled for small families by means of a decomposition result for Pfister factors [15]. In the special case of positive definite forms over the rational numbers, the conjecture is proved for families of any size. These theorems over the rationals provide some insight into the theory of orthogonal designs [5], [6].

I wish to thank A. Wadsworth for helping to clear up several parts of this paper, and for providing some of the key ideas in the third and fourth sections.

1. Realizing (s, t)-families. We follow the notations of [8] and [11]. Throughout the paper, F denotes a field of characteristic not 2. The vector spaces, algebras, and forms all have F as ground field and are finite dimensional. All forms are assumed to be nonsingular.

The key idea for the proofs in this paper is the connection between λ-forms on V and involutions on $\operatorname{End}(V)$. Here, for $\lambda = \pm 1$, a λ-form B on a vector space V is a (nonsingular) bilinear form $B: V \times V \to F$ satisfying: $B(y, x) = \lambda B(x, y)$. Then, a 1-form is equivalent to a quadratic form. An *involution* of an algebra is an F-linear antiautomorphism whose square is the identity.

DEFINITION 1.1. If J is an involution of an algebra A, and $a \in A$ is invertible, define the map J^a by:

$$J^a(x) = a^{-1} \cdot J(x) \cdot a \,, \qquad x \in A \,.$$

If $J(a) = \pm a$, then J^a is also an involution. Two involutions J and J_1 of A are *comparable* if $J_1 = J^a$, for some $a \in A^\times$. In this case, $J(a) = \lambda a$, for some central element λ with $J(\lambda) \cdot \lambda = 1$. If J acts trivially on the center of A, then $\lambda^2 = 1$ and this element λ depends only on J_1 and J. If the center of A is a field, or if it has F-dimension 2, then $\lambda = \pm 1$.

If the involutions J_1 and J of A are comparable and act trivially on the center of A, define J_1 and J to have the *same parity* if $J_1 = J^a$, where $J(a) = a$, and to have *opposite parity* if $J_1 = J^a$, where $J(a) = -a$.

A (nonsingular) λ-form B on V induces an *adjoint involution* I_B on $\operatorname{End}(V)$, defined:

$$B(f(u), v) = B(u, I_B(f)(v))$$

for $u, v \in V$ and $f \in \operatorname{End}(V)$. We sometimes write $\tilde{f}$ for $I_B(f)$. Suppose that $f \in \operatorname{End}(V)$ is invertible, and define the bilinear form B^f by:

$$B^f(u, v) = B(u, f(v)) \,, \qquad u, v \in V \,.$$

The following lemmas are proved in [12; §8].

LEMMA 1.2. *If $I_B(f) = \mu \cdot f$, where $\mu \in F$, then $\mu = \pm 1$ and B^f is a $\lambda\mu$-form with adjoint involution I_B^f.*

LEMMA 1.3. *If I is an involution of* $\mathrm{End}(V)$, *then there is a λ-form B on V whose adjoint involution is I,* (for some $\lambda = \pm 1$). *This form B is unique, up to nonzero scalar multiple.*

If the involution I induces a λ-form on V, then I is called a *λ-involution*. This sign, λ, determines the parity of the involution: a λ-involution and a μ-involution on $\mathrm{End}(V)$ have the same parity iff $\lambda = \mu$.

LEMMA 1.4. *Let D be a quaternion algebra, write D_0 for the pure part of D, and let J_0 denote the usual bar involution: $J_0(d) = \bar{d}$. An involution J on D must be either J_0 or J_0^c, for some $c \in D_0^\times$. If b and c in D_0 anticommute, then $J_0^c(b) = b$. For $c \in D_0^\times$, J_0 and J_0^c are the only involutions of D which send c to $-c$.*

The whole problem we are studying in this paper concerns $\mathrm{Sim}(V, B)$ and its subspaces. We assume the reader is familiar with the definitions and basic properties of $\mathrm{Sim}(V, B)$, as in [11; §1]. We now give a more general approach to thd definition [11; (4.1)] of (s, t)-families.

DEFINITION 1.5. Let (V, B) be a λ-space. Two (nonsingular) subspaces S, T of $\mathrm{Sim}(V, B)$ are *amicable* if $I_B(f)\circ g = I_B(g)\circ f$, for all $f \in S$ and $g \in T$. An *(s, t)-family* on (V, B) is a pair (S, T) of amicable subspaces of $\mathrm{Sim}(V, B)$ with $\dim S = s$, $\dim T = t$, and $1_V \in S$.

Suppose (S, T) is an (s, t)-family, and $S = F1_V \perp S_1$, in the usual way [11]. Then

(1) S_1 is antisymmetric, (i.e., if $f \in S_1$, then $I_B(f) = -f$);
(2) T is symmetric (i.e., if $g \in T$, then $I_B(g) = g$); and
(3) S_1 and T anticommute (i.e., if $f \in S_1$, $g \in T$, then $fg = -gf$).

LEMMA 1.6. (a) *If S, T are amicable subspaces of* $\mathrm{Sim}(V, B)$ *and if $f \in \mathrm{Sim}^\times(V, B)$, then $f\circ S$, $f\circ T$ are amicable and $S\circ f$, $T\circ f$ are amicable.*

(b) *Suppose (S, T) is an (s, t)-family on (V, B). If $f \in S$, $\sigma(f) \neq 0$, then $(f\circ S, f\circ T)$ is an (s, t)-family. If $g \in T$, $\sigma(g) \neq 0$, then $(g\circ T, g\circ S)$ is a (t, s)-family.*

The proof is an easy calculation.

The similarity factor map on $\mathrm{Sim}(V)$ induces quadratic forms on S and T by restriction. We always assume these forms are nonsingular. If σ and τ are quadratic forms, we write

$$(\sigma, \tau) < \mathrm{Sim}(V, B)$$

if there are amicable subspaces S, T of $\mathrm{Sim}(V, B)$, where the quadratic space S is isometric to σ and the quadratic space T is isometric to τ.

Throughout this paper we will work solely with (s, t)-families, because the assumption that $1_V \in S$ facilitates the introduction of Clifford algebra representations. Thus, we assume that the form σ represents 1.

The following lemmas appear in [11; (4.6)]. They remain true in the general case of amicable subspaces.

SHIFT LEMMA 1.7. *Suppose σ, τ, and α are quadratic forms, where* $\dim \alpha \equiv 0 (\mathrm{mod}\ 4)$. *Then, for a λ-space (V, B),*

$$(\sigma \perp \alpha, \tau) < \mathrm{Sim}(V, B) \Longleftrightarrow (\sigma, \tau \perp \langle d\alpha \rangle \alpha) < \mathrm{Sim}(V, B) .$$

LEMMA 1.8. *Suppose σ, τ, α are forms where* $\dim \alpha = 2$. *Let (V, B) be a λ-space with $(\sigma \perp \alpha, \tau) < \mathrm{Sim}(V, B)$. Then, there is a $(-\lambda)$-form B' on V such that $(\sigma, \tau \perp \langle -1 \rangle \alpha) < \mathrm{Sim}(V, B')$.*

Next we recall the correspondence between (s, t)-families and certain Clifford algebra representations. For more details in the case $t = 0$, see [11; §3].

Suppose (S, T) is an (s, t)-family on the λ-space (V, B). Express $S = F \cdot 1_V \perp S_1$. If $f \in S_1$ and $g \in T$ then

$$(f + g)^2 = f^2 + g^2 = (-\sigma(f) + \sigma(g))1_V .$$

Letting σ denote the form on S and τ the form on T, so that $\sigma = \langle 1 \rangle \perp \sigma_1$, this equation shows that the Clifford algebra $C = C(\langle -1 \rangle \sigma_1 \perp \tau)$ has a representation

$$\pi \colon C \longrightarrow \mathrm{End}(V) .$$

Further, note that $I_B(f + g) = -f + g$. Define an involution J on C by using the map $(-1) \perp 1$ on $\langle -1 \rangle \sigma_1 \perp \tau$, and extending it to an antiautomorphism. Then I_B and J are compatible: for $c \in C$,

$$I_B(\pi(c)) = \pi(J(c)) .$$

In the language of [4] in this situation, the form B on V *admits* the action of the algebra-with-involution (C, J).

Conversely, suppose C and J are defined as above, and $\pi \colon C \rightarrow \mathrm{End}(V)$ is a representation. If there is an involution I_B on $\mathrm{End}(V)$ compatible with J, then we do get $(\sigma, \tau) < \mathrm{Sim}(V, B)$.

Main Question 1.9. Suppose σ and τ are quadratic forms, where σ represents 1. Given n and λ, when is there an n-dimensional λ-space (V, B) with $(\sigma, \tau) < \mathrm{Sim}(V, B)$?

The answer to this question is a central topic of this paper. We will first reduce the question to the case $n = 2^m$, then show that the answer depends only on the form $\sigma \perp \langle -1 \rangle \tau$ and on the value of t, and finally obtain the solution (in terms of the Witt invariant) when $\dim(\sigma \perp \langle -1 \rangle \tau) \geqq 2m - 1$.

THEOREM 1.10. *Let $n = 2^m \cdot n_0$, where n_0 is odd, and let σ and τ be quadratic forms. Then, $(\sigma, \tau) < \operatorname{Sim}(V, B)$, for some n-dimensional λ-space (V, B) if and only if $(\sigma, \tau) < \operatorname{Sim}(V', B')$, for some 2^m-dimensional λ-space (V', B').*

Proof. If $(\sigma, \tau) < \operatorname{Sim}(V_1, B_1)$, then we can tensor with any μ-space (V_0, B_0) to get $(\sigma, \tau) < \operatorname{Sim}(V_0 \otimes V_1, B_0 \otimes B_1)$. Therefore, the "if" part is trivial. Suppose $(\sigma, \tau) < \operatorname{Sim}(V, B)$. Let

$$C = C(\langle -1 \rangle \sigma_1 \perp \tau)$$

and let J be the involution as above. Then V is a C-module and B *admits* (C, J). Now apply the decomposition theorem [10; (2.1)] as in [11; (3.12)] to conclude: $V \simeq V_1 \perp \cdots \perp V_r$, where each V_i is a C-submodule of V, and $\dim V_i = 2^k$, for some k. Let B_i be the restriction of the form B to V_i. Then (V_i, B_i) admits the (C, J)-action, so that $(\sigma, \tau) < \operatorname{Sim}(V_i, B_i)$. Since $n = 2^k \cdot r$, we have $k \leqq m$, and tensoring (V_1, B_1) with the 1-space $2^{m-k}\langle 1 \rangle$ produces a λ-space (V', B') which does the job.

REMARK 1.11. Let $C = C(\langle -1 \rangle \sigma_1 \perp \tau)$ and suppose the dimension of an irreducible C-module V_0 is 2^k. Certainly, if $(\sigma, \tau) < \operatorname{Sim}(V, B)$, then $\dim V$ is a multiple of 2^k. Also, using the involution J, we can define a λ-hyperbolic (C, J)-module H_λ by putting a λ-form on the C-module $V_0 \oplus \bar{V}_0^*$. See [4; §4] or [10; §2] for details. Then, $(\sigma, \tau) < \operatorname{Sim}(H_\lambda)$ and the dimension of H_λ is 2^{k+1}. Therefore, the smallest m for which $(\sigma, \tau) < \operatorname{Sim}(V)$ for a 2^m-dimensional λ-space V is either k or $k + 1$.

In order to show that the answer to (1.9) depends only on the form $\beta = \sigma \perp \langle -1 \rangle \tau$ and on the value of t, we need more information on the parity of involutions of a Clifford algebra.

Let U be a quadratic space and $C = C(U)$ be the Clifford algebra. For a splitting $U = R \perp T$, where $\dim R = r$, $\dim T = t$, let J_T be the involution of C extending $(-1) \perp 1$ on $R \perp T$. These J_T are all the involutions of C which preserve U. Any such J_T preserves C_0 and C_1. When $t = 0$, we write J_0 for J_T; when $r = 0$ we write J_1. Then J_0 is bar and J_1 is ε, as in [11; §3] and [8; pp. 107, 139]. Let $y = z(T)$, that is, y is the product (in C) of an orthogonal basis of T.

LEMMA 1.12. (1) *$J_0(y) = (-1)^{t(t+1)/2} \cdot y$; $J_1(y) = (-1)^{t(t-1)/2} \cdot y$.*

(2) If t is even, $J_T = J_0^y$.

If t is odd, $J_T = J_1^y$.

(3) Let $U = R' \perp T'$, where $\dim T' = t'$.

If $\dim U$ is even, then J_T and $J_{T'}$ have the same parity iff $t' \equiv t$ or $r + 1 \pmod 4$.

If $\dim U$ is odd, then J_T and $J_{T'}$ are comparable iff $t \equiv t' \pmod 2$. In this case, $J_{T'} = J_T^a$, for some $a \in C^\times$ with $J_T(a) = (-1)^{(t-t')/2} \cdot a$. Thus, if J_T acts trivially on the center of C, then J_T and $J_{T'}$ have the same parity iff $t \equiv t' \pmod 4$. If J_T acts nontrivially on the center, the element a can be chosen with $J_T(a) = a$.

The proof is a calculation we omit.

PROPOSITION 1.13. *Let σ, τ, n, λ be given, as in question (1.9). The existence of an n-dimensional λ-space (V, B) with $(\sigma, \tau) < \mathrm{Sim}(V, B)$ depends only on the form $\beta = \sigma \perp \langle -1 \rangle \tau$ and on the residue of $t \pmod 4$.*

Proof. Suppose $(\sigma, \tau) < \mathrm{Sim}(V, B)$ for such a λ-space. Let $\beta = \sigma' \perp \langle -1 \rangle \tau'$ be another decomposition of β, where σ' represents 1, and $\dim \tau' = t' \equiv t \pmod 4$. We will show that there is a λ-form B' on V with $(\sigma', \tau') < \mathrm{Sim}(V, B')$.

Writing $\beta = \langle 1 \rangle \perp \beta_1$, the Clifford algebra $C = C(\langle -1 \rangle \beta_1)$ has two involutions J, J' corresponding to the splittings of β. Since $t \equiv t' \pmod 4$, Lemma (1.12) implies that $J' = J^a$ for some $a \in C^\times$ with $J(a) = a$. The λ-involution I_B on $\mathrm{End}(V)$ is compatible with J, using the given representation $\pi: C \to \mathrm{End}(V)$. With $f = \pi(a)$, we see that $I_B(f) = f$, and I_B^f is an involution on $\mathrm{End}(V)$ compatible with J'. Therefore, $B' = B^f$ is the desired λ-form, as in (1.2).

By this proposition, the following definition is valid.

DEFINITION 1.14. Suppose β is a quadratic form which represents 1, and $\dim \beta = s + t$. Then, *β is realized as an (s, t)-family on n-dimensional λ-space* if for some (every) pair of forms σ, τ where $\dim \sigma = s$, $\dim \tau = t$, σ represents 1, and $\beta \simeq \sigma \perp \langle -1 \rangle \tau$, we have $(\sigma, \tau) < \mathrm{Sim}(V, B)$, for some n-dimensional λ-space (V, B).

The question (1.9) has become: Given a form β which represents 1, given s and t with $\dim \beta = s + t$, and given λ and m, when is β realized as an (s, t)-family on 2^m-dimensional λ-space?

When $\dim \beta \geqq 2m - 1$, the answer is given in the next section as Theorem (2.3). When $\dim \beta < 2m - 1$, no general answer is known.

2. Characterizing large (s, t)-families. The next proposition characterizes large (s, t)-families on a λ-space, for an undetermined value of λ. The stronger result, with separation of the cases $\lambda = 1$ and $\lambda = -1$ is stated in (2.3). We will use properties of the Witt invariant listed in [8; p. 121] without further mention.

PROPOSITION 2.1. *Let $\beta = \langle 1\rangle \perp \beta_1$ be a form with $\dim \beta \geqq 2m - 1$, and suppose $\dim \beta = s + t$, where $s \geqq 1, t \geqq 0$. The following statements give necessary and sufficient conditions for β to be realized as an (s, t)-family on some 2^m-dimensional λ-space, for some λ: $\dim \beta \leqq 2m + 2$, and:*

(1) *If $\dim \beta = 2m + 2$, require: $d_\pm\beta = \bar{1}$, $c(\beta) = 1$, and $m \not\equiv t (\mathrm{mod}\ 2)$.*

(2) *If $\dim \beta = 2m + 1$, require: $c(\beta) = 1$.*

(3) *If $\dim \beta = 2m$, require: either $c(\langle b\rangle\beta) = 1$, for some $b \in F^\times$, or $d_\pm\beta = \bar{1}$, $c(\beta) =$ quaternion, and $m \equiv t\ (\mathrm{mod}\ 2)$.*

(4) *If $\dim \beta = 2m - 1$, require: $c(\beta) =$ quaternion.*

Furthermore, β can be realized as an (s, t)-family for both values of λ if β falls in case (4), or in case (3) except when $d_\pm\beta \neq \bar{1}$ and $m \equiv t\ (\mathrm{mod}\ 2)$. In other cases, only one value of λ will work.

Proof. Let $C = C(\langle -1\rangle\beta_1)$. Suppose β is realized on the space (V, B), and let $\pi: C \to \mathrm{End}(V)$ be the given representation. By the structure theory, either C or C_0 is simple, so that $\dim \beta \leqq 2m + 2$, by counting dimensions. Let $z \in C$ be an element of highest degree, and let J be the involution on C induced by the given (s, t)-splitting of β.

(1) $\dim \beta = 2m + 2$. The size of C forces C nonsimple and $C_0 \cong \mathrm{End}(V)$. Therefore, the quadratic invariant $\delta = d_\pm(\langle -1\rangle\beta_1) = d_\pm\beta$ must be $\bar{1}$ [8; p. 111], and $c(\beta) = c(\langle -1\rangle\beta_1) = [C_0] = 1$ in the Brauer group. Furthermore, $\pi(z)$ must be scalar, so $J(z) = z$. Since

$$J(z) = (-1)^t J_0(z) = (-1)^{t+m+1} z\ ,$$

we have $t + m + 1$ is even.

Conversely, given the conditions on β, we have $C \cong C_0 \times C_0$ and $C_0 \cong \mathrm{End}(V)$ for some vector space V of dimension 2^m. This gives a representation $\pi: C \to \mathrm{End}(V)$ carrying z to a scalar. For an (s, t)-splitting $\beta = \sigma \perp \langle -1\rangle\tau$, let J be the corresponding involution on C. Since $m \not\equiv t (\mathrm{mod}\ 2)$, $J(z) = z$, and we do get an involution I on $\mathrm{End}(V)$ compatible with J. Using (1.3) we get a λ-form on V which does the job.

(2) $\dim \beta = 2m + 1$. If β is realized on (V, B), then dimension count shows $C \cong \mathrm{End}(V)$ so that $c(\beta) = [C] = 1$ in the Brauer group. The converse follows as before.

(3) $\dim \beta = 2m$. Suppose β is realized on (V, B). Since C_0 is central simple of dimension 2^{2m-2}, the centralizer D of $\pi(C_0)$ in $\mathrm{End}(V)$ is central simple with $\dim D = 4$, [8; p. 74]. Then D is a quaternion algebra and $c(\beta) = c(\langle -1\rangle\beta_1) = [C_0] = [D]$ in the Brauer group. The element $z \in C$ is central and $J(z) = (-1)^t J_0(z) = (-1)^{t+m} z$. Also, $z^2 = \delta \in F\cdot 1$, where $\bar{\delta} = d_{\pm}(\langle -1\rangle\beta_1) = d_{\pm}\beta$. If $\pi(z)$ is scalar, then $d_{\pm}\beta = \bar{1}$ and $J(z) = z$ so that $m \equiv t (\mathrm{mod}\ 2)$. Otherwise $\pi(z) \in D_0$, the pure quaternions, so that $D \cong (\delta, b/F)$, for some $b \in F^{\times}$. Therefore $c(\langle b\rangle\beta) = 1$.

For the converse, suppose $c(\beta) =$ quaternion, say $[C_0] = [D]$ for a quaternion algebra D. Then $C_0 \otimes D \cong \mathrm{End}(V)$, for some vector space V of dimension 2^m. For an (s, t)-splitting $\beta = \sigma \perp \langle -1\rangle\tau$, let J be the corresponding involution on C and let J° be the restriction of J to C_0. If $d_{\pm}\beta = \bar{1}$, and $m \equiv t (\mathrm{mod}\ 2)$, there is a homomorphism $C \to C_0$ which preserves J, (and maps z to a scalar). Choose any involution K on D, and let $I = J^{\circ} \otimes K$, an involution on $\mathrm{End}(V)$. Then I furnishes a λ-form B on V as claimed. Note that there are choices of K of opposite parity which do the job.

If $c(\langle b\rangle\beta) = 1$, then $[D] = c(\beta) = [(\delta, b/F)]$, where $d_{\pm}\beta = \bar{\delta}$. Therefore, $D \cong (\delta, b/F)$, and we can choose $d \in D_0$, the pure quaternions, with $d^2 = \delta$. Since also $z^2 = \delta$, we extend the natural map $C_0 \to C_0 \otimes D$ to a homomorphism $\pi\colon C \to C_0 \otimes D$ by sending z to $1 \otimes d$. This gives the desired representation. Using (1.4) we can choose an involution K on D with $K(d) = \mu d$ when $J(z) = \mu z$ ($\mu = \pm 1$), and get the involution $I = J^{\circ} \otimes K$ compatible with J. If $m \not\equiv t$ (mod 2) in this case, then $J(z) = -z$ and, by (1.4), there are choices of K of opposite parity which do the job. If $m \equiv t$ (mod 2), then $J(z) = z$, and all choices of K have the same parity.

(4) $\dim \beta = 2m - 1$. The argument is similar to that of (3), but it is easier and omitted. In this case the choice of the involution K on the quaternion algebra D is arbitrary, and both values of λ occur. This completes the proof.

We can refine the argument in (2.1) (4) to determine how the (s, t)-family on (V, B), induced by β, can be enlarged. The next proposition shows that any larger family on (V, B) corresponds to a subform of either $\beta \perp \langle -d_{\pm}\beta\rangle$ or $\beta \perp \langle -d_{\pm}\beta\rangle\langle x, y, -xy\rangle$, where, for some x, y: $c(\beta) = [(x, y/F)]$, a quaternion algebra.

Suppose (V, B) is a fixed 2^m-dimensional λ-space, and (S, T) is an (s, t)-family on (V, B), with $s + t = 2m - 1$. Then, as usual let $C = C(\langle -1\rangle\sigma_1 \perp \tau)$ and $C \otimes D = \mathrm{End}(V)$, for a quaternion algebra D. The adjoint involution I_B extends the involutions J on C and some involution K on D. If z is an element of highest degree in C, then z anticommutes with S_1 and T. If $J(z) = -z$, then $(S + Fz, T)$ is an $(s+1, t)$-family, while if $J(z) = z$, then $(S, T + Fz)$ is an $(s, t+1)$-family. Further, if $g \in D$, then zg anticommutes with S_1 and T, so if $K(g) = \pm g$, then zg can be added to either S or T to produce a larger family. Knowing all the involutions on D by (1.4), we can get a precise statement of the possibilities:

PROPOSITION 2.2. *Let (V, B), S, T, D be as above. All the families here are families on (V, B). Write D_0 for the set of pure quaternions in D.*

(1) *If $m \equiv t \pmod 2$, $(S + Fz, T)$ is an $(s+1, t)$-family.*

If $m \not\equiv t \pmod 2$, $(S, T + Fz)$ is an $(s, t+1)$-family.

(2) *Suppose $K = bar$.*

If $m \equiv t \pmod 2$, $(S, T + zD_0)$ is an $(s, t+3)$-family.

If $m \not\equiv t \pmod 2$, $(S + zD_0, T)$ is an $(s+3, t)$-family.

(3) *Suppose $K \neq bar$. Then D_0 splits up as $D_0 = D_1 \oplus D_2$, where $K(d) = -d$ for $d \in D_1$, and $K(d) = d$ for $d \in D_2$.*

If $m \equiv t \pmod 2$, $(S + zD_2, T + zD_1)$ is an $(s+2, t+1)$-family.

If $m \not\equiv t \pmod 2$, $(S + zD_1, T + zD_2)$ is an $(s+1, t+2)$-family.

(4) *Any enlargement of (S, T) to a family on (V, B) must be contained in one of the families listed above.*

Proof. First note that $J(z) = (-1)^t J_0(z) = (-1)^{t+m-1} \cdot z$, so part (1) follows. Parts (2), (3) are derived from similar sign computations, and the facts that $\dim D_1 = 1$, $\dim D_2 = 2$, (by (1.4)). Since $2m + 2$ is the largest possible dimension (2.1) (1), the families in (2) and (3) are maximal. For the maximality of the families in (1), note that no element $f \in \operatorname{End}(V)$ can anticommute with S_1, T, and z, since $s + t - 1$ is even. For (4), if f lies in an enlargement of (S, T), and f is orthogonal to S and T, then f anticommutes with S_1 and T, so that $zf = d \in D$. Since $d^2 \in F$ and $K(d) = \pm d$, we conclude that either $f \in Fz$, $f \in zD_1$, or $f \in zD_2$.

The main difficulty in characterizing the forms β which can be realized as (s, t)-families is to separate the cases $\lambda = 1$ and $\lambda = -1$ in (2.1). This can be done by first determining which pairs (s, t) with $s + t = 2m + 2$ can occur as the sizes of families on some 2^m-dimensional λ-space. We first state the final result.

CHARACTERIZATION THEOREM 2.3. *Let β be a quadratic form with $2m - 1 \leqq \dim \beta \leqq 2m + 2$, and suppose $\dim \beta = s + t$, where $s \geqq 1$, $t \geqq 0$. The following cases give necessary and sufficient conditions for β to be realized as an (s, t)-family on some 2^m-dimensional quadratic space.*

(1) $\dim \beta = 2m + 2$: *require $d_{\pm}\beta = \bar{1}$, $c(\beta) = 1$, and $m \equiv t - 1$*

(mod 4).

(2) $\dim \beta = 2m + 1$: *require* $c(\beta) = 1$ *and* $m \equiv t$ *or* $t - 1$ (mod 4).

(3) $\dim \beta = 2m$: *require,*
if $m \equiv t$ (mod 4): *either* $c(\langle b\rangle\beta) = 1$, *for some* $b \in F^{\times}$, *or* $d_{\pm}\beta = \bar{1}$ *and* $c(\beta) =$ *quaternion;*
if $m \equiv t + 1$ (mod 4): $c(\langle b\rangle\beta) = 1$, *for some* $b \in F^{\times}$;
if $m \equiv t + 2$ (mod 4): $d_{\pm}\beta = \bar{1}$ *and* $c(\beta) =$ *quaternion;*
if $m \equiv t + 3$ (mod 4): $c(\langle b\rangle\beta) = 1$, *for some* $b \in F^{\times}$.

(4) $\dim \beta = 2m - 1$: *require* $c(\beta) =$ *quaternion.*
Furthemore, the conditions for the case $\lambda = -1$ *are obtained from these by cycling all the congruences by* 2 (mod 4).

PROPOSITION 2.4. *Let* β *be a quadratic form of dimension* $2m+2$, *and let* (V_i, B_i) *be* λ*-spaces of dimension* 2^m, *for* $i = 1, 2$. *If* β *is realized as an* (s_i, t_i)*-family in* $\mathrm{Sim}(V_i, B_i)$, *for* $i = 1, 2$, *then* $t_1 \equiv t_2$ (mod 4).

Proof. Let $C = C(\langle -1\rangle\beta_1)$ as usual. From the given representations $\pi_i: C \to \mathrm{End}(V_i)$, we see by dimension count that C is split and the V_i are irreducible C-modules. The two nonisomorphic irreducibles differ only by ν, the main automorphism of C, so replace π_1 by $\pi_1 \circ \nu$ if necessary to assume $V_1 \cong V_2$ as C-modules. We identify these modules, to get a single representation $\pi: C \to \mathrm{End}(V)$, which realizes β as an (s_i, t_i)-family on (V, B_i). Let J_1, J_2 be the involutions on C and let I_1, I_2 be the adjoint involutions on $\mathrm{End}(V)$. Since $\pi(z)$ must be scalar, when z is a highest degree element of C, we have $J_1(z) = J_2(z) = z$. By hypothesis, I_1 and I_2 are λ-involutions, so $I_2 = I_1^f$ for some invertible $f \in \mathrm{End}(V)$ with $I_1(f) = f$. Since $\pi: C_0 \cong \mathrm{End}(V)$, choose $a \in C_0^{\times}$ with $\pi(a) = f$. Then $J_1(a) = a$ and $J_2 = J_1^a$. Therefore, J_1 and J_2 have the same parity and (1.12) (3) shows $t_1 \equiv t_2$ (mod 4).

COROLLARY 2.5. *Suppose* (V_i, B_i) *are* λ*-spaces of dimension* 2^m, *for* $i = 1, 2$, *and* (V_i, B_i) *admits an* (s_i, t_i)*-family, where* $s_i + t_i = 2m + 2$. *Then* $t_1 \equiv t_2$ (mod 4) *and* $s_1 \equiv s_2$ (mod 4).

Proof. Say these families come from forms β_i of dimension $2m + 2$. After tensoring up to the algebraic closure of F, β_1 and β_2 become isometric, and (2.4) gives the result.

Recall that the Clifford algebra construction of [12; (9.9)] produces an $(m + 1, m + 1)$-family (σ, σ) on a 2^m-dimensional Pfister space (C, N_1). Then, if there is any (s, t)-family on a 2^m-dimensional quadratic space, where $s + t = 2m + 2$, (2.5) implies $s \equiv m + 1 \equiv t \pmod 4$. Then, $s \equiv t \pmod 8$.

Conversely, given such (s, t), an (s, t)-family does exist on any m-fold Pfister space, since we can shift the $(m + 1, m + 1)$-family by 4's, using (1.7). Also, by (1.8) there is an $(m + 3, m - 1)$-family on 2^m-dimensional (-1)-space and we can apply similar arguments:

PROPOSITION 2.6. *Suppose $s + t = 2m + 2$, where $s \geqq 1$, $t \geqq 0$. An (s, t)-family exists on some 2^m-dimensional λ-space iff*

$$t \equiv m + \lambda \pmod 4) .$$

COROLLARY 2.7. *In the situation of* (2.2), *if $\lambda = 1$, then $K = bar$ occurs iff $m - t \equiv 2$ or $3 \pmod 4$.*

Proof. If $K =$ bar and $m \equiv t \pmod 2$, then there is an $(s, t + 3)$-family. By (2.6), $t + 3 \equiv m + 1 \pmod 4$, so that $m - t \equiv 2 \pmod 4$. The other three cases are similar.

We can now combine this separation of the cases $\lambda = 1$ and $\lambda = -1$ with (2.1) to prove the characterization stated in (2.3).

Proof of (2.3). (4) follows from (2.1), and (1) follows from (2.1) (1) and (2.5). For (2), suppose $\dim \beta = 2m + 1$. If β is realized as an (s, t)-family on (V, B), it cannot be maximal by (2.2). It embeds in either an $(s + 1, t)$ or an $(s, t + 1)$-family, and (2.6) forces either $t \equiv m + 1$ or $t + 1 \equiv m + 1 \pmod 4$. Conversely, given such β, (2.1) implies it is realized on some 2^m-dimensional λ-space, so again the family enlarges and (2.6) implies $\lambda = 1$.

(3) Suppose $\dim \beta = 2m$. Then, the equivalence follows immediately in the cases where both values of λ work in (2.1). Otherwise, $m - t$ is even and $d_{\pm}\beta \neq \bar{1}$. Then, by (2.1), β can be realized as an (s, t)-family, for unique λ, iff $c(\langle b \rangle \beta) = 1$, for some $b \in F^{\times}$. Claim: $\lambda = 1$ iff $m \equiv t \pmod 4$.

Since $d_{\pm}\beta \neq \bar{1}$, (2.2) implies that this (s, t)-family enlarges to an (s', t')-family, where $s' + t' = 2m + 2$. Since $t' \equiv m + 1 \pmod 4$ and $m - t$ is even, we have $t' = t + 1$. Viewing the $(s + 1, t + 1)$-family as an enlargement of an $(s - 1, t)$-subfamily, we see from (2.2) that $K \neq$ bar. Then, (2.7) settles the claim.

The conditions in the case $\lambda = -1$ follow by application of (1.8). This completes the proof of the theorem.

By direct application of (2.6), we can see which (s, t) are possible parameters for a family on a 2^m-dimensional quadratic space,

$(s \geqq 1, t \geqq 0)$: if $s + t = 2m + 2$, require $t \equiv m + 1 \pmod 4$; if $s + t = 2m + 1$, require $t \equiv m$ or $m + 1 \pmod 4$; if $s + t \leqq 2m$, always.

Therefore, given t, the maximum possible s for which an (s, t)-family can exist on 2^m-dimensional quadratic space is:

$$s = \rho_t(2^m) = \begin{cases} 2m + 1 - t & \text{if} \quad m \equiv t \\ 2m - t & \text{if} \quad m \equiv t + 1 \\ 2m - t & \text{if} \quad m \equiv t + 2 \\ 2m + 2 - t & \text{if} \quad m \equiv t + 3 \end{cases} \pmod 4 ,$$

unless the given value is less than one. In that case, no such (s, t)-family exists. This determines the Hurwitz functions $\rho_t(n)$ introduced in [11; (4.2)], by a simpler method. Recall that $\rho_0(n) = \rho(n)$ is the original Hurwitz-Radon function [8; pp. 131, 137].

We can also generalize all the arguments in §8 of [12], obtaining in particular the uniqueness result corresponding to (8.6):

THEOREM 2.8. *Suppose V_1, V_2 are 2^m-dimensional quadratic spaces with $(\sigma, \tau) < \mathrm{Sim}(V_i)$, an (s, t)-family, for $i = 1, 2$. If $s \geqq \rho_{t+1}(2^m)$, then V_1 and V_2 must be similar. Moreover this bound on s is sharp.*

The proof is left to the interested reader. Note that $s + t \geqq 2m + 1$ is sufficient for the conclusion of the theorem. For the sharpness of the bound, note that $\rho_{t+1}(2^m) = 1 + \rho_t(2^{m-1})$.

3. The Pfister factor conjecture. Suppose (V, q) is a quadratic space of dimension 2^m. Then (2.2) implies that if (V, q) admits a space of similarities of dimension $2m - 1$, then it admits an (s, t)-family where $s + t = 2m + 2$. By shifting via (1.7) we obtain an $(m + 1, m + 1)$-family. Therefore, the Pfister factor conjecture, stated in [12; (7.1)], is reduced to the following:

Conjecture PC(m) 3.1. Any form q over F with $\dim q = 2^m$, having an $(m + 1, m + 1)$-family in $\mathrm{Sim}(q)$, is similar to a Pfister form.

This conjecture is known true for any field, when $m \leqq 5$ [13]. In this section we give new proofs for the cases $m = 4, 5$, and we apply the uniqueness result (2.8) and the shift lemma (1.7) to find fields where the conjecture holds for all m.

LEMMA 3.2. *Let q be a quadratic form with* $\dim q = 2^m$. *The following statements are equivalent*:

(1) *q is similar to a Pfister form.*

(2) $(\sigma, \sigma) < \mathrm{Sim}(q)$, *for some form σ with* $\dim \sigma = m + 1$.

(3) *$(m+1)\boldsymbol{H}$ can be realized as a family in* $\mathrm{Sim}(q)$.

Proof. $(1) \Rightarrow (2)$ is the Clifford construction of [12; §9]; and $(2) \Rightarrow (1)$ follows from (2.8), since this construction gives $(\sigma, \sigma) < \mathrm{Sim}(p)$ for an m-fold Pfister form p. $(2) \Rightarrow (3)$ is trivial. Given (3), suppose $(m+1)\boldsymbol{H}$ is realized as an (s, t)-family $(\sigma, \tau) < \mathrm{Sim}(q)$. Then $\sigma \perp \langle -1 \rangle \tau \cong (m+1)\boldsymbol{H}$. If $s = t = m+1$, then $\sigma \cong \tau$ and we have (2). Otherwise, either σ or τ must be isotropic. But then, by [11; (3.15)] or [13; §3], q must be hyperbolic. In this case, q is certainly Pfister, and we have (1).

LEMMA 3.3. *Suppose* $\dim \sigma = \dim \tau = m+1$ *and* σ *represents* 1. *Then*: $(\sigma, \tau) < \mathrm{Sim}(q)$ *for some* 2^m*-dimensional quadratic space* q *if and only if* $d_{\pm}\sigma = d_{\pm}\tau$ *and* $c(\sigma) = c(\tau)$.

Proof. By (2.3) (1) and some calculation.

PROPOSITION 3.4.
(1) $\langle 1, a \rangle < \mathrm{Sim}(q) \Leftrightarrow (\langle 1, a \rangle, \langle 1, a \rangle) < \mathrm{Sim}(q) \Leftrightarrow \langle\langle a \rangle\rangle \mid q$.
Furthermore, $(\langle 1, a \rangle, \tau) < \mathrm{Sim}(\langle\langle a \rangle\rangle) \Leftrightarrow \tau < \langle 1, a \rangle$.
(2) $\langle 1, a, b \rangle < \mathrm{Sim}(q) \Leftrightarrow (\langle 1, a, b \rangle, \langle 1, a, b \rangle) < \mathrm{Sim}(q) \Leftrightarrow \langle\langle a, b \rangle\rangle \mid q$.
Furthermore, $(\langle 1, a, b \rangle, \tau) < \mathrm{Sim}(\langle\langle a, b \rangle\rangle) \Leftrightarrow \tau < \langle 1, a, b \rangle$.

Proof. The first equivalences in (1) and (2) follow from [11; (3.13)]. For the second ones, expand τ to be maximal (as in (2.2)), and apply (3.3) plus the fact [8; p. 124] that two forms of dimension $\leqq 3$ are isometric if and only if they have the same 'dim,' 'det,' and Witt invariant.

These Pfister factor results settle the conjecture $PC(m)$ when $m \leqq 3$. To handle the cases $m = 4, 5$ we use an inductive method due to Wadsworth.

PROPOSITION 3.5. *Suppose* $PC(m-1)$ *is true over* F. *Let* q *be a form with* $\dim q = 2^m$. *If an isotropic form* β *of dimension* $2m+2$ *can be realized as a family in* $\mathrm{Sim}(q)$, *then* q *is similar to a Pfister form.*

Proof (Wadsworth [14]). We are given an (s, t)-family $(\sigma, \tau) < \mathrm{Sim}(q)$, where $s + t = 2m+2$ and $\sigma \perp \langle -1 \rangle \tau$ is isotropic. If either σ or τ is isotropic, then q is hyperbolic [11; (3.15)] and we are done. In any case, σ and τ represent a common value, say $\sigma \simeq \langle a \rangle \perp \sigma'$ and $\tau \simeq \langle a \rangle \perp \tau'$. Shift subforms of σ' and τ' by (1.7) to get an $(m+1, m+1)$-family $(\langle a \rangle \perp \sigma_0, \langle a \rangle \perp \tau_0) < \mathrm{Sim}(q)$. Then, by (1.6) scale to assume σ_0 represents 1. By (3.3), $\langle a \rangle \perp \sigma_0$ and $\langle a \rangle \perp \tau_0$ have the same invariants; therefore, $d_\pm\sigma_0 = d_\pm\tau_0$ and $c(\sigma_0) = c(\tau_0)$. Then, by (3.3) again, an unsplittable (σ_0, τ_0)-space must have dimension 2^{m-1}, and the decomposition theorem implies that $q \simeq q_1 \perp q_2$ where $\dim q_i = 2^{m-1}$ and $(\sigma_0, \tau_0) < \mathrm{Sim}\,(q_i)$. The uniqueness result (2.8) shows that q_1 and q_2 are similar: $q \simeq q_1 \otimes \langle 1, d\rangle$, for some d. Finally, by $PC(m-1)$, q_1 is similar to a Pfister form, and we are done.

COROLLARY 3.6. *$PC(4)$ is always true.*

Proof. We know $PC(3)$ is true. Suppose $(\sigma, \tau) < \mathrm{Sim}\,(q)$ is a (5, 5)-family on 16-space. Then $\beta = \sigma \perp \langle -1\rangle\tau$ has $\dim\beta = 10$, $d_\pm\beta = \bar{1}$, $c(\beta) = 1$, and therefore β is isotropic [9; p. 123]. Then, q is similar to a Pfister form, by (3.5).

PROPOSITION 3.7. *Suppose $PC(m-1)$ is true over F. Let q be a form with $\dim q = 2^m$, and let $(\sigma, \tau) < \mathrm{Sim}\,(q)$ be an (s, t)-family with $s + t = 2m + 2$. If σ has a 4-dimensional subform of determinant $\bar{1}$, then q is similar to a Pfister form.*

Proof (Wadsworth [14]). Assume $m \geqq 5$ and shift if necessary to assume $s \geqq 6$, and then scale via (1.6) to get the good subform inside σ_1: say $\sigma_1 \simeq \langle a, b, c, abc, d, \cdots\rangle$. Now, shift $\langle a, b, c, d\rangle$ over to τ. In the resulting family (σ', τ'), both σ' and τ' represent abc, and (3.5) yields the result.

COROLLARY 3.8. *$PC(5)$ is always true.*

Proof. We know $PC(4)$ is true. Let $(\sigma, \tau) < \mathrm{Sim}\,(q)$ be a (10, 2)-family on 32-space. Then $\beta = \sigma \perp \langle -1\rangle\tau$ has $\dim\beta = 12$, $d_\pm\beta = \bar{1}$, $c(\beta) = 1$. If β is isotropic, we are done by (3.5). Suppose β is anisotropic and refer to [9; p. 123]. Write $\tau = \langle -a_0\rangle\langle 1, -b\rangle$, so that $\beta = \langle a_0\rangle\langle 1, -b\rangle \perp \sigma$. Following Pfister's argument for dimension 12, $\beta \simeq \varphi_1 \perp \varphi_2 \perp \varphi_3$, $\dim\varphi_i = 4$, $d\varphi_i = \bar{1}$, and $\varphi_1 \simeq \langle a_0, a_1\rangle \otimes \langle 1, -b\rangle$. Therefore, $\sigma \simeq \langle a_1\rangle\langle 1, -b\rangle \perp \varphi_2 \perp \varphi_3$, and we are done by (3.7).

For arbitrary fields, the invariants we have been using do not classify isometry of forms of dimension $\geqq 4$; but for some fields they work nicely. If $I^3F = 0$, then 'dim,' 'det,' and Witt invariant do classify forms [2]. For such a field, Lemmas (3.2) and (3.3) immediately imply that $PC(m)$ is true over F, for all m. However, these fields often have $u(F) \leqq 4$ (e.g., when I^2F is linked, [8; p. 319]), so that any 5-dimensional form is isotropic and $PC(m)$ is trivial.

If I^3F is torsion free (e.g., if F is any global field) then quadratic forms over F are classified by 'dim,' 'det,' Witt invariant, and total signature [2]. Therefore, if F is such a field and if the form q under consideration in (3.1) has all its represented values totally positive (i.e., if $q < n\langle 1\rangle$, for some n), then the conjecture (3.1) is true.

PROPOSITION 3.9. *If I^3F is torsion free and F has at most two orderings, then the conjecture $PC(m)$ is true over F, for all m.*

Proof. If F is nonreal, the remarks above give the result. Suppose F has exactly one ordering. Let $(\sigma, \tau) < \mathrm{Sim}\,(q)$ be the given $(m + 1, m + 1)$-family. We may assume $m \geqq 4$. If $\mathrm{sgn}\,(\sigma) = \mathrm{sgn}\,(\tau)$, then, as noted above, $\sigma \simeq \tau$ and we are done. Otherwise, using (1.6), we assume $\mathrm{sgn}\,(\sigma) > \mathrm{sgn}\,(\tau)$. Over the real closure $\hat{F}$, these (diagonalized) forms become: $\hat{\sigma} \simeq a_1\langle 1\rangle \perp a_2\langle -1\rangle$ and $\hat{\tau} \simeq b_1\langle 1\rangle \perp b_2\langle -1\rangle$. Any shift operation on $(\hat{\sigma}, \hat{\tau})$ in this diagonalization can be lifted to a shift of (σ, τ) over F. We show that appropriate shifts will lead to equal signatures.

Certainly $a_1 + a_2 = b_1 + b_2 = m + 1 \geqq 5$. Since $\beta = \sigma \perp \langle -1\rangle\tau$ has $\dim \beta$ even, $d_{\pm}\beta = \bar{1}$, $c(\beta) = 1$, we know $\hat{\beta} \in I^3\hat{F}$, forcing $\mathrm{sgn}\,(\hat{\beta}) \equiv 0 \pmod 8$, as in [8; p. 117]. Then $a_1 - a_2 = \mathrm{sgn}\,(\sigma) \equiv \mathrm{sgn}\,(\tau) = b_1 - b_2 \pmod 8$, so that $a_i \equiv b_i \pmod 4$. Since $\mathrm{sgn}\,(\sigma) > \mathrm{sgn}\,(\tau)$, we get $a_1 \geqq b_1 + 4$ and $a_2 + 4 \leqq b_2$. If $a_1 \geqq 5$, shift $4\langle 1\rangle$ from $\hat{\sigma}$ to $\hat{\tau}$ and shift $3\langle 1\rangle \perp \langle -1\rangle$ back. Otherwise, $a_1 = 4$ and $a_2 \geqq 1$, and we shift $3\langle 1\rangle \perp \langle -1\rangle$ from $\hat{\sigma}$ to $\hat{\tau}$ and shift $4\langle -1\rangle$ back. In each case the difference of the signatures has been decreased by 8. Repeating this process yields the result.

When F has two orderings, a similar but much more complicated procedure leads to the equality of both signatures. The details are omitted.

The methods above might lead to a proof of the conjecture (3.1) whenever I^3F is torsion free and F has a finite number of orderings. However, even in the case of three orderings, no satisfactory proof is known.

Next we will describe two properties that a field can have, and prove the conjecture $PC(m)$ for any field enjoying both properties.

The Clifford invariant map $\bar{\gamma}\colon I^2F/I^3F \to B(F)$ is conjectured [9] to be injective always, (see [8; p. 117]). This condition says: if β is a quadratic form with $\dim \beta$ even, $d_{\pm}\beta = \bar{1}$, and $c(\beta) = 1$, then $\beta \in I^3F$.

In [3] it is shown that I^3F is linked if and only if every antisotropic $\psi \in I^3F$ has a simple decomposition: $\psi \simeq \varphi_1 \perp \cdots \perp \varphi_r$, where each φ_i is similar to a 3-fold Pfister form

Note that if I^2F is linked, then F automatically satisfies both of these properties [3]. In particular, these conditions hold when F is a global field.

THEOREM 3.10. *If $\bar{\gamma}$ is injective for F and if I^3F is linked, then the conjecture $PC(m)$ is true over F, for all m.*

Proof. Using induction on m, assume $PC(m-1)$ is true over F, and suppose $\dim q = 2^m$ and $(\sigma, \tau) < \mathrm{Sim}(q)$ is an $(m+1, m+1)$-family. If $\beta = \sigma \perp \langle -1 \rangle \tau$ is isotropic, we are done by (3.5). Otherwise, β is anisotropic and the hypotheses on F imply that $\beta \in I^3F$ has a simple decomposition. In particular $\dim \beta = 2m + 2 \equiv 0 \pmod 8$, so that $t \equiv m + 1 \equiv 0 \pmod 4$. Then we can shift to assume $t = 0$, and $\sigma = \beta$. Good subforms are easy to find from the simple decomposition of σ, and (3.7) proves the result.

REMARK. The tensor construction [11; §2] yields subspaces of $\mathrm{Sim}(\langle\langle a_1, \cdots, a_m \rangle\rangle)$, which are essentially given in their simple decompositions. Therefore, any such subspace can be shifted via (1.7) to give a family (σ, σ), as in the Clifford construction.

4. (s, t)-families when $s + t$ is small.

As mentioned in [11; (3.13)] and in [13; §3], there is a close connection between subspaces of similarities and Pfister factors. We now extend these ideas to (s, t)-families which can be realized as families on a 4-dimensional space, (see (4.7)). The odd factor conjecture for such families immediately follows. In the next theorem, $D(q)$ is the set of (nonzero) values represented by the form q, and $G(q)$ is the set of similarity factors of q, that is, $x \in G(q)$ if $\langle x \rangle q \simeq q$.

THEOREM 4.1.

(1, 1): $(\langle 1 \rangle, \langle x \rangle) < \mathrm{Sim}(q) \Leftrightarrow x \in G(q)$.

(2, 0): $\langle 1, a \rangle < \mathrm{Sim}(q) \Leftrightarrow \langle\langle a \rangle\rangle \mid q$.

(2, 1): $(\langle 1, a \rangle, \langle x \rangle) < \mathrm{Sim}(q) \Leftrightarrow \langle\langle a \rangle\rangle \mid q$ *and* $x \in G(q)$.

(2, 2): $(\langle 1, a \rangle, \langle 1, x \rangle) < \mathrm{Sim}(q) \Leftrightarrow \langle\langle a \rangle\rangle \mid q$ *and* $\langle\langle x \rangle\rangle \mid q$.

(3, 0): $\langle 1, a, b \rangle < \mathrm{Sim}(q) \Leftrightarrow \langle\langle a, b \rangle\rangle \mid q$.

Note. Not every (2, 2)-family is included here. But, we can reduce $(\langle 1, a \rangle, \langle x, y \rangle)$ to $(\langle 1, a \rangle, \langle 1, xy \rangle)$ provided $\langle 1, a \rangle$ and $\langle x, y \rangle$ represent a common value, by scaling the family by this value via (1.6).

To prove this theorem, we need to invoke a decomposition theorem due to Wadsworth.

THEOREM 4.2 (Wadsworth [15]). *Let φ be a Pfister form, and q a form where $\varphi | q$.*

(1) *Suppose $c \in G(q)$ but $c \notin G(\varphi)$. Then there is a decomposition*

$$(*) \qquad q \simeq q_1 \perp \cdots q_k \, ,$$

where $\dim q_i = 2 \cdot \dim \varphi$, $\varphi | q_i$, *and* $c \in G(q_i)$.

(2) *Suppose $\langle\!\langle b \rangle\!\rangle | q$ but $\langle\!\langle b \rangle\!\rangle \nmid \varphi$. Then there is a decomposition* (*) *as above, where* $\dim q_i = 2 \cdot \dim \varphi$, $\varphi | q_i$, *and* $\langle\!\langle b \rangle\!\rangle | q_i$.

Proof of (4.1). The (2, 0) and (3, 0) statements are proved in [11; (3.13)] and [13; Theorem 4], and the other ($\Rightarrow$) statements quickly follow. After appying (4.2), the three remaining assertions of the theorem are settled by the following lemma.

LEMMA 4.3.

(1) *If* $\dim q = 2$ *and* $x \in G(q)$, *then* $(\langle 1 \rangle, \langle x \rangle) < \mathrm{Sim}\,(q)$.

(2) *If* $\dim q = 4$, $\langle\!\langle a \rangle\!\rangle | q$, *and* $x \in G(q)$, *then* $(\langle 1, a \rangle, \langle x \rangle) < \mathrm{Sim}\,(q)$.

(3) *If* $\dim q = 4$, $\langle\!\langle a \rangle\!\rangle | q$, *and* $\langle\!\langle x \rangle\!\rangle | q$, *then* $(\langle 1, a \rangle, \langle 1, x \rangle) < \mathrm{Sim}\,(q)$.

Proof. First, scale q to assume q represents 1. (1) Since $(q, q) < \mathrm{Sim}\,(q)$ and q represents 1 and x, we are done. The proof of (2) is similar to that of (3) and is omitted. (3) If $\bar{a} = \bar{x}$, then $(\langle 1, a \rangle, \langle 1, x \rangle) < \mathrm{Sim}\,(\langle\!\langle a \rangle\!\rangle)$ and, since $\langle\!\langle a \rangle\!\rangle | q$, we are done. Suppose $\bar{a} \neq \bar{x}$. Now $q \simeq \langle\!\langle a, b \rangle\!\rangle$ for some b, and $\langle 1, x \rangle < q$, so that $x \in D(\langle a \rangle \perp \langle b, ab \rangle)$. Then $x = x_1 + x_2$, where $x_1 \in D(\langle a \rangle) \cup \{0\}$ and $x_2 \in D(\langle b, ab \rangle)$, (here $x_2 \neq 0$ since $\bar{a} \neq \bar{x}$). Therefore, $q \simeq \langle\!\langle a, x_2 \rangle\!\rangle$ and $\langle 1, x \rangle < \langle 1, a, x_2 \rangle$, so we are done since $(\langle 1, a, x_2 \rangle, \langle 1, a, x_2 \rangle) < \mathrm{Sim}\,(q)$.

REMARK 4.4. There is a different approach to (4.1) following Dieudonné [1]. He essentially proves the (1, 1)-family result by a clever application of the Witt extension theorem to find a decomposition of q into 2-planes. His argument can be directly generalized to cover the (2, 1)-family result as well.

Theorem (4.1) helps settle the odd factor conjecture for small familes

Odd factor conjecture 4.5 [12; (7.4)]. Suppose α, ω are quadratic forms and $\dim \omega$ is odd. Then, for an (s, t)-family:

$$(\sigma, \tau) < \mathrm{Sim}\,(\alpha \otimes \omega) \Rightarrow (\sigma, \tau) < \mathrm{Sim}\,(\alpha) \, .$$

COROLLARY 4.6. *The odd factor conjecture is true for* (s, t)-

families when $s+t \leqq 3$, *and also for* $(2, 2)$-*families* (σ, τ) *when* σ *and* τ *represent a common value.*

Proof. Apply (4.1), the note after (4.1), and the odd factor results for similarity factors and for Pfister factors, as in [13; §3, statement B].

PROPOSITION 4.7. *Suppose* (σ, τ) *is a pair with* $(\sigma, \tau) < \mathrm{Sim}(\varphi)$ *for some* φ *of dimension* 4. *Then the odd factor conjecture is true for* (σ, τ).

Proof. First, (4.6) applies if $s+t \leqq 3$, so suppose $s+t \geqq 4$. Also, by (1.6), assume $s \geqq t$. We may suppose φ represents 1, so φ is Pfister, since $s \geqq 2$. If $s \geqq 3$, then (3.4) (2) shows that either $s=4$, $t=0$, and $\sigma \cong \varphi$; or $s=3$, and $\tau < \sigma < \varphi$. Moreover in these cases: $(\sigma, \tau) < \mathrm{Sim}(q)$ iff $\varphi | q$, and we are done as in (4.6).

The remaining case is $s=t=2$, say $(\sigma, \tau) = (\langle 1, a\rangle, \langle x, y\rangle)$. If $\overline{axy} = \bar{1}$, then: $(\sigma, \tau) < \mathrm{Sim}(q)$ iff $(\langle 1, a\rangle, \langle x\rangle) < \mathrm{Sim}(q)$; and again (4.6) settles this case. Suppose $\overline{axy} \neq \bar{1}$. Then knowing the maximal families of $\mathrm{Sim}(\varphi)$, as in (2.2) and (3.4) (2), we see that (σ, τ) lies inside a $(3, 3)$-family $(\gamma, \gamma) < \mathrm{Sim}(\varphi)$. Since σ, τ are 2-planes in the 3-space γ, they must meet: they represent a common value. We are now done, by (4.6).

The odd factor conjecture for arbitrary fields is known only for a few more cases; namely, for (s, t)-families where $s+t$ is very large, as in (2.8) and [12; (8.7)]. The large gap between these results, and the failure of any attempts to generalize (4.7) lead us to the suspicion that the conjecture is false in general.

5. The odd factor conjecture. The conjecture will be establised for the forms $q \simeq n\langle 1\rangle = \langle 1, 1, \cdots, 1\rangle$ over the rational field $\boldsymbol{Q}$ (or any global field with at most one ordering). This result has been used in the theory of orthogonal designs [5], [6]. For more general forms over $\boldsymbol{Q}$, the odd factor conjecture remains an open question.

THEOREM 5.1. *Suppose* $n = 2^m \cdot n_0$, *where* n_0 *is odd. Let* σ, τ *be forms over* $\boldsymbol{Q}$, *where* σ *represents* 1. *Then*:

$$(\sigma, \tau) < \mathrm{Sim}(n\langle 1\rangle) \Leftrightarrow (\sigma, \tau) < \mathrm{Sim}(2^m\langle 1\rangle) .$$

LEMMA 5.2. *Let* q *be a positive definite form over* $\boldsymbol{Q}$ *of dimension* n, *with* $(\sigma, \tau) < \mathrm{Sim}(q)$ *an* (s, t)-*family. If* $s \geqq 3$ *and* $n \equiv 0 \pmod 8$, *then* $q \simeq n\langle 1\rangle$.

Proof. Suppose $\langle 1, a, b\rangle < \sigma$. Then, $q \simeq \langle\langle a, b\rangle\rangle \otimes \gamma$, for some γ with $\dim \gamma$ even. But, for any $a, b, x, y \in \boldsymbol{Q}^+$, $\langle\langle a, b\rangle\rangle \otimes \langle x, y\rangle \simeq 8\langle 1\rangle$, the only anisotropic 3-fold Pfister form over $\boldsymbol{Q}$. Therefore, $q \simeq n\langle 1\rangle$ as claimed.

Proof of (5.1). Suppose $(\sigma, \tau) < \mathrm{Sim}\,(n\langle 1\rangle)$ is an (s, t)-family. By (1.6) we may assume $s \geqq t$. By the decomposition theorem, $n\langle 1\rangle \simeq q_1 \perp \cdots \perp q_r$, where $\dim q_i = 2^k$, and $(\sigma, \tau) < \mathrm{Sim}\,(q_i)$. If $k \leqq 2$, we are done by (4.7), so assume $k \geqq 3$. If $s \geqq 3$, the lemma shows $q_1 \simeq 2^k\langle 1\rangle$, and the result follows since $m \geqq k$. If $s + t \leqq 3$, we are done by (4.6). When $s = t = 2$, the form $\alpha = \sigma \perp \tau$ has $\alpha \perp \langle d\alpha\rangle\alpha \simeq 8\langle 1\rangle$, since it is Pfister. Hence, $(\sigma, \tau) < (\alpha, \alpha) < \mathrm{Sim}\,(8\langle 1\rangle)$ and the result follows since $m \geqq 3$.

The standard trick of averaging a bilinear form to get an invariant form can be applied to Clifford algebras.

PROPOSITION 5.3. *Let F be an ordered field, σ, τ positive definite forms over F with $\sigma = \langle 1\rangle \perp \sigma_1$, and $C = C(\langle -1\rangle\sigma_1 \perp \tau)$ the associated Clifford algebra. If C has a representation on a vector space V, then there is a positive definite form q on V with $(\sigma, \tau) <$ $\mathrm{Sim}\,(V, q)$.*

Proof. Let $\sigma \simeq \langle 1, a_2, \cdots, a_s\rangle$, $\tau = \langle b_1, \cdots, b_t\rangle$ and let $\{e_2, \cdots, e_s, f_1, \cdots, f_t\}$ be the corresponding generators of C. Then, $e_i^2 = -a_i$, and $f_j^2 = b_j$. The set $\{e_\Delta f_\Gamma \mid \Delta \subseteqq \{2, \cdots, s\}$ and $\Gamma \subseteqq \{1, \cdots, t\}\}$ is an F-basis of C. Let B_0 be any positive definite form on V and use the C-action on V to define:

$$B(u, v) = \sum \frac{1}{a_\Delta b_\Gamma} B_0(e_\Delta f_\Gamma(u), e_\Delta f_\Gamma(v))\ ,$$

for $u, v \in V$. The sum runs over all $\Delta \subseteqq \{2, \cdots, s\}$ and $\Gamma \subseteqq \{1, \cdots, t\}$. Then B is also positive definite, and calculation shows that $I_B(e_i) = -e_i$ and $I_B(f_j) = f_j$. Therefore I_B is compatible with the involution of C and $(\sigma, \tau) < \mathrm{Sim}\,(V, B)$.

REMARK. Using (5.1), (5.2), (5.3), and (2.3) it is a straightforward exercise to determine exactly which forms σ, τ over $\boldsymbol{Q}$ have $(\sigma, \tau) < \mathrm{Sim}\,(n\langle 1\rangle)$. This result answers some questions raised by A. Geramita and W. Wolfe in the theory of orthogonal designs [5], [16], [6].

Next, we describe an attempt at proving the odd factor conjecture for all forms over $\boldsymbol{Q}$. We seem to need a strong version of the Pfister factor conjecture which is not always true. The next result is true over any field where I^3F is a principal ideal.

PROPOSITION 5.4. *Suppose σ, τ are forms over $\boldsymbol{Q}$, where σ represents* 1, *such that: whenever $(\sigma, \tau) < \mathrm{Sim}\,(q)$ is unsplittable, then q is similar to a Pfister form. Then, the odd factor conjecture over $\boldsymbol{Q}$ is true for (σ, τ).*

Proof. By (4.7), we may assume that the unsplittable form q has dimension at least 8. Suppose $(\sigma, \tau) < \mathrm{Sim}\,(\alpha \otimes \omega)$, where $\dim \omega$ is odd. The decomposition theorem gives $\alpha \otimes \omega \simeq q_1 \perp \cdots \perp q_r$, where $\dim q_i = 2^m$ and $(\sigma, \tau) < \mathrm{Sim}\,(q_i)$ is unsplittable. By hypothesis, each q_i is similar to a Pfister form. If all the q_i are similar, say $q_i \simeq \langle a_i \rangle \psi$, then $\psi | \alpha \otimes \omega$. By the Pfister factor results [13; §3], $\psi | \alpha$ and we get $(\sigma, \tau) < \mathrm{Sim}\,(\alpha)$ as claimed. Suppose that the q_i are not all similar.

The only m-fold Pfister forms over $\boldsymbol{Q}$, for $m \geqq 3$, are $2^m\langle 1 \rangle$ and $2^{m-1}\boldsymbol{H}$, (an easy fact, [8; p. 127]). Therefore each q_i has $2^{m-1}\langle 1 \rangle$ as a tensor factor, and $2^{m-1}\langle 1 \rangle | \alpha \otimes \omega$. The Pfister factor results [13; §3] imply $2^{m-1}\langle 1 \rangle | \alpha$. Replacing α by $\langle -1 \rangle \alpha$ if necessary, we can express $\alpha \simeq l \cdot 2^{m-1}\langle 1 \rangle \perp l' \cdot 2^{m-1}\boldsymbol{H}$, for some integers $l, l' \geqq 0$. Dimension count shows l is even, and α is a sum of copies of $2^m\langle 1 \rangle$ and $2^{m-1}\boldsymbol{H}$. Since the q_i are not all similar, both of these spaces admit (σ, τ) as an (s, t)-family. Therefore, $(\sigma, \tau) < \mathrm{Sim}\,(\alpha)$ as claimed.

The next step should be to prove that strong version of the Pfister factor conjecture over $\boldsymbol{Q}$. We can settle several cases:

PROPOSITION 5.5. *Let F be a field where I^2F is linked. Suppose the form β can be realized as an (s, t)-family on* Sim (V, q) *making (V, q) unsplittable. Then q must be similar to a Pfister form, except possibly in the case: $c(\beta) \neq \bar{1}$, $d_{\pm}\beta \neq 1$, and* $\dim \beta \equiv 2t \pmod 4$.

Proof. By (3.10), the Pfister factor conjecture (3.1) is true over F. Suppose $\dim q = 2^m$. If $\dim \beta \geqq 2m - 1$, the methods of §2 show that q admits an $(m + 1, m + 1)$-family, and then q must be similar to a Pfister form.

Let $C = C(\langle -1 \rangle \beta_1)$. If the C-module V is not irreducible, then C must be of hyperbolic type [11; (3.6), (3.12)], and (V, q) is hyperbolic. Otherwise, V is irreducible. Since I^2F is linked, $c(\beta)$ is quaternion. By computing dimensions of irreducible C-modules, it turns out that $\dim \beta \geqq 2m - 1$ except in the case when $\dim \beta$ is even, $c(\beta) \neq 1$, $d_{\pm}\beta = \bar{\delta} \neq \bar{1}$. In that case, $\dim \beta = 2m - 2$, and C is simple with center $Z \cong F(\sqrt{\delta})$. Then C is a central simple Z-

algebra acting on the Z-vector space V, so that $C \otimes_Z D \cong \mathrm{End}_Z(V)$, for some central simple Z-algebra D. Dimension count shows D is a quaternion algebra. Since $[C] = [D]$ in the Brauer group $B(Z)$, we have $D \cong D' \otimes_F Z$, where D' is a quaternion algebra over F.

The adjoint involution I on $\mathrm{End}_F(V)$ induces the given (s, t)-involution J on C and also induces an involution K on D. Suppose $J(z) = -z$; that is, $m \equiv t \pmod 2$. Then K is an involution of the "second kind" on D and there exists an F-form D' of D preserved by K, [7; p. 40]. Therefore, there are elements $h_1, h_2 \in D$ with $h_1h_2 + h_2h_1 = 0$, $h_i^2 \in F^\times$, and $K(h_i) = \pm h_i$. These $h_i \in \mathrm{End}(V)$ commute with C. Now, pull out one element f of the given (s, t)-family with $I(f) = -f$, and replace it with $\{fh_1, fh_2\}$. This gives a family of size $2m - 1$, forcing q to be Pfister, as before.

The smallest bad case is that of a $(2, 2)$-family $(\langle 1, a\rangle, \langle x, y\rangle)$ where $C = C(\langle -a, x, y\rangle)$ is a division algebra. This occurs iff $\overline{axy} \neq \bar{1}$ and $\langle 1, a, -x, -y\rangle$ is anisotropic. We can show that (5.5) actually fails over $\boldsymbol{Q}$ in this case.

Let J be the involution for this splitting, and define a form B_0 on C by: $B_0(u, v) = l_1(J(u)\cdot v)$, for $u, v \in C$, as in [12; §9]. The basis of C derived from the generators $\{e_2, f_1, f_2\}$, as in (5.3), is an orthogonal basis, and: $(C, B_0) \cong \langle\langle a, x, y\rangle\rangle$. If $(\langle 1, a\rangle, \langle x, y\rangle) < \mathrm{Sim}(V, B)$, where $\dim V = 8$, we can identify the left C-modules C and V, and see that the form B on V becomes B_0^c, for some $c \in C^\times$ with $J(c) = c$. Here, $B_0^c(u, v) = B_0(u, vc)$.

For any choice of c, the 8×8 symmetric matrix of the form B_0^c can be computed. We chose $c = 1 - f_1 - z$ and diagonalized the resulting matrix to find: $(\langle 1, a\rangle, \langle x, y\rangle) < \mathrm{Sim}(q)$, where

$$q \simeq \langle\langle a\rangle\rangle \otimes \langle 1, -x(x-1), y(x-1)(x-1+axy), -xy(x-1+axy)\Gamma\rangle ,$$

where $\Gamma = (x-1)^2 - 2(x+1)axy + (axy)^2$, (assuming all these values are nonzero). This form is Pfister iff $\Gamma \in D(\langle\langle a\rangle\rangle)$. Examples of non-Pfister behavior are quickly found. Using $a = 1$, $x = 16/7$, $y = 2/7$, we see that $(\langle 1, 1\rangle, \langle 7, 14\rangle) < \mathrm{Sim}(q)$, where $q \simeq \langle\langle 1\rangle\rangle \otimes \langle 1, -1, 19, 19\cdot 47\rangle$. This example shows that (5.5) cannot be improved; but still no counter-examples to the odd factor conjecture are known.

From the construction in [11; (4.9)] it follows that, if $(\sigma, \tau) < \mathrm{Sim}(q)$ and if $(\alpha, \beta) < \mathrm{Sim}(p)$, where $\alpha = \langle 1\rangle \perp \alpha_1$ and $\beta = \langle 1\rangle \perp \beta_1$, then $(\sigma \perp \alpha_1, \tau \perp \beta_1) < \mathrm{Sim}(q \otimes p)$. We apply this to the $(2, 2)$-family obtained above, using $p = 2^m\langle 1\rangle$, and $\alpha = \beta = (m+1)\langle 1\rangle$, when $m \geqq 1$, to get:

$$((m+2)\langle 1\rangle, \langle 7, 14\rangle \perp m\langle 1\rangle) < \mathrm{Sim}(\psi)$$

where $\psi \simeq 2^{m+1}\boldsymbol{H} \perp 2^{m+2}\langle 1\rangle$. Furthermore, by (2.3)(3), this family cannot be realized on a space of smaller dimension.

For example, when $m = 2$, we can shift $\langle 1, 1, 7, 14\rangle$ to the left, to obtain an 8-plane $\sigma \simeq 6\langle 1\rangle \perp \langle 7, 14\rangle$ with $\sigma < \mathrm{Sim}\,(8\boldsymbol{H} \perp 16\langle 1\rangle)$, but σ not embeddable in $\mathrm{Sim}\,(8\boldsymbol{H})$ or $\mathrm{Sim}\,(16\langle 1\rangle)$. This provides an explicit counterexample to the conjecture (7.3) of [12].

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