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ON SUBDIRECTLY IRREDUCIBLE COMMUTATIVE SEMIGROUPS

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We give a characterization of finitely generated commutative semigroups which are finitely subdirectly irreducible.

We call a semigroup S finitely subdirectly irreducible, or just irreducible, if it has the following property: when S is a subdirect product of finitely many semigroups S_i, one of the projections $S \to S_i$ must be an isomorphism. These semigroups are of interest because every finitely generated commutative semigroup is a subdirect product of finitely many irreducible semigroups. (The factors in this decomposition are then finitely generated and commutative as well as irreducible.)

Let S be a finitely generated commutative semigroup. If S is irreducible then S is either cancellative or a nilsemigroup or what we call a subelementary semigroup, i.e., the disjoint union $S = N \cup C$ of a nilsemigroup N which is also an ideal of S, and a subsemigroup $C \neq \varnothing$ every element of which is cancellative in S. The first two cases are easily dealt with and our new results are in the subelementary case. This case reduces to the other two if N or C is trivial. If $S = N \cup C$ is subelementary with N, C nontrivial, then the irreducibility of S is equivalent to four simple conditions of an elementary nature. A second characterization is also given, as follows. If S is subelementary, then S can be completed to an elementary semigroup (=a subelementary semigroup whose cancellative part C is a group); this does not affect irreducibility. Elementary semigroups can in turn be constructed by coextension techniques; in the case under consideration the construction is in terms of a group, a finite nilsemigroup and a factor set. A characterization of irreducible semigroups is also given in terms of this construction.

These results specialize immediately to a characterization of finitely generated commutative semigroups which are subdirectly irreducible. These semigroups must be finite and classify into groups, nilsemigroups and elementary semigroups. In the elementary case, our two characterizations and construction from groups and nilsemigroups are still new. In particular they go far deeper than Schein's results in [14], as one could expect since our results are also considerably less general.

The proofs of our results depend on techniques of congruence construction which were essentially developed by Schein in [14].

The paper is arranged in three sections. The first section contains whatever basic facts about subelementary semigroupe are needed in the paper; the reader is referred to [6] and [5] for more details. Our study of irreducible semigroups is conducted in §§2 and 3: the easier results are given in §2; §3 deals with the subelementary case.

We denote by $\boldsymbol{Z}$ the additive group of all integers; by $\boldsymbol{Z}(p^n)$ a cyclic group of order p^n; by N the additive semigroup of nonnegative integers. Otherwise the notation is as in [2].

The results were announced in [4].

1. Subelementary semigroups.

1. A *subelementary* semigroup is a commutative semigroup S which is the disjoint union $S = N \cup C$ of an ideal N which is a nilsemigroup, and a subsemigroup $C(\neq \varnothing)$ every element of which is cancellative in S. If C is a group S is an *elementary* semigroup; the terminology was introduced in this case by Ponizovskiĭ [13]. In every subelementary semigroup $S \cup N = C$, the zero element of N is also a zero element of S and the identity element of C, if any, is an identity element of S. A finite commutative semigroup is subelementary if and only if it is elementary, if and only if it has identity and zero elements and no other idempotent.

A subelementary semigroup can always be completed into an elementary semigroup. More generally, let S be a commutative semigroup and $C \neq \varnothing$ be a subsemigroup of S, every element of which is cancellative in S. A semigroup of fractions $C^{-1}S$ can then be constructed as follows. The elements of $C^{-1}S$ are all fractions s/c with $s \in S$, $c \in C^1$, with $s/c = t/d$ if and only if $ds = ct$ [one may require $c \in C$ without changing $C^{-1}S$]; multiplication is well-defined by $(s/c)(t/d) = st/cd$. The homomorphism $s \mapsto s/1$ of S into $C^{-1}S$ is injective; it is convenient to identify s and $s/1$, so that S becomes a subsemigroup of $C^{-1}S$. If $S = C$ is cancellative, then $C^{-1}S = S^{-1}S$ is the group of fractions, or universal group, of S.

If $S = N \cup C$ is subelementary, then $C^{-1}S$ is elementary (with group part $C^{-1}C$; see [6]).

2. Any commutative nilsemigroup N can be partially ordered, by: $x \leqq y$ if and only if $x = uy$ for some $u \in N^1$. Sice each $u \in N$ is nilpotent, $ux < x$ holds for all $x \in N\backslash 0$. If $S = N \cup C$ is subelementary, then $x \leqq y$ implies $sx \leqq sy$ for all $s \in S$. However, when $x \in N$, $x \neq 0$, no two elements of Cx are comparable in N; more generally, two elements $x \neq y$ of N such that $cx = dy \neq 0$ for some

$c, d \in C^1$ are never comparable. If indeed $y = ux$ for some $u \in N$, there is a greatest n with $u^n x \neq 0$, and then $cu^n x = du^n y = du^{n+1}x = 0 = c0$, which is impossible since c is cancellative in S.

When $S = N \cup C$ is subelementary, the relation $\equiv$ on S defined by: $a \equiv b$ if and only if $ca = db$ for some $c, d \in C^1$, is a congruence on S. The class of $a \in S$ modulo $\equiv$ is the *orbit* of a and will be denoted by Ω_a; the quotient semigroup $S/\equiv$ is the *semigroup of orbits* of S and will be denoted by Ω^1. We see that C itself is an orbit, so that Ω^1 is a nilsemigroup with identity adjoined; we let Ω denote the semigroup of nilpotent orbits. The terminology comes from the elementary case. If $S = N \cup G$ is elementary, the group G acts on S by multiplication; we see that $a \equiv b$ if and only if $a = gb$ for some $g \in G$, so that the orbits in S are precisely the orbits under the action of G. In general, S and $C^{-1}S$ have isomorphic semigroups of orbits: more precisely, every orbit of $C^{-1}S$ intersects S, and this yields all the orbits of S (see [6]).

We have seen that, when $x \in N$, no two elements of Ω_x are comparable. The projection $x \mapsto \Omega_x$, $N \mapsto \Omega$, is order-preserving. In fact, $x < y$ in N implies $\Omega_x < \Omega_y$ in Ω (otherwise $\Omega_x = \Omega_z$ contains comparable elements). In particular, x is a minimal element of N if and only if Ω_x is a minimal element of Ω.

3. Additional properties can be obtained when the subelementary semigroup $S = N \cup C$ is finitely generated. We state these as:

PROPOSITION 1.1. *Let $S = N \cup C$ be a finitely generated subelementary semigroup. Then C is finitely generated; $C^{-1}S$ is finitely generated; Ω is finite; N has finite height, in particular is nilpotent; the set M of all minimal elements of N is a nonempty union of orbits* (the union of all minimal elements of Ω); *in particular, for all $c \in C$, $m \in M$ if and only if $cm \in M$.*

Proof. Since N is an ideal, C is generated by all the generators of S that lie in C, hence is finitely generated. So is $C^{-1}S$, which is generated by S and the inverses (in $C^{-1}S$) of the elements of C. Next, Ω^1 is a homomorphic image of S and hence is also finitely generated; since Ω^1 is a nilsemigroup with identity adjoined, it must be finite. The implication $x > y$ implies $\Omega_x > \Omega_y$ shows that no chain in N has more elements than Ω, and thus N has finite height h. This implies $N^h = 0$: if $x_1, x_2, \cdots, x_h \in N$ and $x_1 x_2 \cdots x_h \neq 0$, then $x_1 > x_1 x_2 > \cdots > x_1 x_2 \cdots x_h > 0$ would be a chain of length $h + 1$. Since N has finite height, we have $M \neq \varnothing$ (unless $N = 0$); since $m \in M$ if and only if Ω_m is minimal, M is the union of all minimal orbits. When $c \in C$, $m \equiv cm$ and hence $m \in M$ if and only if $cm \in M$.

Note that in the above N itself need not be finitely generated (i.e., finite). For instance, let $S = N \cup G$, where $N = \{0\} \cup \{x_i;\ i \in \boldsymbol{Z}\}$ be a null semigroup and $G = \{a^i;\ i \in \boldsymbol{Z}\} \cong \boldsymbol{Z}$, with G acting on N by: $a^i x_j = x_{i+j}$. We see that S is elementary, and generated by a, a^{-1} and x_0; but N is not finitely generated.

4. An elementary semigroup $S = N \cup G$ is *homogeneous* in case G acts regularly on $S \backslash 0$, i.e., $gx = x \neq 0$ implies $g = 1$. If G is finite, this means that all nonzero orbits of S have the same order (namely, the order of G). These semigroups were called equisected in [4]. By extension we call a subelementary semigroup $S = N \cup C$ *homogeneous* if its elementary semigroup $C^{-1}S$ is homogeneous; equivalently, if $cx = dx \neq 0$ implies $c = d$ when $c, d \in C^1$.

When $S = N \cup G$ is homogeneous, the extension theory in [5], [11] provides a construction of S in terms of G and Ω, which is particularly satisfactory in its cohomological aspects [7]. We complete this section with a brief account of the construction itself.

Let Ω be a commutative nilsemigroup, G be an abelian group, and $\sigma = (\sigma_{\alpha,\beta})$ be a family of elements of G, with $\sigma_{\alpha,\beta}$ defined whenever $\alpha, \beta \in \Omega$ and $\alpha\beta \neq 0$. Call σ a *factor set* on Ω with values in G when

$$\begin{cases} \sigma_{\alpha,\beta} = \sigma_{\beta,\alpha} & \text{whenever} \quad \alpha\beta \neq 0\,, \\ \sigma_{\alpha,\beta}\sigma_{\alpha\beta,\gamma} = \sigma_{\alpha,\beta\gamma}\sigma_{\beta,\gamma} & \text{whenever} \quad \alpha\beta\gamma \neq 0\,. \end{cases}$$

When σ is a factor set, an elementary semigroup $S = [\Omega, G, \sigma]$ can be built as follows. The group part of S is G. The nil part N of S is the set consisting of an element 0 and of all pairs (g, α) with $g \in G$, $\alpha \in \Omega$, $\alpha \neq 0$. Multiplication in N is defined thus: 0 is a zero element, and

$$(g, \alpha)(h, \beta) = \begin{cases} (g\sigma_{\alpha,\beta}h,\ \alpha\beta) & \text{if} \quad \alpha\beta \neq 0\,, \\ 0 & \text{if} \quad \alpha\beta = 0\,. \end{cases}$$

The action of G on N is given by: $g0 = 0$, $g(h, \alpha) = (gh, \alpha)$. It is not hard to see that S is indeed a homogeneous elementary semigroup, with semigroup of orbits Ω^1.

Conversely, let $S = N \cup G$ be homogeneous elementary, with semigroup of orbits Ω^1. Pick one element p_α in each nonzero nilpotent orbit $\alpha \in \Omega$. Since S is homogeneous, every element $x \neq 0$ of N can be written uniquely as $x = gp_\alpha$ with $g \in G$, $\alpha \in \Omega$, $\alpha \neq 0$. In particular, when $\alpha\beta \neq 0$ in Ω, we have $p_\alpha p_\beta = \sigma_{\alpha,\beta} p_{\alpha\beta}$ for some unique $\sigma_{\alpha,\beta} \in G$. It is not hard to see that σ is a factor set and that $S \cong [\Omega, G, \sigma]$.

In the above we call σ a *factor set associated with S*. Note that σ depends on the choice of the elements p_α and hence is not uniquely determined by S. It is easy to see that any two factor sets associated with S must differ by a factor set of the form $\sigma_{\alpha,\beta} = u_\alpha + u_\beta - u_{\alpha\beta}$ $(\alpha\beta \neq 0)$, where $u = (u_\alpha)_{\alpha \in \Omega\backslash 0}$ is a family of elements of G; we call a factor set of this form *trivial*.

A homogeneous elementry semigroup $S = N \cup G$ *splits* in case the projection $f: N \to \Omega$ is a retraction, i.e., there exists a homomorphism $p: \Omega \to N$ such that fp is the identity on Ω. If S splits, we may choose $p_\alpha = p\alpha$ in the above; this yields $\sigma_{\alpha,\beta} = 0$ whenever $\alpha\beta \neq 0$, so that $S \cong [\Omega, G, 0]$. Conversely, it is easy to see that $[\Omega, G, \sigma]$ splits if and only if σ is trivial (as defined above).

2. Irreducible semigroups.

1. Recall that a semigroup S is a subdirect product of the semigroups $(S_i)_{i \in I}$ if there is an injective homomorphism $S \to \prod_{i \in I} S_i$ such that the induced projection $S \to S_i$ is surjective for all $i \in I$. The projections $S \to S_i$ induce congruences $\underline{C}_i$ $(i \in I)$ on S whose intersection is the equality on S; if conversely the equality on S is the intersection of congruences $(\underline{C}_i)_{i \in I}$ on S, then S is a subdirect product of the quotient semigroup $(S/\underline{C}_i)_{i \in I}$. A semigroup S is subdirectly irreducible if it has more than one element and if, in every subdirect decomposition of S into semigroups $(S_i)_{i \in I}$, some projection $S \to S_i$ must be an isomorphism; equivalently, S is subdirectly irreducible if S is not trivial and the equality on S is completely $\cap$-irreducible, i.e., is not the intersection of congruences on S all different from the equality. A classic theorem of Birkhoff implies that every commutative semigroup is a subdirect product of subdirectly irreducible [commutative] semigroups (see for instance [3]).

We call a [commutative] semigroup S *finitely subdirectly irreducible* if it is not trivial and if, in every subdirect decomposition of S into finitely many semigroups $(S_i)_{i \in I}$, some projection $S \to S_i$ must be an isomorphism; equivalently, if S is not trivial and the equality on S is $\cap$-irreducible, i.e., is not the intersection of finitely many congruences on S all different from the equality. A subdirectly irreducible semigroup is evidently irreducible, but not conversely; for instance, $\boldsymbol{Z}$ is irreducible yet is a subdirect product of all $\boldsymbol{Z}(p)$ with p prime.

PROPOSITION 2.1. *Every finitely generated commutative semigroup S is a subdirect product of finitely many irreducible semigroups.*

Proof. One form of Rédei's theorem [2] states that the congruences on a finitely generated free commutative semigroup F satisfy the ascending chain condition. Since S is isomophic to a quotient of some such F, the congruences on S also satisfy the ascending chain condition. An easy Noetherian induction then implies that every congruence on S is the intersection of finitely many $\cap$-irreducible congruences. [If this is not true, there is a congruence $\underline{C}$ on S which is maximal with the property of not lying in the set R of finite intersections of $\cap$-irreducible congruences. In particular $\underline{C}$ is not $\cap$-irreducible, and is the intersection of finitely many congruences $\underline{C}_i$ greater than $\underline{C}$. By the maximality of $\underline{C}$, every $\underline{C}_i$ is in R. However, R is closed under finite intersections, so that $\underline{C} \in R$, a contradiction.] In particular, the equality on S is the intersection of finitely many $\cap$-irreducible congruences $\underline{C}_i$; one may assume that no $S/\underline{C}_i$ is trivial. Since $\underline{C}_i$ is $\cap$-irreducible, the equality S/C_i on is $\cap$-irreducible, and $S/\underline{C}_i$ is irreducible; and S is a subdirect product of the finitely many semigroups $S/\underline{C}_i$.

By contrast, we shall see that, when S is finitely generated but not finite, a subdirect decomposition of S into subdirectly irreducible semigroups necessarily has infinitely many terms.

The *irreducible semigroups in the decomposition of* S *in* 2.1 *are* homomorphic images of S and therefore are also finitely generated [and commutative].

2. Another subdirect decomposition was given in [6]. When S is a finitely generated commutative semigroup, Propositions 1.6, 1.7 of [6] imply that S is a subdirect product product of finitely many semigroups that are cancellative, nil or subelementary. Hence:

PROPOSITION 2.2. *A finitely generated commutative irreducible semigroup is cancellative, nil or subelementary.*

The proof in [6] used primary decomposition in the Noetherian ring $C[S]$. Since 2.2 is fundamental for what follows, we give ar alternate proof, which uses congruences much as in [9].

Assume that S is finitely generated and commutative, so that the congruences on S satisfy the ascending chain condition, and is irreducible, so that the equality on S is $\cap$-irreducible. Take any element c of S. For each $k > 0$, define a congruence $\underline{C}_k$ on S by: $x\underline{C}_k y$ if and only if $c^k x = c^k y$. Then $\underline{C}_1 \subseteqq \underline{C}_2 \subseteqq \underline{C}_3 \cdots$ and hence $\underline{C}_n = \underline{C}_{n+1}$ for some $n > 0$. Let $\underline{R}$ be the Rees congruence of the ideal $c^n S$, i.e., $x\underline{R}y$ if and only if $x = y$ or $x, y \in c^n S$. If $x, y \in c^n S$ and $x\underline{C}_1 y$, then $cx = cy$; also $x = c^n u$, $y = c^n v$ for some $u, v \in S$; hence $u\underline{C}_{n+1}v$, so that $u\underline{C}_n v$ and $x = y$. It follows that $\underline{R} \cap \underline{C}_1$ is the equality. Therefore, either $\underline{C}_1$ is the equality, in which case $a = b$, or $\underline{R}$ is the equality, in which case $c^n S$ is a trivial ideal of S, so that S has a zero element and $c^n = 0$. Thus, either c is cancellative in S, or S has a zero element and c is nilpotent. Note that a nilpotent element cannot be cancellative in S, and that (when S has a zero) the nilpotent elements form an ideal of S. It is then immediate that S is cancellative or nil or subelementary.

3. The following result will be used in the cancellative and subelementary cases. Assume that $S \neq \{1\}$ is commutative and has a subsemigroup C, every element of which is cancellative in S. Then:

PROPOSITION 2.3. *S is irreducible if and only if $C^{-1}S$ is irreducible.*

Proof. Let $\underline{A}$ be a congruence on $C^{-1}S$ which is the equality on S. When $a, b \in C^{-1}S$, there exists $c \in C^1$ such that $ca, cb \in S$. If $a\underline{A}b$, then $ca\underline{A}cb$, so that $ca = cb$ and $a = b$. Therefore $\underline{A}$ is the equality. [In other words, S is dense in $C^{-1}S$, as defined in [8].]

Now assume that S is irreducible. Let the equality on $C^{-1}S$ be the intersection of finitely many congruences $\underline{A}_i$. The equality on S is then the intersection of the finitely many congruences on S induced by the congruences $\underline{A}_i$; therefore some $\underline{A}_i$ must be the equality on S; by the above, some $\underline{A}_i$ is then the equality on $C^{-1}S$. Thus $C^{-1}S$ is irreducible.

For the converse we construct from each congruence $\underline{A}$ on S a congruence $\underline{A}^*$ on $C^{-1}S$, thus: let $a\underline{A}^*b$ if and only if $ca, cb \in S$ and $ca\underline{A}cb$ for some $c \in C^1$; it is immediate that $\underline{A}^*$ is indeed a congruence. We see that $\underline{A}^*$ is the equality on $C^{-1}S$ if and only if $\underline{A}$ is the equality on S. Furthermore let $(\underline{A}_i)_{i \in I}$ be a finite family of congruences on S, and $\underline{A} = \bigcap_{i \in I} \underline{A}_i$. Assume $a\underline{A}_i^*b$ for all i. For each i, then $c_ia, c_ib \in S$ and $c_ia\underline{A}_ic_ib$ for some $c_i \in C^1$. The product c of all c_i then satisfies $ca, cb \in S$ and $ca\underline{A}_icb$ for all i. Therefore $a\underline{A}^*b$. Conversely, $a\underline{A}^*b$ implies $a\underline{A}_i^*b$ for all i, since $\underline{A} \subseteqq \underline{A}_i$. Thus, $(\bigcap_{i \in I} \underline{A}_i)^* = \bigcap_{i \in I} \underline{A}_i^*$.

Now assume that $C^{-1}S$ is irreducible. Let the equality on S be the intersection of finitely many congruences $\underline{A}_i$. By the above the equality on $C^{-1}S$ is the intersection of the finitely many congruences $\underline{A}_i^*$. Therefore some $\underline{A}_i^*$ must be the equality on $C^{-1}S$ and the corresponding $\underline{A}_i$ is the equality on S. Thus S is irreducible.

PROPOSITION 2.4. *Assume that S is not trivial. Then S is irreducible if and only if S^0 is irreducible, if and only if S^1 is irreducible.*

Proof. Assume that S does not have a zero. Let $\underline{A}$ be a congruence on S^0 whose restriction to S is the equality. If $\underline{A}$ is not the equality, then $0\underline{A}z$ for some $z \in S$; for all $x \in S$, $xz\underline{A}x0 = 0\underline{A}z$ and $xz = z$; this contradicts the hypothesis on S. Therefore $\underline{A}$ is the equality. As in the proof of 2.3, it follows that when S is irreducible then so is S^0.

For each congruence $\underline{A}$ on S define a congruence $\underline{A}^0$ on S^0 by: $x\underline{A}^0y$ if and only if $x = y = 0$ or $x, y \in S$, $x\underline{A}y$. It is clear that this contruction preserves intersections. Furthermore, $\underline{A}_0$ is the equality on S^0 if and only if $\underline{A}$ is the equality on S. As in the proof of 2.3, it follows that when S^0 is irreducible then so is S. The proof is similar for S^1.

The same arguments show that, when S is not trivial, S is subdirectly irreducible if and only if S^0 is subdirectly irreducible, and similarly for S^1. In 2.3, if S is subdirectly irreducible then so is $C^{-1}S$. (The converse is true in all cases considered hereafter but we do not have a general proof.)

4. We now investigate the three kinds of finitely generated irreducible semigroups provided by 2.2. The first two kinds are easily disposed of.

PROPOSITION 2.5. *Let $S \neq \{1\}$ be a cancellative, finitely generated commutative semigroup. Then S is irreducible if and only if it is isomorphic to $\boldsymbol{Z}$, $\boldsymbol{Z}(p^n)$* (with p prime), *or to a subsemigroup of $\boldsymbol{N}$.*

Proof. By 2.3, S is irreducible if and only if $S^{-1}S$ is. Now $S^{-1}S$ is a finitely generated abelian group, and is irreducible if and only if it is isomorphic to $\boldsymbol{Z}$ or to $\boldsymbol{Z}(p^n)$ (with p prime). If $S^{-1}S \cong \boldsymbol{Z}(p^n)$ then S is finite and hence $S = S^{-1}S \cong \boldsymbol{Z}(p^n)$. If $S^{-1}S \cong \boldsymbol{Z}$, then S is isomorphic to a subsemigroup T of $\boldsymbol{Z}$ such that $T - T = \boldsymbol{Z}$. If T does not contain negative integers, or does not contain positive integers, then it is isomorphic to a subsemigroup of $\boldsymbol{N}$.

Assume that T contains both positive and negative integers. Note that T contains arbitrarily large positive integers and arbitrarily large negative integers. Since $T - T = \boldsymbol{Z}$ contains 1, there also exists $a \in T$ with $a + 1 \in T$; we can always add a sufficiently large integer $c \in T$ to a, and therefore may assume $a > 0$. Then T contains every integer $k \geqq a^2$: indeed $k = qa + r$ with $0 \leqq r < a, q \geqq a > r$ and therefore $k = (q - r)a + r(a + 1) \in T$. [This argument also shows that if T does not contain negative integers then $\boldsymbol{N}\backslash T$ is finite.]

For any $b \in T$, T contains all integers $m \geqq b + a^2$; if we let b be negative sufficiently large, this includes -1 and $+1$, and therefore $T = \boldsymbol{Z}$.

This proof also shows that every subsemigroup T of N is finitely generated. As above, we may assume that $T - T = \boldsymbol{Z}$, so that T contains integers a and $a + 1$, and hence all integers $k \geqq a^2$, which are generated by a and $a + 1$. Therefore T is generated by all elements $l < a^2$ of T.

5. Now let $S = N \neq 0$ be a finitely generated commutative nilsemigroup; this forces N to be finite. Since N only has finitely many congruences, it is irredudible if and only if it is subdirectly irreducible. A characterization of these irreducible semigroups can therefore be found in [14] (Corollary 4.6.1). We give below a slight restatement of this result and, for the reader's convenience, a direct proof. Call a congruence $\underline{C}$ on the nilsemigroup N *pure* in case $\{0\}$ is a class modulo $\underline{C}$. We denote by $\underline{P}$ the upper Teissier congruence of $\{0\}$ [15]; $\underline{P}$ is the greatest pure congruence on S, and is given by: $x\underline{P}y$ if and only if $\{u \in N^1;\ ux = 0\} = \{u \in N^1;\ uy = 0\}$.

PROPOSITION 2.6 (Schein [14]). *A commutative nilsemigroup N of finite height is irreducible if and only if $N \neq 0$ and its congruence $\underline{P}$ is the equality.*

Proof. Assume $N \neq 0$ throughout, so that the set M of minimal elements of N is nonempty. Assume now that N is irreducible. To each $m \in M$ there corresponds the ideal $\{0, m\}$ of N. If M has two or more elements, the Rees congruences of the corresponding ideals intersect into the equality on N; since N is irreducible this cannot happen, and hence M has exactly one element m. Now 0 and m are not equivalent modulo $\underline{P}$; the intersection of $\underline{P}$ and the Rees congruence of $\{0, m\}$ is therefore the equality; hence $\underline{P}$ is the equality.

Conversely, assume that $\underline{P}$ is the equality. Since all elements of M are equivalent modulo $\underline{P}$, we again have $M = \{m\}$. Let $\underline{C}$ be a congruence on N which is not the equality: there exist $x, y \in N$ with $x\underline{C}y$, $x \neq y$. Since $x\underline{P}y$ then does not hold, there exists $u \in N^1$ with, say, $ux \neq 0$, $uy = 0$. Since each nonzero element of N lies above some minimal element, we have $ux \geqq m$ and there exists $v \in N^1$ such that $vux = m$. It follows that $m = vux\underline{C}vuy = 0$. Thus 0 and m are equivalent modulo every congruence on N different from the equality; therefore the equality on N is not an intersection of such congruences.

COROLLARY 2.7. *An irreducible finite commutative nilsemigroup has precisely one minimal element.*

The converse of 2.7 does not hold. This is shown by:

EXAMPLE 2.8. Let $N = \{a, b, m, 0\}$, with multiplication given by: $a^2 = ab = b^2 = m$ and all other products are 0 (in particular $N^3 = 0$ and N is a semigroup). We see that N is a finite commutative nilsemigroup. The order relation on N is: $a, b, m > 0$, $a > m$, $b > m$; there are no other comparable pairs. In particular m is the sole minimal element of N. But we see that $\{u \in N^1; ua = 0\} = \{m, 0\} = \{u \in N^1; ub = 0\}$; by 2.6, N is not irreducible.

3. The subelementary case. 1. We now consider the third kind of irreducible semigroups in 2.2: throughout this section, $S = N \cup C$ is a finitely generated subelementary semigroup.

If C is trivial, then $S = N^1$ is finite, and 2.4, 2.6 tell when S is irreducible. The case when $N = 0$ is similarly taken care of by 2.4, 2.5. In what follows we further assume that $N \neq 0$ and C is not trivial. [In view of 2.3 we could also assume that S is in fact elementary; but this is of no particular advantage for most of the proofs.] Then it follows from 1.1 that C is finitely generated, N has finite height, and $M \neq \varnothing$.

2. We denote by $\underline{M}$ the congruence on N defined by: $x\underline{M}y$ if and only if, for all $u \in N^1$, $ux \in M$ is equivalent to $uy \in M$ and implies $ux = uy$; $\underline{M}$ is the intersection of the upper Teissier congruences of the elements of M [15]. We call N *weakly irreducible* in case $\underline{M}$ is the equality. Since every nonzero element of N has a minimal multiple, we see that $\underline{M}$ is a pure congruence. In particular, $\underline{M} \subseteq \underline{P}$; if N is irreducible then by 2.6 it is weakly irreducible. The converse does not hold: the null semigroup with three elements is weakly irreducible, but, by 2.7, not irreducible. However, we have the following result (which will not be used later):

PROPOSITION 3.1. *A commutative nilsemigroup N of finite height is irreducible if and only if it is weakly irreducible and has only one minimal element.*

Proof. Both conditions are necessary, as we have seen. Now assume that N is weakly irreducible and has only minimal element m. Take $x, y \in N$ with $x \neq y$. Then $x\underline{M}y$ does not hold, so that there exists $u \in N^1$ such that, say, $ux = m$, $uy \neq m$. If $uy \neq 0$, then uy lies above some minimal element, hence $uy > m$; hence $m = vuy$

for some $v \in N$; since $vux = vm = 0$, we see that $x\underline{P}y$ does not hold. If $uy = 0$ then again $x\underline{P}y$ does not hold. This proves that $\underline{P}$ is the equality on N, and the result follows from 2.6.

3. We now return to S and study what the irreducibility of S implies for N, C and the action of C on N.

LEMMA 3.2. *If S is irreducible then N is weakly irreducible.*

Proof. We see that $\underline{M}$ is the equality on the ideal $M \cup \{0\}$ of S. We can extend $\underline{M}$ to a congruence $\underline{M}'$ on S, namely: $s\underline{M}'t$ if and only if either $s = t \in C$ or $s, t \in N$, $s\underline{M}t$. Then the intersection of $\underline{M}'$ and the Rees congruence of the ideal $M \cup \{0\}$ is the equality on S; therefore $\underline{M}'$ is the equality, and N is weakly irreducible.

LEMMA 3.3. *If S is irreducible then M is an orbit.*

Proof. By 1.1, M is always a union of orbits; hence it suffices to show that $C^1m \cup C^1n \neq \emptyset$ for all $m, n \in M$. But if $C^1m \cap C^1n = \emptyset$, the Rees congruences of the ideals $C^1m \cup \{0\}$, $C^1n \cup \{0\}$ of S have the equality on S as intersection and S is not irreducible.

LEMMA 3.4. *If S is irreducible then it is homogeneous.*

Proof. The proof is somewhat similar to the proof of Theorem 3.6 in [14]. Let $a, b \in C$, $x \in N$ be such that $ax = bx \neq 0$.

Since $x \neq 0$ there exists a minimal element $n \leqq x$, i.e., $n = ux$ for some $u \in N^1$; $ax = bx$ then implies $an = bn$. For each $m \in M$ there exist, by 3.3, elements $c, d \in C^1$ with $cm = dn$; then $cam = dan = dbn = cbm$, and therefore $am = bm$, for all $m \in M$.

Let $\underline{C}$ be the congruence on S defined by: $s\underline{C}t$ if and only if $a^i b^j s = a^j b^i t$ for some $i, j \geqq 0$ (with $a^0 = b^0 = 1 \in S^1$). We see that $\underline{C}$ is a pure congruence on S. If furthermore $s, t \in M$ (so that $as = bs$, $at = bt$) and $s\underline{C}t$, then, for some $i, j \geqq 0$, $a^{i+j}s = a^i b^j s = a^j b^i t = a^{i+j}t$, and hence $s = t$. Therefore $\underline{C}$ is the equality on the ideal $M \cup \{0\}$ of S. The intersection of $\underline{C}$ and the Rees congruence of $M \cup \{0\}$ is then the equality; since S is irreducible, $\underline{C}$ is the equality. But $a\underline{C}b$ (since $ba = ab$), and thus $a = b$—which proves that S is homogeneous.

LEMMA 3.5. *If is S irreducible then so is C.*

Proof. The proof is somewhat similar to that of Theorem 5.3 in [14]. For each congruence $\underline{A}$ on C we construct a congruence

$\underline{A}^*$ on S thus: $s\underline{A}^*t$ if and and only if $as = bt$ for some $a, b \in C$ such that $a\underline{A}b$. It is immediate that $\underline{A}^*$ is a pure congruence on S, whose restriction to C contains $\underline{A}$.

Now assume that the equality on C is the intersection of two congruences $\underline{A}, \underline{B}$ on C. Let $s, t \in S$ satisfy $s\ \underline{A}^* \cap \underline{B}^*\ t$; then $as = bt$, $cs = dt$ for some $a, b, c, d \in C$ with $a\underline{A}b$, $c\underline{B}d$. If $s = 0$, then $t = 0 = s$. Assume $s \neq 0$. Then $ads = bdt = bcs$. Therefore $ad = bc$ (trivially if $s \in C$, by 3.4 if $s \in N$). It follows that $ad\underline{A}bd$, $ad = bc\underline{B}bd$, $ad = bd$ and $a = b$. Then $as = bt = at$ and again $s = t$. Thus $\underline{A}^* \cap \underline{B}^*$ is the equality on S. Therefore $\underline{A}^*$ or $\underline{B}^*$ is the equality on S; this implies that $\underline{A}$ or $\underline{B}$ is the equality on C.

4. It turns out that the necessary properties given by these four lemmas are together sufficient; namely:

THEOREM 3.6. *Let $S = N \cup C$ be a finitely generated subelementary semigroup, with $N \neq 0$ and C nontrivial. Then S is irreducible if and only if N is weakly irreduible, C is irreducible, the minimal elements of N form an orbit and S is homogeneous.*

Proof. Assume that S has all four properties. Let $\underline{C}$ be a congruence on S which is not the equality; then $s\underline{C}t$ for some $s, t \in S$, $s \neq t$. If $s, t \in C$, then for any $m \in M$ we have $sm, tm \in M$ (by 1.1), $sm \neq tm$ (since S is homogeneous) and $sm\underline{C}tm$, so that $\underline{C}$ is not the equality on M. If $s \in C$, $t \in N$, then for any $m \in M$ we have $sm \in M$, $tm = 0$ and hence $sm\underline{C}0$; since C is not trivial and S is homogeneous, there exists $c \in C$ such that $cs \neq s$, hence $csm \neq sm$ and as above $csm \in M$, $csm\underline{C}0\underline{C}sm$ (since $cs\underline{C}ct \in N$); again $\underline{C}$ is not the equality on M. Finally, assume $s, t \in N$. Since N is weakly irreducible, there exists $u \in N^1$ such that, say, $us \in M$ and either $ut \in M$, $ut \neq us$ or $ut \notin M$. In the first case $\underline{C}$ is not the equality on M. In case $ut \notin M$, either $ut = 0$ or $ut \neq 0$. If $ut = 0$ then $m = us\underline{C}ut = 0$; as before, $cm \neq m$ for some $c \in C$, since S is homogeneous and C nontrivial, and $cmCc0 = 0\underline{C}m$, with $m, cm \in M$. If $ut \neq 0$ (and still $ut \notin M$) then $ut > m$ for some $m \in M$, i.e., $m = vut$ for some $v \in N$; hence $m = vut\underline{C}vus = 0$ since $us \in M$, $v \in N$, and as above $\underline{C}$ is not the equality on M. Thus we have proved that if $\underline{C}$ is not the equality on S then $\underline{C}$ is not the equality on M.

Now let $\underline{A}, \underline{B}$ be congruences on S different from the equality. By the above there exist $x, y, z, t \in M$ such that $x\underline{A}y$, $z\underline{B}t$, $x \neq y$, $z \neq t$. Take $m \in M$. Since M is an orbit, there exist $a, b, c, d \in C^1$ such that $ax, by, cz, dt \in C^1m$; then any $r \in abcdC$ satisfies $rx, ry, rz, rt \in Cm$, $rx\underline{A}ry$, $rz\underline{B}rt$, $rx \neq ry$, $rz \neq rt$; in other words we may as well assume that $x, y, z, t \in Cm$, for some $m \in M$.

Then let $\underline{A}'$ be the congruence on C defined by: $a\underline{A}'b$ if and only if $am\underline{A}bm$. Since $x, y \in Cm$ we see that $\underline{A}'$ is not the equality. Similarly the congruence $\underline{B}'$ on C defined by: $a\underline{B}'b$ if and only if $am\underline{B}bm$, is not the equality on C. Since C is irreducible, $\underline{A}' \cap \underline{B}'$ is not the equality on C and there exist $a, b \in C$ such that $a \neq b$, $am\underline{A}bm$, $am\underline{B}bm$. Since S is homogeneous, $am \neq bm$, which shows that $\underline{A} \cap \underline{B}$ is not the equality on S.

In the course of this proof we have shown:

COROLLARY 3.7. *Let S be as in Theorem 3.6. If S is irreducible, every congruence on S other than the equality identifies two elements of M.*

5. Combining Theorem 3.6 with the previous results (2.5, 2.6, 2.4) we obtain all irreducible, finitely generated commutative semigroups. The list is as follows:

I: a cancellative semigroup, isomorphic to $\boldsymbol{Z}(p^n)$ (with P prime), to $\boldsymbol{Z}$, or to a subsemigroup of $\boldsymbol{N}$;

II: a finite commutative nilsemigroup whose congruence $\underline{P}$ is the equality;

III: $\{0, 1\}$;

a cancellative semigroup, as in I, with zero adjoined;

a finite nilsemigroup, as in II, with identity adjoined;

a finitely generated subelementary semigroup $S = N \cup C$, such that N is weakly irreducible, C is as in I, M is an orbit and S is homogeneous.

6. This result is similar to the result of Schein [14] which gives all subdirectly irreducible, finitely generated commutative semigroups. The list of these semigroups can be simply obtained from the above by adding the requirement that the cancellative semigroups in I and III be isomorphic to $\boldsymbol{Z}(p^n)$ only (cf. [14]).

It is easy to prove this result directly (without referring to [14]). First we show:

LEMMA 3.8. *Let S be as in Theorem 3.6. If S is subdirectly irreducible then so is C.*

Proof. Assume that S is subdirectly irreducible (in particular, irreducible), and that the equality on C is the intersection of congruences $\underline{C}_i (i \in I)$ on C. Pick $m \in M$ and let $\underline{C}_i^*$ be the relation on S defined by: $s\underline{C}_i^*t$ if and only if $s = t$ or $s = am$, $t = bm$ for some

$a, b \in C$ such that $a\underline{C}_i b$. Since S is is homogeneous, $bm = cm$ implies $b = c$ when $b, c \in C$, and it follows that $\underline{C}_i^*$ is an equivalence relation. Since $xCm = 0$ for all $x \in N$ we see that $\underline{C}_i^*$ is in fact a congruence.

Let $\underline{C}^*$ be the intersection of all $\underline{C}_i^*$. Asume $s\underline{C}^*t$, $s \neq t$. Then $s, t \in Cm$, say $s = am$, $t = bm$ for some $a, b \in C$. For each i we also have $s = cm$, $t = dm$ for some $c, d \in C$ with $c\underline{C}_i d$. Since S is homogeneous, $am = cm$, $bm = dm$ implies $a = c$, $b = d$ and therefore $a\underline{C}b$. Since this holds for all i, it follows that $a = b$, which contradicts $s \neq t$. Therefore $\underline{C}^*$ is the equality on S. This implies that some $\underline{C}_i^*$ is the equality on S. The corresponding $\underline{C}_i$ is then the equality, since $a\underline{C}_i b$, $a \neq b$ implies $am\underline{C}_i^* bm$, $am \neq bm$.

Now let S be a finitely generated commutative semigroup. Assume that S is irreducible. If $S = C$ is cancellative and isomorphic to either $\boldsymbol{Z}$ or a subsemigroup of N, then S is not subdirectly irreducible. By the remark following 2.4, S is not subdirectly irreducible if $S = C^0$ with C as above; and Lemma 3.8 says that S is not subdirectly irreducible if $S = N \cup C$ is subelementary with C as above. The remaining semigroups in our list of irreducible semigroups are all finite: this is evident in all but last case, in which S is homogeneous elementary (since $S \cong \boldsymbol{Z}(p^n)$ is a group), so that all nonzero orbits have p^n elements; then S is finite since, by 1.1, it only has finitely many orbits. Now a finite irreducible semigroup only has finitely many congruences and is therefore subdirectly irreducible. Thus the subdirectly irreducible, finitely generated commutative semigroups obtain from our list of irreducible semigroups by adding the requirement that the nontrivial cancellative semigroups in cases I and III be isomorphic to $\boldsymbol{Z}(p^n)$ only. Finiteness is an equivalent requirement.

In particular this yields the following theorem of Mal'cev [12]:

COROLLARY 3.9. *A subdirectly irreducible commutative semigroup which is finitely generated must be finite.*

Equivalently, finitely generated commutative semigroups are residually finite. Other proofs of this result were given by Carlisle [1] and Lallement [10].

7. Our last result is a sharpening of Theorem 3.6 in the elementary case. By 2.3 this is, theoretically, the only case we need consider in Theorem 3.6 [although in practice it is not immediately evident that if $C^{-1}N \subsetneqq C^{-1}S$ is weakly irreducible then so is N]. If furthermore S is subdirectly irreducible in Theorem 3.6, then we

have seen that S is finite, and therefore elementary.

Let $S = N \cup G$ be a finitely generated elementary semigroup, with $N \neq 0$ and G nontrivial. If S is irreducible, then, by 3.6, S is homogeneous. If conversely S is homogeneous, then we saw in §1 that S can be reconstructed from G and from the finite nilsemigroup Ω of nilpotent orbits of S. Looking at the other conditions in Theorem 3.6, we see that M is an orbit if and only if Ω has only one minimal element (by 1.1); that G is irreducible if and only if $G \cong \boldsymbol{Z}$ or $G \cong \boldsymbol{Z}(p^n)$. The last condition, that N be weakly irreducible, can be expressed as follows:

LEMMA 3.10. *Let $S = N \cup G$ be an elementary semigroup such that $N \neq 0$, G is not trivial, and Ω is finite with only one minimal element μ. Then N is weakly irreducible if and only if any factor set σ associated with S satisfies the condition: when $\alpha, \beta \in \Omega$ and $\alpha \underline{M} \beta$, $\alpha \neq \beta$, then the function $\sigma_{\alpha,\upsilon}\sigma^{-1}_{\beta,\upsilon}$ is not constant on $T = \{\upsilon \in \Omega;\ \upsilon\alpha = \mu\} = \{\upsilon \in \Omega;\ \upsilon\beta = \mu\}$.*

Proof. Let σ be any factor set associated with S. Then $S \cong [\Omega, G, \sigma]$; we may as well assume that $S = [\Omega, G, \sigma]$. Then N consists of 0 and all pairs (g, α) with $g \in G$, $\alpha \in \Omega$, $\alpha \neq 0$. Also, $(g, \alpha) \in M$ if and only if $\alpha = \mu$.

Let $\alpha, \beta \in \Omega$ be such that $\alpha \underline{M} \beta$, $\alpha \neq \beta$. From the definition of $\underline{M}$ we see that $\alpha, \beta \neq 0, \mu$ (since $\alpha \neq \beta$) and that $T = \{\upsilon \in \Omega;\ \upsilon\alpha = \mu\} = \{\upsilon \in \Omega;\ \upsilon\beta = \mu\}$ (since $\upsilon\alpha, \upsilon\beta \neq \mu$ when $\upsilon = 1 \in \Omega^1$). Furthermore $T \neq \varnothing$, since $\alpha, \beta > \mu$. Take and assume that N is weakly irreducible. Let $a = (\sigma_{\beta,\upsilon}, \alpha)$, $b = (\sigma_{\alpha,\upsilon}, \beta) \in N$. We have $a \neq b$ (since $\alpha \neq \beta$). Therefore $a \underline{M} b$ does not hold. However, for all $u \in N^1$, $ua \in M$ if and only if $ub \in M$ (since $\alpha \underline{M} \beta$ in Ω). Hence there exists $u \in N^1$ such that $ua, ub \in M$, $ua \neq ub$. We cannot have $u = 0, 1$; hence $u = (g, \tau)$ with $g \in G$, $\tau \in \Omega$, $\tau \neq 0$. We have $\tau \in T$ (since $ua \in M$). Also

$$(g\sigma_{\beta,\upsilon}\sigma_{\alpha,\tau}, \mu) = ua \neq ub = (g\sigma_{\alpha,\upsilon}\sigma_{\beta,\tau}, \mu)\ ,$$

so that $\sigma_{\alpha,\tau}\sigma^{-1}_{\beta,\tau} \neq \sigma_{\alpha,\upsilon}\sigma^{-1}_{\sigma,\upsilon}$. Since $\tau, \upsilon \in T$, the condition in the lemma is necessary.

Conversely, assume this condition holds. Take $a, b \in N$, $a \neq b$. If either of a, b is zero or minimal then $a \underline{M} b$ cannot hold in N. Now assume that neither of a, b is zero or minimal, so that $a = (g, \alpha)$, $b = (h, \beta)$, where $\alpha, \beta > \mu$. If $\alpha = \beta$, then $g \neq h$; we see on the formula which gives the multiplication on N (in §1) that $au \neq bu$ whenever $au, bu \neq 0$; therefore $a \underline{M} b$ does not hold. Assume $\alpha \neq \beta$. If $\alpha \underline{M} \beta$ does not hold in Ω, then $a \underline{M} b$ does not hold in N. Now assume that $\alpha \underline{M} \beta$ holds in Ω. As before $T \neq \varnothing$. Take $\upsilon \in T$ and let $u = (1, \upsilon)$, so that $au = (g\sigma_{\alpha,\upsilon}, \mu)$, $bu = (h\sigma_{\beta,\upsilon}, \mu) \in M$. If $\sigma_{\alpha,\upsilon}\sigma^{-1}_{\beta,\upsilon} \neq hg^{-1}$ then

$au \neq bu$ and $a\underline{M}b$ does not hold. If $\sigma_{\alpha,\upsilon}\sigma^{-1}_{\beta,\upsilon} - hg^{-1}$ then by the hypothesis there exists $\tau \in T$ such that $\sigma_{\alpha,\tau}\sigma^{-1}_{\beta,\tau} \neq hg^{-1}$; then $t = (1, \tau)$ is (as above) such that $at, bt \in M$, $at \neq bt$, and hence $a\underline{M}b$ does not hold. In no case does $a\underline{M}b$ hold when $a \neq b$, which proves that N is weakly irreducible.

Combining 3.10 with our previous results we obtain a construction of all our irreducible semigroups in terms of groups ($\boldsymbol{Z}$ and $\boldsymbol{Z}(p^n)$) and finite nilsemigroups.

Lemma 3.10 shows that the properties inherited by Ω when S is irreducible are even weaker than these inherited by N. Specifically, Example 2.8 is a finite nilsemigroup which has only one minimal element but is not irreducible, and hence is not weakly irreducible (by 3.1). However it is the semigroup of nilpotent orbits of a finite irreducible semigroup: the factor sets with values in any group $G \cong \boldsymbol{Z}(p^n)$ are easy to calculate: they are all families $\sigma_{a,a}, \sigma_{a,b}, \sigma_{b,a}, \sigma_{b,b} \in G$ such that $\sigma_{a,b} = \sigma_{b,a}$; the condition in 3.10 is that $\sigma_{a,a}\sigma^{-1}_{b,a} \neq \sigma_{a,b}\sigma^{-1}_{b,b}$ and with $G \neq 1$ as above there is always a factor set σ with this property.

In general, however, the condition in 3.10 does put restrictions on Ω (beyond the uniqueness of the minimal element). For instance note that if S splits in 3.10 then N cannot be weakly irreducible unless Ω is: for we may let $\sigma = 0$ and then the function $\sigma_{\alpha,\upsilon}\sigma^{-1}_{\beta,\upsilon}$ has to be constant on T in case $\alpha\underline{M}\beta$, $\alpha \neq \beta$. In other words, if S is irreducible and splits, then Ω is weakly irreducible. In particular, if Ω is not weakly irreducible, then 3.10 requires of Ω that its factor sets with values in G are not all trivial.

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