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ON CHARACTERIZATIONS AND INTEGRALS OF GENERALIZED NUMERICAL RANGES

MOSHE GOLDBERG AND E. G. STRAUS

Let $c=(\gamma_1, \cdots, \gamma_n)$ be given. The generalized numerical range of an $n\times n$ matrix A, associated with c, is the set $W_c(A)=\{\Sigma\gamma_j(Ax_j, x_j)\}$ where $(x_1, \cdots, x_n)$ varies over orthonormal systems in C^n. Characterizations of this range, for real c, are given. Next, we study integrals of the form $\int W_c(A)d\mu(c)$ where $\mu(c)$ is a measure defined on a domain in R^n. The above characterizations are used to study the inclusion $\int W_c(A)d\mu(c)\subset\lambda W_{c'}(A)$. We determine those λ, for which this inclusion holds for all $n\times n$ matrices A. Such relations lead to more elementary ones, when the integral reduces to a finite linear combination of ranges. In particular, we obtain the inclusion relations of the form $W_c(A)\subset \lambda W_{c'}(A)$ which hold for all A.

1. Introduction. The generalized numerical range of an $n\times n$ complex matrix A, associated with a fixed vector $c=(\gamma_1, \cdots, \gamma_n)\in C^n$, is the set of complex numbers

$$(1.1)\qquad W_c = W_{(\gamma_1,\cdots,\gamma_n)}(A) = \left\{\sum_{j=1}^{n}\gamma_j(Ax_j, x_j) : (x_1, \cdots, x_n)\in \Lambda_n\right\},$$

where Λ_n is the set of all orthonormal n-tuples of vectors is C^n. We call W_c a generalized range since for $c=(1, 0, \cdots, 0)$ it reduces to the classical range

$$W(A) = \{(Ax, x) : ||x|| = 1\}.$$

It is clear from (1.1) that W_c remains invariant under permutations of the components of c; that is, W_c depends on the unordered set $\{\gamma_1, \cdots, \gamma_n\}$ rather than on c.

Westwick, [5], has shown that if c is a real vector then W_c is convex, but if $c\in C^n$ with $n\geqq 3$, then $W_c(x)$ may fail to be convex even for normal A. For this reason we restrict our attention, in this paper, to generalized numerical ranges with *real* coefficients.

Our first purpose is to characterize the sets W_c. In §2 we show that

$$W_c(A) = \{\mathrm{tr}\,(HA) : H\in\mathscr{H}_c\},$$

where $\mathscr{H}_c$ is a class of Hermitian matrices depending on c.

In §3 we define integrals of the form $\int_{\mathscr{D}} W_c(A)d\mu(c)$ where $\mathscr{D}$

is a domain in $\boldsymbol{R}^n$ and $\mu(c)$ is a nonnegative measure on $\mathscr{D}$. Since the sets W_c are convex, such integrals are convex as well, and we may define them in terms of their support functions.

Finally, using the above characterization of W_c, we investigate inclusion relations of the form

$$(1.2) \qquad \int_{\mathscr{D}} W_c(A) d\mu(c) \subset \gamma W_{c'}(A), \quad \lambda = \text{constant}\,,$$

which hold, uniformly, for all $A \in \boldsymbol{C}_{n\times n}$, i.e., for all n-square matrices. If the measure $\mu(c)$ is concentrated on a finite number of vectors c, then (1.2) is reduced to inclusion relations involving finite linear combinations of generalized numerical ranges. Such relations were considered in earlier works [2, 3].

In particular, for given vectors c, c' we obtain necessary and sufficient conditions under which

$$W_c(A) \subset \lambda W_{c'}\,, \quad \forall A \in \boldsymbol{C}_{n\times n}\,.$$

2. Characterization of generalized ranges. For any vector $c = (\gamma_1, \cdots, \gamma_n)$ consider the diagonal matrix

$$C = \operatorname{diag}(c) = \operatorname{diag}(\gamma_1, \cdots, \gamma_n)\,,$$

and construct the class of matrices

$$\mathscr{U}_c = \operatorname{conv}\{UCU^*\colon U \text{ unitary}\}\,,$$

where conv denotes the convex hull.

Since we restrict attention to $c \in \boldsymbol{R}^n$ it is evident that the elements of $\mathscr{U}_c$ are Hermitian.

Using $\mathscr{U}_c$ we have the following characterization of ranges with real coefficients.

THEOREM 1. *If $c \in \boldsymbol{R}^n$ then*

$$W_c(A) = \{\operatorname{tr}(HA)\colon H \in \mathscr{U}_c\}\,.$$

Proof. It follows from the definition of $W_c(A)$ in (1.1) that

$$W_c(A) = \{\operatorname{tr}(CU^*AU)\colon U \text{ unitary}\}\,.$$

Thus

$$(2.1) \qquad W_c(A) = \{\operatorname{tr}((UCU^*))A)\colon U \text{ unitary}\}\,,$$

which implies that

$$W_c(A) \subset \{\operatorname{tr}(HA)\colon H \in \mathscr{U}_c\}\,.$$

For the converse inclusion let

$$H = \sum_i \lambda_i(U_i C U_i^*);\ \lambda_i \geqq 0\ , \quad \sum_i \lambda_i = 1\ ,$$

be an arbitrary element of $\mathscr{U}_c$. By the convexity of W_c and by (2.1) we have

$$\operatorname{tr}(HA) = \sum \lambda_i \operatorname{tr}((U_i C U_i^*)A) \in W_c(A)\ .$$

So,

$$\{\operatorname{tr}(HA): H \in \mathscr{U}_c\} \subset W_c(A)\ ,$$

and the theorem follows.

We introduce two definitions which lead to another characterization of $W_c(A)$.

DEFINITION 1. (i) A real vector $c = (\gamma_1, \cdots, \gamma_n)$ is called *ordered* if

$$\gamma_1 \geqq \gamma_2 \geqq \cdots \geqq \gamma_n\ .$$

(ii) We say that c, c' satisfy $c' \prec c$ if there exists a doubly stochastic matrix S (i.e., a matrix with nonnegative entries whose row sums and columns sums equal 1), such that $c' = Sc$.

In Theorem 5 of [3] we proved the following.

LEMMA 1. *For ordered c, c' we have $c' \prec c$ if and only if*

$$\sum_{j=1}^{l} \gamma'_j \leqq \sum_{j=1}^{l} \gamma_j\ , \quad l = 1, \cdots, n\ ,$$

with equality for $l = n$.

DEFINITION 2. Let $c \in \boldsymbol{R}^n$, and let $\Lambda_l (1 \leqq l \leqq n)$ be the set of all orthonormal l-tuples of vectors in $\boldsymbol{C}^n$. We define $\mathscr{H}_c$ to be the class of all Hermitian matrices H for which

$$\text{(2.2)} \qquad \sum_{j=1}^{l} (Hx_j, x_j) \leqq \sum_{j=1}^{l} \gamma_j\ , \quad \forall (x_1, \cdots, x_l) \in \Lambda_l\ , \quad l = 1, \cdots, n\ ,$$

with equality for $l = n$.

Let $e_1, \cdots, e_n$ be the standard basis of $\boldsymbol{C}^n$. Note that if $\Sigma\gamma_j = 0$ (which is the case assumed in § 3), then the equality for $l = n$ in (2.2) implies that

$$\sum_{j=1}^{n} (He_j, e_j) = \Sigma\gamma_j = 0\ ;$$

i.e., all members of $\mathscr{H}_c$ have trace 0.

LEMMA 2. *If c is ordered then $\mathscr{H}_c = \mathscr{U}_c$.*

Proof. Take a unitary matrix U and orthonormal vectors $x_1, \cdots, x_l$, $(1 \leqq l \leqq n)$. Since the vectors $y_j = U^* x_j$, $j = 1, \cdots, l$, are orthonormal as well, it is not hard to verify that

$$\sum_{j=1}^{l} (UCU^* x_j, x_j) = \sum_{j=1}^{l} (Cy_j, y_j) \leqq \gamma_1 + \cdots + \gamma_l , \quad C = \operatorname{diag}(c) , \tag{2.3}$$

with equality for $l = n$. Therefore, if

$$H = \sum_i \lambda_i U_i C U_i^* , \quad \left(\lambda_i \geqq 0 , \quad \sum_i \lambda_i = 1 \right),$$

is any (Hermitian) matrix in $\mathscr{U}_c$, we find by (2.3) that

$$\sum_{j=1}^{l} (Hx_j, x_j) = \sum_{j=1}^{l} \sum_i \lambda_i (U_i C U_i^* x_j, x_j) \leqq \sum_i \lambda_i \sum_{j=1}^{l} \gamma_j = \sum_{j=1}^{l} \gamma_j ,$$

with equality for $l = n$. So, by Definition 2, $H \in \mathscr{H}_c$, and consequently $\mathscr{U}_c \subset \mathscr{H}_c$.

Conversely, take any $H \in \mathscr{H}_c$. Since H is Hermitian, it is unitarily similar to a real diagonal matrix, i.e., there exists a unitary V such that

$$C' \equiv V^* H V = \operatorname{diag}(\gamma_1', \cdots, \gamma_n') , \tag{2.4}$$

where we may assume that $c' = (\gamma_1', \cdots, \gamma_n')$ is ordered. Using (2.2) and the orthonormal vectors $x_j = Ve_j$, $j = 1, \cdots, l$, we find that

$$\sum_{j=1}^{l} \gamma_j' = \sum_{j=1}^{l} (C' e_j, e_j) = \sum_{j=1}^{l} (V^* H V e_j, e_j) = \sum_{j=1}^{l} (Hx_j, x_j) \leqq \sum_{j=1}^{l} \gamma_j ,$$

with equality for $l = n$. That is, by Lemma 1, $c' \prec c$. Hence, there exists a doubly stochastic matrix S such that $c' = Sc$. Now recall that doubly stochastic matrices are convex combinations of permutation matrices P_σ. In particular $S = \Sigma_\sigma \lambda_\sigma P_\sigma$. Thus

$$c' = \sum_{\sigma \in S_n} \lambda_\sigma P_\sigma c ; \lambda_\sigma \geqq 0 , \quad \Sigma \lambda_\sigma = 1 , \tag{2.5}$$

where S_n is the symmetric group. Since for every B, $P_\sigma B P_\sigma^*$ has both the rows and columns of B permuted according to σ, we have

$$\operatorname{diag}(P_\sigma c) = P_\sigma \operatorname{diag}(c) P_c^* = P_\sigma C P_\sigma^* . \tag{2.6}$$

So, by (2.5), (2.6),

$$C' = \operatorname{diag}(c') = \sum_\sigma \lambda_\sigma \operatorname{diag}(P_\sigma c) = \sum_\sigma \lambda_\sigma P_\sigma C P_\sigma^* . \tag{2.7}$$

From (2.4) and (2.7) we obtain

$$H = VC'V^* = \sum_\sigma \lambda_\sigma[(VP_\sigma)C(VP_\sigma)^*] = \sum_\sigma \lambda_\sigma(U_\sigma C U_\sigma^*) , \tag{2.8}$$
$$\lambda_\sigma \geqq 0 , \quad \Sigma\lambda_\sigma = 1 ,$$

where $U_\sigma \equiv VP_\sigma$ are, of course, unitary. Hence, $H \in \mathscr{U}_c$, i.e., $\mathscr{H}_c \subset \mathscr{U}_c$ and the proof is complete.

Theorem 1 together with Lemma 2 imply a second characterization of generalized numerical ranges with real coefficients.

THEOREM 2. *If c is ordered then*

$$W_c(A) = \{\mathrm{tr}\,(HA) \colon H \in \mathscr{H}_c\} .$$

Another simple consequence of the last lemma and the convexity of $\mathscr{U}_c$ is that for ordered c, $\mathscr{H}_c$ is convex.

At this point we recall the definition of the k-numerical range, $(1 \leqq k \leqq n)$, given by Halmos [1, § 167], which after a convenient normalization becomes

$$W_k(A) = \left\{\frac{1}{k}\,\mathrm{tr}\,(PAP) \colon P = \text{orthogonal projection of rank } k\right\} .$$

It can be verified that $W_k(A)$ may be written as

$$W_k(A) = \left\{\frac{1}{k}\sum_{j=1}^{k}(Ax_j, x_j) \colon (x_1, \cdots, x_k) \in \Lambda_k\right\} .$$

Hence we see that

$$W_k(A) = W_{c_k}(A) , \quad \text{with } c_k = \frac{1}{k}(e_1 + \cdots + e_k) .$$

That is, the k-numerical range is a special case of the generalized numerical range.

The matrices $\mathscr{H}_{c_k}$ are those Hermitian matrices which satisfy Definition 2 with $c = c_k$. Using this definition one can show that

$$\mathscr{H}_{c_k} = \left\{\text{Hermitian } H \colon 0 \leqq H \leqq \frac{1}{k}I,\ \mathrm{tr}\,(H) = 1\right\} .$$

Thus Theorem 2 generalizes the result

$$W_k(A) = \left\{\mathrm{tr}\,(HA) \colon 0 \leqq H \leqq \frac{1}{k}I,\ \mathrm{tr}\,(H) = 1\right\} , \quad k = 1, \cdots, n$$

of Fillmore and Williams [1, Theorem 1.2].

3. Integrals of generalized ranges. In this section we are interested in linear combinations, or more generally, in integrals of the sets $W_c(A)$, where A is arbitrary but fixed, and c varies in some domain of $\boldsymbol{R}^n$.

Let $c = (\gamma_1, \cdots, \gamma_n)$ be a real vector with $\gamma \equiv \Sigma\gamma_j \neq 0$, and consider the vector $b = (\beta_1, \cdots, \beta_n)$ defined by

$$b = c - \left(\frac{\gamma}{n}, \cdots, \frac{\gamma}{n}\right).$$

We have $\Sigma\beta_j = 0$ and

$$B \equiv \operatorname{diag}(b) = \operatorname{diag}(c) - \frac{\gamma}{n}I = C - \frac{\gamma}{n}I.$$

So, by Theorem 1,

$$\begin{aligned} W_b(A) &= \{\operatorname{tr}(UBU^*A) \colon U \text{ unitary}\} \\ &= \left\{\operatorname{tr}\left[U\left(C - \frac{\gamma}{n}I\right)U^*A\right] \colon U \text{ unitary}\right\} = W_c(A) - \left\{\frac{\gamma}{n}\operatorname{tr}(A)\right\}. \end{aligned}$$

This argument suggests that it is convenient to restrict attention to those vectors c for which $\Sigma\gamma_j = 0$. The limitation merely involves a translation of the ranges by multiples of the trace, or, equivalently, the restriction to matrices of trace 0.

Since W_c is invariant under permutations of the γ_j, we may assume that each vector c in our domain is ordered. Hence, we consider the set of ordered vectors c with $\Sigma\gamma_j = 0$, which form a conical subset $\mathscr{C}$ of an $(n-1)$-dimensional subspace of $\boldsymbol{R}^n$.

We are ready now to study integrals of $W_c(A)$ relative to an arbitrary measure μ on $\mathscr{C}$, that is integrals of the form

$$J_\mu = J_\mu(A) = \int_{\mathscr{C}} W_c(A)d\mu(c). \tag{3.1}$$

One way of defining the integral in (3.1) is by carrying linear sums, over partitions of $\mathscr{C}$, to the limit. Alternatively, one realizes that J_μ, being an integral of the convex sets W_c, is a convex set as well. Hence J_μ may be characterized by its support function (e.g., [4] part V),

$$u(J_\mu, \theta) = \sup_{z \in J_\mu} \operatorname{Re}(ze^{-i\theta}), \quad 0 \leqq \theta < \pi$$

In order to evaluate $u(J_\mu, \theta)$, we consider the support functions of our closed convex integrands W_c. We have

$$u(W_c, \theta) = u(c, \theta) = \max_{z \in W_c} \operatorname{Re}(ze^{-i\theta}), \quad 0 \leqq \theta < \pi.$$

Since $u(c, \theta)$ is a linear function of c in the sense that

$$u(\lambda W_c + \lambda' W_{c'}, \theta) = \lambda u(c, \theta) + \lambda' u(c', \theta) \ , \quad \forall \lambda, \lambda' \geqq 0 \ ,$$

we have

$$u(J_\mu, \theta) = u\Big(\int W_c d\mu(c), \theta\Big) = \int u(W_c, \theta) d\mu(c) = \int u(c, \theta) d\mu(c) \ .$$

Of course, the measure μ may be concentrated at a finite number of points $c_1, \cdots, c_m \in \mathscr{C}$. In this case the integral J_μ reduces to the finite linear combination

$$\mu(c_1) W_{c_1}(A) + \cdots + \mu(c_m) W_{c_m}(A) \ .$$

Since $W_{\lambda c} = \lambda W_c$ for scalar λ, we shall avoid integration over proportional vectors of $\mathscr{C}$. This can be achieved by restricting integration to the domain

$$\mathscr{D} = \{c : c = (\gamma_1, \cdots, \gamma_n),\ \Sigma\gamma_j = 0 \ , \quad \gamma_1 = 1\} \ ,$$

which is the bounded set of all vectors in $\mathscr{C}$ with $\gamma_1 = 1$.

The above concept of integration can be extended in order to consider the integral

$$\mathscr{H}_\mu \equiv \int_{\mathscr{D}} \mathscr{H}_c d\mu(c) \ . \tag{3.2}$$

We recall that the integrands $\mathscr{H}_c$ are convex sets in the ($n^2 - 1$ real dimensional) space $\boldsymbol{H}$ of Hermitian matrices of trace 0. It follows that $\mathscr{H}_\mu$ is also a convex set in $\boldsymbol{H}$. Again, the convexity of $\mathscr{H}_c$ and $\mathscr{H}_\mu$ implies that the integral may be defined in terms of the support functions of $\mathscr{H}_c$. Here, in analogy to the previous case, the support function of $\mathscr{H}_c$ assigns to each point H_1 on the unit sphere of $\boldsymbol{H}$, the distance from the origin O of $\boldsymbol{H}$ to the plane of support of $\mathscr{H}_c$ perpendicular to the direction $\overrightarrow{OH_1}$.

Having the integrals J_μ and $\mathscr{H}_\mu$ defined we state our main result.

THEOREM 3. *Let μ be a nonnegative measure on $\mathscr{D}$, and let $c' \neq 0$ be an ordered vector with $\Sigma\gamma'_j = 0$. Then*

$$\int_{\mathscr{D}} W_c(A) d\mu(c) \subset \lambda W_{c'}(A) \ , \quad \forall A \in \boldsymbol{C}_{n\times n} \ , \tag{3.3}$$

if and only if $\lambda \geqq \eta(c')$ or $\lambda \leqq \zeta(c')$ where

$$\eta(c') = \max_{1 \le l < n} \int_{\mathscr{D}} \frac{\gamma_1 + \cdots + \gamma_l}{\gamma'_1 + \cdots + \gamma'_l} d\mu(c) \ , \tag{3.4a}$$

$$\zeta(c') = \min_{1 \le l < n} \int_{\mathscr{D}} \frac{\gamma_1 + \cdots + \gamma_l}{\gamma'_n + \cdots + \gamma'_{n-l+1}} d\mu(c) \ . \tag{3.4b}$$

Proof. In the proof of Lemma 8 of [3] we have shown that if $c' \neq 0$ with $\Sigma\gamma'_j = 0$, then

$$(3.5) \quad \gamma'_1 + \cdots + \gamma'_l > 0\,, \quad \gamma'_n + \cdots + \gamma'_{n-l+1} < 0;\ l = 1, \cdots, n-1\,.$$

This establishes that η, ζ of (3.4) are well defined and since μ is a nonnegative measure we see that $\eta \geqq 0$, $\zeta \leqq 0$.

Next we show that $\lambda \geqq \eta(c')$ or $\lambda \leqq \zeta(c')$ imply (3.3). For this purpose we use the definition of $\mathscr{H}_\mu$, Theorem 2, and the linearity of the trace to evaluate the set on the left of (3.3):

$$(3.6) \quad \int_{\mathscr{D}} W_c(A)d\mu(c) = \int_{\mathscr{D}} \{\mathrm{tr}\,(HA) \colon H \in \mathscr{H}_c\} d\mu(c)$$
$$= \left\{\mathrm{tr}\,(HA) \colon H \in \int_{\mathscr{D}} \mathscr{H}_c d\mu(c)\right\} = \{\mathrm{tr}\,(HA) \colon H \in \mathscr{H}_\mu\}\,.$$

Now choose λ with $\lambda \geqq \eta(c')$. Since $\lambda \geqq 0$, the vector $\lambda c'$ remains ordered. Hence, by Theorem 2,

$$(3.7) \qquad \lambda W_{c'}(A) = W_{\lambda c'}(A) = \{\mathrm{tr}\,(HA) \colon H \in \mathscr{H}_{\lambda c'}\}\,.$$

From (3.6), (3.7) we see that in order to prove (3.3) it suffices to show that

$$(3.8) \qquad \mathscr{H}_\mu \subset \mathscr{H}_{\lambda c'}\,.$$

Thus, let H_0 be a matrix in $\mathscr{H}_\mu$. Then by (3.2), there exist elements $H_c \in \mathscr{H}_c$ for all $c \in \mathscr{D}$, such that

$$H_0 = \int_{\mathscr{D}} H_c d\mu(c)\,.$$

The matrices H_c satisfy Definition 2, and since μ is a nonnegative measure on $\mathscr{D}$, it follows that for l-tuples $x_1, \cdots, x_l$ in Λ_k we have

$$(3.9) \qquad \sum_{j=1}^{l} (H_0 x_j, x_j) = \int_{\mathscr{D}} \sum_{j=1}^{l} (H_c x_j, x_j) d\mu(c)$$
$$\leqq \int_{\mathscr{D}} (\gamma_1 + \cdots + \gamma_l) d\mu(c);\ l = 1, \cdots, n\,,$$

with equality for $l = n$. Since $\Sigma\gamma_j = \Sigma\gamma'_j = 0$, the above equality for $l = n$ implies

$$(3.10\text{a}) \qquad \sum_{j=1}^{n} (H_0 x_j, x_j) = 0 = \lambda \sum_{j=1}^{n} \gamma'_j\,.$$

For $1 \leqq l < n$ we use the assumption $\lambda \geqq \eta$ to obtain from (3.9) that

(3.10b) $\sum_{j=1}^{l}(H_0x_j, x_j)$

$$\leqq (\gamma'_1 + \cdots + \gamma'_l)\int_{\mathscr{D}} \frac{\gamma_1 + \cdots + \gamma_l}{\gamma'_1 + \cdots + \gamma'_l} d\mu(c) \leqq \lambda(\gamma'_1 + \cdots + \gamma'_l) .$$

By Definition 2, the relations (3.10) mean that $H_0 \in \mathscr{H}_{\lambda c'}$. Hence, (3.8) holds, and consequently the inclusion in (3.3) follows.

For $\lambda \leqq \zeta$ the situation is slightly different. Consider the vector $c'' \equiv (-\gamma'_n, \cdots, -\gamma'_1)$. Since c' is ordered, c'' is too. Also, the condition $\lambda \leqq \zeta(c')$ becomes

$$(3.11) \qquad -\lambda \geqq -\zeta(c') = -\min_{1\leqq l<n}\int_{\mathscr{D}} \frac{\gamma_1 + \cdots + \gamma_l}{\gamma'_n + \cdots + \gamma'_{n-l+1}} d\mu(c)$$

$$= \max_{1\leqq l<n}\int_{\mathscr{D}} \frac{\gamma_1 + \cdots + \gamma_l}{-\gamma' - \cdots - \gamma'_{n-l+1}} d\mu(c) = \eta(c'') .$$

Hence, by the previous part of the proof, we have that

$$(3.12) \qquad \int_{\mathscr{D}} W_c(A)d\mu(c) \subset -\lambda W_{c''}(A) , \quad \forall A \in \boldsymbol{C}_{n\times n} .$$

But $-\lambda c''$ is merely a reordering of $\lambda c'$. Thus, the set on the right of (3.12) satisfies

$$-\lambda W_{c''}(A) = W_{-\lambda c''}(A) = W_{\lambda c'}(A) = \lambda W_{c'}(A) ,$$

and we obtain (3.3).

To complete the proof we have to show that if $\zeta < \lambda < \eta$, then (3.3) does not hold for some $A \in \boldsymbol{C}_{n\times n}$. First assume $0 \leqq \lambda < \eta$. That is, for some l, $1 \leqq l < n$,

$$(3.13) \qquad \lambda(\gamma'_1 + \cdots + \gamma'_l) < \int_{\mathscr{D}} (\gamma_1 + \cdots + \gamma_l)d\mu(c) .$$

Consider the matrix $A_l = I_l \oplus O_{n-l}$. A simple computation shows that for an ordered vector c, the range $W_c(A_l)$ is a real interval with right end-point $\gamma_1 + \cdots + \gamma_l$. Then, the left side of (3.3) represents a real interval with right end-point

$$\int_{\mathscr{D}} (\gamma_1 + \cdots + \gamma_l)d\mu(c) ,$$

which, by (3.13), exceeds the right end-point $\lambda(\gamma'_1 + \cdots + \gamma'_l)$ of $W_{\lambda c'}(A_l)$.

Finally, if $\zeta(c') < \lambda < 0$, then (3.11) implies that $0 < -\lambda < \eta(c'')$ where $c'' = (-\gamma'_n \cdots, -\gamma'_1)$. Thus by the above example the inclusion

$$\int_{\mathscr{D}} W_c(A_l)d\mu(c) \subset -\lambda W_{c''}(A_l) = \lambda W_{c'}(A_l)$$

fails to hold, and the theorem follows.

We remember of course, that we restricted integration to the domain $\mathscr{D}$ for convenience only. Therefore, if so desired, $\mu(c)$ can be extended to the domain $\mathscr{C}$, and Theorem 3 remains valid.

If μ is concentrated at a finite number of vectors $c_1, \cdots, c_m \in \mathscr{C}$, then Theorem 3 characterizes all λ for which

$$\sum_{i=1}^{m} \mu(c_i) W_{c_i}(A) \subset \lambda W_{c'}(A) , \quad \forall A \in \boldsymbol{C}_{n\times n} .$$

A result of this type is given in Theorem 1 of [2].

Of particular interest is the case where μ is concentrated at a single vector $c'' \in \mathscr{C}$. That is,

$$\int_{\mathscr{D}} W_c(A) d\mu(c) = W_{c''}(A) ,$$

and η, ζ of (3.13) are given now by

$$\eta(c') = \max_{1\leq l<n} \frac{\gamma_1'' + \cdots + \gamma_l''}{\gamma_1' + \cdots + \gamma_l'} ; \zeta(c') = \min_{1\leq l<n} \frac{\gamma_1'' + \cdots + \gamma_l''}{\gamma_n' + \cdots + \gamma_{n-l+1}'} . \tag{3.14}$$

Thus, from Theorem 3 we conclude,

COROLLARY. *Let $c' \neq 0$ and c'' be ordered vectors with $\Sigma\gamma_j' = \Sigma\gamma_j'' = 0$. Then*

$$W_{c''}(A) \subset \lambda W_{c'}(A) , \quad \forall A \in \boldsymbol{C}_{n\times n}$$

if and only if $\lambda \geqq \eta(c')$ or $\lambda \leqq \zeta(c')$ where η, ζ are given in (3.14).

This result was proved differently in Theorem 8 of [3].

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UNIVERSITY OF CALIFORNIA, LOS ANGELES
LOS ANGELES, CA 90024