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SEMIGROUPS WITH IDENTITY ON PEANO CONTINUA

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A continuum is cell-cyclic if every cyclic element is a finite dimensional cell. We show that any finite dimensional cell-cyclic Peano continuum X admits a commutative semigroup with zero and identity, and apply this to show that if X is also homogeneous it is a point.

In [12] we showed that each cell-cyclic Peano continuum (locally connected metric continuum every cyclic element of which is a finite dimensional cell) X admits a semilattice (commutative idempotent topological semigroup). We now extend this result to show that X admits a commutative semigroup with identity and zero, and then apply this to homogeneous continua. Our extension is a partial answer to a question first raised by R. J. Koch in [6].

A semilattice is also a partially ordered Hausdorff topological space in which every two elements have a greatest lower bound and the function $(x, y) \to glb\{x, y\}$ is continuous. For $A \subset S$, let $L(A) = \{z: z \leqq x$ for some $x \in A\}$ and $M(A) = \{y: x \leqq y$ for some $x \in A\}$. A set A is *increasing* if $M(A) = A$. An *arc chain* is a totally ordered subset of a semilattice whose underlying space is an arc. We reserve I for the unit interval under min multiplication, and T for the quotient semilattice obtained by identifying (0, 0) and (1, 0) in $\{0, 1\} \times I$. Note that I^n and T^n, under coordinatewise multiplication, are semilattices with identity on the n-cell, with zero in the boundary and interior respectively.

Let X be a cell-cyclic Peano continuum. We use the cyclic element notation and results of Whyburn [10] and Kuratowski and Whyburn [8], slightly modified in the following way. In X we say a set A separates a and b if each arc from a to b meets A. $C(p, q)$ denotes the *cyclic chain from p to q* and is $\{x \in X \mid$ some arc from p to q contains $x\}$. An subcontinuum A of X is an *A-set* if each arc in X having end points in A is contained in A. Cyclic elements and cyclic chains are A-sets. Given a point x and an A-set A, if $x \notin A$ there is a unique element $y \in A$ such that y separates each element of A from x. Denote this y by $P(A, x)$. If $x \in A$ set $P(A, x) = x$. Then for a fixed A-set A the function $x \to P(A, x)$ is a monotone retraction of X onto A mapping $X \backslash A$ into $Fr(A) = \{x \in A \mid x \notin D^0$ for any cyclic element D of $A\} \cup \{$cut points of $A\}$. A set M is *nodal* in X if $M \cap (X \backslash M)^*$ contains at most one point. A point is an *end point* of X if it has a basis of neighborhoods having one point boundary. A *node* of X is either (i) a true cyclic element which is a nodal set or (ii) an endpoint. By $\mathrm{Com}(x, A)$ we mean the component of x in A. The interior of A is denoted by A^o.

I. Preliminary results.

THEOREM 1.1. [12]. *Any cell-cyclic Peano continuum admits the structure of a semilattice.*

We note that in the proof of 1.1 given in [12], I^n and T^n, as defined above, could have been used for the semilattice structures on the individual cyclic elements. Thus the structure may be so constructed that each cyclic element is a semilattice with identity; also the zero may be chosen to by any predetermined point.

The following is an unpublished result due to Phyrne Bacon. We include a proof for completeness.

THEOREM 1.2. *Let X be a compact semilattice and C an arc chain contaimining* 0. *If Π_C is defined by* $\Pi_C(x) = \sup\{a \in C \mid x \in M(a)\}$, *then*

(i) *Π_C is a homomorphism from X onto C*

(ii) *Π_C is continuous iff whenever $x, y \in C$ and $x < y$ then $y \in M(x)^o$.*

Proof. X compact implies Π_C is well-defined. For (i), first note that Π_C is order preserving. Let $x, y \in X$ and suppose $\Pi_C(x) \leqq \Pi_C(y)$. Since Π_C is order preserving we have $\Pi_C(xy) \leqq \Pi_C(x)$. If $\Pi_C(xy) < \Pi_C(x)$, then there exists $z \in C$ such that $\Pi_C(xy) < z < \Pi_C(x)$. Thus $x \in M(z)$ and $xy \notin M(z)$. But $\Pi_C(x) < \Pi_C(y)$ and $x \in M(z)$ implies $y \in M(z)$. We conclude $xy \in M(z)$, a contradiction. Thus

$$\Pi_C(xy) = \Pi_C(x) = \Pi_C(x)\Pi_C(y) .$$

By symmetry, if $\Pi_C(y) \leqq \Pi_C(x)$ then the same conclusion is reached, and Π_C is a homomorphism.

For (ii), suppose whenever $x, y \in C$ and $x < y$, then $y \in M(x)^o$. For each $x \in C$ define $V(x) = X \backslash M(x)$. Then each $V(x)$ is open, and we claim that $x < y$ implies $V(x)^* \subset V(y)$. First note that $M(M(x)^o)$ is open by the continuity of multiplication, contains $M(x)^o$, and is contained in $M(x)$. Thus $M(M(x)^o) = M(x)^o$, and $M(x)^o$ is increasing. So if $x < y$, then $y \in M(x)^o$, and $M(y) \subseteqq M(M(x)^o) = M(x)^o$. Thus $V(y) = X \backslash M(y)$ contains $X \backslash M(x)^o = [X \backslash M(x)]^* = V(x)^*$. Since C is an arc chain, $\inf\{a \in C \mid x \in V(a)\} = \sup\{a \in C \mid x \in M(a)\} = \Pi_C(x)$. Thus a proof like that for Urysohn's lemma [3] shows Π_C is continuous. This completes the proof.

It is implicit in results of Lawson [9] that if X is a semilattice on a finite dimensional Peano continuum, then (i) each point of X lies on an arc chain C containing 0, and (ii) if $x < y$ in C, then $y \in M(x)^o$. We conclude

COROLLARY 1.3. *Each point of a finite dimensional Peano continuum X lies on an arc chain C containing* 0 *and there is a homomorphic retraction of X onto C.*

THEOREM 1.4. *Any finite dimensional cell-cyclic chain $C(p, q)$ admits a semilattice with identity. Moreover, if $q \in Fr(C(p, q))$ then q can be chosen to be the identity.*

Proof. Note that the true cyclic elements of $C(p, q)$ form a countable collection $\{D_i\}$. We consider two cases:

Case 1. Some true cyclic element D_0 of $C(p, q)$ contains q. Then D_0 admits a semilattice structure with zero $a = P(D, p) \neq q$ and identity e. Moreover if $q \in Fr(C(p, q))$ then $q \in Fr(D_0)$, so we may choose $e = q$. By 1.1, $C(p, a)$ admits a semilattice in which each cyclic element D_i is a semilattice with identity e_i and zero $P(D_i, p)$. In each D_i there is an arc chain T_i from e_i to $b_i = P(D_i, q)$ and also an arc chain T_0 in D_0 from e to a and a homomorphism $h: D_0 \to T_0$ which is a retraction. Let $f_i: T_0 \to T_i$ be an onto homomorphism for each i. Now define a semilattice structure $*$ on $C(p, q)$ to agree with those on $C(p, a)$ and D_0 and such that if $x \in C(p, a)$ and $y \in D$ then

$$x*y = y*x = \begin{cases} x \text{ if } x \text{ is a cut point of } C(p, a) \\ x \cdot f_i(h(y)) \text{ if } x \in D_i \end{cases}$$

This obviously idempotent and commutative. Associativity and continuity follow since h and f_i are homomorphisms and continuous. Note that e is an identity for $*$.

Case 2. q is not in any true cyclic element of $C(p, q)$. Then there is a sequence $\{c_i\}$ of distinct cut points of $C(p, q)$ such that $\{c_i\} \to q$ and c_{i+1} separates c from q. This implies

$$C(p, q) \setminus \bigcup_{i=1}^{\infty} C(c_i, c_{i+1}) = \{q\} .$$

Endow each $C(c_i, c_{i+1})$ with a semilattice structure as in 1.1 so that c_i is the zero of $C(c_i, c_{i+1})$ and each cyclic element D_j is a semilattice with zero $P(D_j, p)$ and identity e_j, and let T_j be a (possibly degenerate) arc chain in D_j from e_j to $P(D_j, q)$. Let S_i be an arc chain in $C(c_i, c_{i+1})$ from c_i to c_{i+1} and let $h_i: C(c_i, c_{i+1}) \to S_i$ be a homomorphism and retraction. For each $i, j \in Z^+$, let $f_{i,j}: S_i \to T_j$ be an onto homomorphism. Now define an operation $*$ on $C(p, q)$ to agree with that on each $C(c_i, c_{i+1})$ and such that if $x \in C(c_m, c_{m+1})$ and $y \in C(c_n, c_{n+1})$ then

$$y*x = x*y = \begin{cases} x \text{ if } x \text{ is a cut point and } n = m+1 \\ x \text{ if } n > m+1 \\ xf_{n,j}(h_n(y)) \text{ if } x \text{ is not a cut point} \\ \qquad \text{(i.e., } x \in D_j \text{ for some } j \in Z^+\text{) and} \\ \qquad n = m+1 \\ xy \text{ if } n = m \end{cases}$$

Define q to be an identity for $C(p, q)$.

This is obviously idempotent and commutative. The proof of associativity is similar to that in Case 1 except in the following case: Suppose $x \in C(c_n, c_{n+1})$, $y \in C(c_{n+1}, c_{n+2})$ and $z \in C(c_{n+2}, c_{n+3})$. If x is a cut point, then $x*y*z = x$ in any order, and if y is a cut point then $x*y*z = x*y$ in any order. If neither is a cut point then $x \in D_z$ and $y \in D_k$ for some true cyclic elements D_j and D_k. So

$$(x*y)*z = x*y = xf_{n+1,j}(h_{n+1}(y)) .$$

Now $x*(y*z) = x*(yf_{n+2,k}(h_{n+2}(z)) = xf_{n+1,j}(h_{n+1}(yf_{n+2,k}(h_{n+2}(z))))$. But $h_{n+1}(yf_{n+2,k}(h_{n+2}(z))) = h_{n+1}(y)h_{n+1}(f_{n+2,k}(h_{n+2}(z)))$ since h_{n+1} is a homomorphism. Also $h_{n+1}(y) \leqq P(D_k, q) = h_{n+1}(f_{n+2,k}(h_{n+2}(z)))$ since $S_{n+1} \cap D_k$ is an arc chain with maximum element $P(D_k, q)$ and T_k is an arc chain with minimum element $P(D_k, q)$. It follows that

$$x*(y*z) = xf_{n+1,j}(h_{n+k}(y)) = x*y = (x*y)*z .$$

Suppose $x_n \to x$ and $y_n \to y$. If $x \neq q \neq y$, then one can prove $x_n*y_n \to xy$ using the continuity of the functions h_i and $f_{n,j}$ and the fact that the cyclic chains $C(c_i, c_{i+1})$ meet only at cut points. If $x = q \neq y$ and $y \in C(c_i, c_{i+1})$ then eventually $c_{i-1} \leqq y_n \leqq c_{i+2}$ and $c_{i+4} \leqq x_n$ so that $x_n*y_n = y_n \to y = xy$. If $x = q = y$ and if $W(x_n, y_n)$ denotes the smaller of i and j where $x_n \in C(c_i, c_{i+1})$ and $y_n \in C(c_j, c_{j+1})$ then $W(x_n, y_n) \to \infty$ as $n \to \infty$. Since $x_n*y_n \in C(c_{W(x_n,y_n)}, c_{W(x_n,y_n)+1})$ and since $C(p, q)$ is locally connected we conclude that $x_n*y_n \to q = xy$. This completes the proof.

We note that in Case 2, if c_{n+1} separates x from p and c_n separates y from q then $x*y = y$.

II. Ruled continua.

DEFINITION 2.1. Suppose X is a topological space and $E \subseteq X$,

$0 \in X$. Let $A = \{[0, e]: e \in E\}$ be a collection of arcs in X satisfying:

(i) $X = U\{[0, e]: e \in E\}$.

(ii) $[0, e] \cap [0, f]$ is a proper subarc of each when e and f are distinct elements of E.

(iii) For each $e \in E$, there is a unique $[0, e] \in A$.

(iv) If $x_\alpha \rightarrow x$ then $[0, x_\alpha] \rightarrow [0, x]$ in the sense of lim sup-lim inf convergence.

Then A is said to be a *ruling* of X and X is said to be a *ruled space* with zero 0. The concept of a ruled space was introduced by Eberhart in his dissertation [4]. Spaces admitting a stronger type of ruling have been studied by Koch and McAuley [7]. We note that if X is ruled then for each $x \in X$ there is a unique arc $[0, x]$ which is contained in every $[0, e]$ containing x.

DEFINITION 2.2. A metric d is *radially convex* with respect to a partial order $\leqq$ on X if $x \leqq y$, $y \leqq z$ and $y \neq z$ imply $d(x, y) < d(x, z)$.

LEMMA 2.3. *Let X be a compact metric ruled space. Define $x \leqq y$ iff $x \in [0, y]$. Then $\leqq$ is a closed partial order on X. Moreover X admits a metric radially convex with respect to this order, so that if $r \leqq d(0, e)$ there is a unique $x(r) \in [0, e]$ such that $d(0, x(r)) = r$.*

Proof. This is clearly a partial order; that it is closed follows from property iv) of ruled spaces. By a result of Carruth [2], X admits a metric radially convex with respect to this order. The lemma now follows.

THEOREM 2.4. *Any cell-cyclic Peano continuum X admits a ruling, and 0 may be chosen to be any point of X.*

Proof. By 1.1, X admits a semilattice with zero 0 chosen arbitrarily. As in the proof of 1.1 given in [12], for each true cyclic element D of X let h_D denote the homeomorphism from I^n or T^n to D used to define this semilattice. Set $E = \mathrm{Fr}(X)\backslash(\{\text{cut points of } X\} U\{0\})$. For each $e \in E$ and each true cyclic element D of $C(0, e)$ define $T(D, e)$ to be the image under h_D of the straight line segment $[h_D^{-1}(P(D, 0)), h_D^{-1}(P(D, e))]$ in I^n or T^n. Then define $[0, e] = (\cup\{T(D, e): D \in C(0, e)\}) \cup \{\text{cut points of } C(0, e)\}$. Then $[0, e]$ is a metric, compact (since $C(0, e)\backslash[0, e]$ is open in $C(0, e)$) order dense chain in the semilattice X and hence an arc. We now show the four conditions are satisfied.

(i) $X = U\{0, e]: e \in E\}$. If $x \in X \backslash E$ then x is either an interior point of some cyclic element D of X or a cut point of X. If x is an interior point of D then $x \in h_D(h_D^{-1}(P(D, 0),\ h_D^{-1}(e)])$ for some $e \in \mathrm{Fr}(D)$. If $e \in E$ then $x \in T(D, e) \subseteqq [0, e]$. If $e \notin E$ then choosing an end element e' of a component of $X \backslash \{e\}$ other than the one containing 0, $x \in [0, e']$

(ii) and (iii) are clear

(iv) If $e_\alpha \rightarrow e$, then $[0, e_\alpha] \rightarrow [0, e]$. This follows from the fact that $[0, e_\alpha] \subseteqq L(e_\alpha)$ and from techniques like those in [12]. We omit the details.

THEOREM 2.5. *Any cell-cyclic Peano continuum with a nodal cyclic element admits a commutative semigroup with identity and zero.*

Proof. Let X be a cell cyclic Peano continuum and suppose $X = C \cup D$, where $C \cap D = \{0\}$ and D is a true cyclic element. Then C is a cell-cyclic Peano continuum and hence admits a ruling $A = \{[0, e]: e \in E\}$ with zero 0 and a radially convex metric. Let h be a homeomorphism from I^n or T^n to D, depending on whether 0 is in the boundary or interior of D, and define a semilattice with identity e on D using h. Then there is in D an arc chain S from 0 to e and a retraction $f: D \rightarrow S$ which is a homomorphism. Moreover we may assume that S is radially convex so that for $x, y \in S$, $d(0, xy) = \min\{d(0, x), d(0, y)\}$. Without loss of generality we may assume $d(0, e)$ is maximal among $\{d(0, x) \mid x \in X\}$. Now define a semigroup on X by

$$y*x = x*y = \begin{cases} 0 \text{ if } x, y \in C \\ xy \text{ if } x, y \in D \\ \text{The point in } [0, x] \text{ of distance } r = \\ \min\{d(0, x), d(0, f(y))\} \text{ from } 0 \text{ if } x \in C,\ y \in D\,. \end{cases}$$

Associativity is obvious in all cases except the following: Suppose $x \in C$ and $y, z \in D$. Then $(x*y)*z$ is the point in $[0, x]$ of distance $\min\{d(0, x), d(0, f(y)), d(0, f(z))\}$ from 0, whereas $x*(y*z)$ is the point in $[0, x]$ of distance $\min\{d(0, x), d(0, f(y, z))\}$ from 0. But $d(0, f(yz)) = d(0, f(y)f(z)) = \min\{d(0, f(y)), d(0, f(z))\}$ so $(x*y)*z = x*(y*z)$. Continuity follows from the properties of ruled spaces and the fact that f is continuous. It is clear that e is an identity and 0 a zero. This completes the proof.

We conjecture that any X as in 2.5 admits a semilattice with identity. In fact, if X can be embedded in a plane then X can be embedded in a two-cell N and ruled in such a way that $X \cap Fr(N)$ is one of the arcs ruling X. One can now apply a theorem from

Eberhart's dissertation to show that X admits a semilattice with identity.

III. Cell-cyclic Peano continua without a nodal cyclic element. The goal of this section is a result like 2.5 for finite dimensional cell-cyclic Peano continua without a nodal cyclic element.

LEMMA 3.1. *Let X be a cell-cyclic Peano continuum. Then there exist two sequences $\{p_i\}$ and $\{q_i\}$ in $Fr(X)$, with p_1 and q_1 chosen arbitrarily, such that*

(i) *If we set $H_n = \bigcup_{i=1}^{n} C(p_i, q_i)$, then for each $n > 1$, $\{p_n\} = C(p_n, q_n) \cap H_{n-1}$*

(ii) *If we set $H = \bigcup_{n=1}^{\infty} H_n$, then each point of $X \backslash H$ is an end point of X, and so $H^* = X$.*

(iii) *The diameter of the components of $S \backslash H_n$ tends to 0 uniformly with $1/n$.*

Proof. This was proved by Whyburn ([10], p. 73) without the condition that $\{p_i\}$ and $\{q_i\}$ are in $Fr(X)$. We show this condition can also be assumed. Whyburn's proof considers a dense sequence $\{r_i\}$ and sets $p_1 = r_1$, $q_1 = r_2$. Clearly these may be chosen arbitrarily in $Fr(X)$. In Whyburn's proof, for $j > 1$ q_j is the r_i of smallest index such that $r_i \notin H_{j-1}$ and $p_j = P(H_{j-1}, q_j)$. Thus $p_j \in Fr(X)$. If $q_j \notin Fr(X)$, then q_j is an interior point of some true cyclic element D. Let q'_j be any point in $Fr(D)$ other than $P(D, p_j)$. Then $C(p_j, q_j) = C(p_j, q'_j)$, so we may assume $q_j \in Fr(X)$. The lemma follows.

Now let X be a finite dimensional cell-cyclic Peano continuum without a nodal cyclic element. Then X has at least 2 end points ([10], p. 77); let 0 and 1 denote end points of X. Let $\{p_i\}$, $\{q_i\}$, $\{H_n\}$, and H be as described in 3.1, with $p_1 = 0$, $q_1 = 1$. Each $C(p_i, q_i)$ admits a semilattice with zero p_i and identity q_i by 1.3. We now define inductively an algorithm for defining a semilattice with identity on H.

Let $\{c_j\}$ be the sequence of cut points of $C(0, 1)$ converging to 1 such that c_{j+1} separates c_j from 1 used in 1.3 to define the semilattice on $C(0,1)$. Let n_1 be one more than the smallest i such that c_i separates p_2 from 1 in X. Set $Q_1 = C(p_2, q_2)$, $P_1 = [\mathrm{Com}\,(1, C(0, 1) \backslash \{c_{n_1}\})]^*$, and $R_1 = [\mathrm{Com}\,(0, C(0, 1) \backslash \{c_{n_1}\}]^*$. Let T_1 be an arc chain from p_2 to q_2 in Q_1 and S_1 be an arc chain from c_{n_1} to 1 in $C(0, 1)$. Let $f_1: S_1 \rightarrow T_1$ be a continuous onto homomorphism such that $f_1^{-1}(q_2) = M(c_{n_1+1}) \cap S_1$, and let $h_1: P_1 \rightarrow T_1$ be the continuous onto homomorphism obtained by composing f_1 and a homomorphic retraction r_1 of $C(c_{n_1}, 1)$ onto S_1. We now define a semilattice $*$ on $H_2 = C(p_1, q_1) \cup C(p_2, q_2) = H_1 \cup Q_1$ by

$$x*y = y*x = \begin{cases} xy \text{ if } x, y \in C(0, 1) = P_1 \cup R_1 \text{ or } x, y \in Q_1 \\ xp_2 \text{ if } x \in R_1,\ y \in Q_1 \\ h_1(x)y \text{ if } x \in P_1,\ y \in Q_1 \end{cases}$$

where juxtaposition means whichever of the previously defined operations on H_1 or Q_1 fits the context.

Associativity is clear in all cases except when $r \in R_1$, $p \in P_1$, $q \in Q_1$. In this case $r*(p*q) = r*(h_1(p)q) = rp_2$, whereas

$$(r*p)*q = (rp)p_2 = r(pp_2) = rp_2$$

by the note at the end of Section I. Continuity is easily checked since P_1, Q_1 and R_1 meet only at cut points of X. Note that any point in $C(c_{n_2+1}, 1)$ acts as an identity for any point in

$$[\mathrm{Com}\,(0, H_2\backslash\{c_{n_1}\})]^*\ ,$$

and 1 acts as an identity for all of H_2.

Suppose that a semilattice structure with zero 0 and identity 1 has been defined on H_{k-1} so that the structure agrees with those on $C(P_i, q_i)$ for each $i \leqq k$. Also suppose $c_{n_{k-1}} \in \{c_i\}$ has been chosen so that any element of $[\mathrm{Com}\,(1, H_{k-1}\backslash\{c_{n_{k-1}+1}\})]^*$ acts as an identity for any element $[\mathrm{Com}\,(0, H_{k-1}\backslash\{c_{n_{k-1}}\})]^*$.

Let n_k be one more than the smallest integer greater than n_{k-1} such that c_{n_k} separates p_{k+1} from 1. Set $Q_k = C(p_{k+1}, q_{k+1})$, $P_k = C(c_{n_k}, 1) = [\mathrm{Com}\,(1, H_k\backslash\{c_{n_k}\})]^*$, and $R_k = [\mathrm{Com}\,(0, H_k\backslash\{c_{n_k}\})]^*$. Let T_k be an arc chain from P_{k+1} to q_{k+1} in Q_k and $S_k = S_1 \cap P_k$. Let $f_k: S_k \rightarrow T_k$ be a continuous onto homomorphism such that

$$f_k^{-1}(q_{k+1}) = M(c_{n_k+1}) \cap S_k$$

in P_k, and let $h_k: P_k \rightarrow T_k$ be a continuous onto homomorphism obtained by composing f_k and the homomorphic retraction $r_k = r_1 | P_k$ of $C(c_{n_k}, 1) = P_k$ onto S_k. We now define a semilattice $*$ with identity 1 on H_k by

$$x*y = y*x = \begin{cases} xy \text{ if } x, y \in H_{k-1} \text{ or } x, y \in Q_k \\ xp_k \text{ if } x \in R_k,\ y \in Q_k \\ h_k(x)y \text{ if } x \in P_k,\ y \in Q_k \end{cases}$$

where juxtaposition means whichever of the previously defined operations on H_k or Q_k fits the context.

Again associativity is clear in all cases except when $r \in R_k$, $p \in P_k$, $q \in Q_k$. In this case $r*(p*q) = r*(h_k(p)q) = rp_k$, whereas $(r*p)*q = (rp)p_k = r(pp_k)$ since the operation on H_{k-1} is associative. But $p \in [\mathrm{Com}\,(1, H_{k-1}\backslash\{c_{n_{k-1}+1}\})]^*$ and $p_k \in [\mathrm{Com}\,(0, H_{k-1}\backslash\{c_{n_{k-1}}\})]^*$ so by hypothesis $pp_k = p_k$, and $r*(p*q) = (r*p)*q$. Continuity is again easily checked. Again any point in $[\mathrm{Com}\,(1, H_k\backslash\{c_{n_k+1}\})]^*$ acts as an identity for any element $[\mathrm{Com}\,(0, H_k\backslash\{c_{n_k}\})]^*$. By induction we have proved the following:

LEMMA 3.2. *Each H_n admits a semilattice with zero 0 and identity 1 so that the operations agree whenever possible.*

LEMMA 3.3. *The function $P(H_n, \cdot)\colon H \to H_n$ is a retraction and a homomorphism for each n.*

Proof. It has been previously noted that each $P(H_n, \cdot)$ is a retraction. To show that each is a homomorphism it suffices to show that the restriction of $P(H_n, \cdot)$ to H_{n+1} is a homomorphism, since $P(H_n, \cdot)$ is the composition of this restriction and $P(H_{n+1}, \cdot)$. Let $x, y \in H_{n+1} = H_n \cup Q_n$. If $x, y \in Q_n$ then

$$P(H_n, x)*P(H_n, y) = p_n*p_n = p_n = P(H_n, x*y)$$

since $x*y \in Q_n$. If $x \in Q_n$, $y \in H_n$ then there are two cases. If $y \in P_n$ then $P(H_n, x)*P(H_n, y) = p_n*y = p_n$ since $p_n \in R_{n-1}$ by definition and any element of P_n acts as an identity for any element of R_{n-1}. However $P(H_n, x*y) = P(H_n, x*h_n(y)) = p_n$ since $x*h_n(y) \in Q_n$. If $y \in R_n$ then $P(H_n, x)*P(H_n, y) = p_n*y = x*y = P(H_n, x*y)$. This completes the proof of the lemma.

LEMMA 3.4. *Let X be as above and let $x, y \in X$, and suppose $\{x_n\}$, $\{y_n\}$ are sequences in H such that $x_n \to x$, $y_n \to y$. Then there exists $z \in X$ such that $\{x_n*y_n\} \to z$, where $*$ denotes the operation on any H_n containing x_n and y_n, and z is independent of the choice of the sequences.*

Proof. We distinguish four cases.

Case I. $x = y = 1$. From the definition of multiplication on H, if a, $b \in P_k = [\mathrm{Com}\,(1, H\backslash\{c_k\})]^*$ then $a*b \in P_k$. Now $\{P_k\}$ forms a neighborhood basis at the end point 1. Since both $\{x_n\}$ and $\{y_n\}$ are eventually in each P_k, $\{x_n*y_n\}$ is eventually in each P_k and hence $\{x_n*y_n\} \to 1$.

Case II. x, y, and 1 all distinct. Let N be an integer so large that $P(H_0, x)$ and $P(H_0, y)$ are in $\mathrm{Com}\,(0, H_0\backslash\{c_N\})$ and that the diameter of any component of $X\backslash H_N < d(x, y)/2$. This implies $\mathrm{Com}\,(x, X\backslash H_N)$ and $\mathrm{Com}\,(y, X\backslash H_N)$ are disjoint open sets, and we may assume $x_N \in \mathrm{Com}\,(x, X/H_N)$ and $y_n \in \mathrm{Com}\,(y, X\backslash H_N)$ for all n. Also we may assume $d(x_n, y_n) > d(x, y)/2$ for all n. We now show

$x_n*y_n = P(H_N, y_n)*P(H_N, y_n)$ for all n. The statement is obvious if $x_n, y_n \in H_N$. Suppose it is true whenever $x_n, y_n \in H_m$ for some $m \geqq N$, and let $x_n, y_n \in H_{m+1} = H_m \cup Q_m$. If $x_n \in Q_m$ and $y_n \in H_m$ then $x_n*y_n = p_m*y_n$. By hypothesis, since $p_m, y_n \in H_m$ then

$$p_m*y_n = P(H_N, p_m)*P(H_N, y_n) .$$

But $P(H_N, p_m) = P(H_N, x_n)$ since $Q_m \subset [\mathrm{Com}\,(x_n, X\backslash H_N)]^*$. Thus $x_n*y_n = P(H_N, x_n)*P(H_N, y_n)$. By symmetry the statement is true if $x_n \in H_m$ and $y_n \in Q_m$. The statement is obvious if both $x_n, y_n \in H_m$, whereas the case $x_n, y_n \in Q_m$ is impossible for it implies $d(x_n, y_n) < d(x, y)/2 < d(x_n, y_n)$.

We know H_N is a semilattice and hence

$$x_n*y_n = P(H_N, x_n)*P(H_N, y_n) \longrightarrow P(H_N, x)*P(H_N, y)$$

since $P(H_N, \cdot)$ is continuous.

Case III. $x = y \neq 1$

(a) $x = y \notin H$. Then $x = y$ is an end point of X and $\{U_i\} = \{[\mathrm{Com}\,(x, X\backslash H_i)]^*\}$ is a neighborhood basis at $x = y$. We show that if U_i is fixed and if $x_n, y_n \in U_i \cap H_N$ then $x_n*y_n \in U_i \cap H_N$, for any N. Note the statement is true for $N \leqq i$. Suppose it is true whenever $x_n, y_n \in U_i \cap H_m$ for some $m \geqq i$, and let

$$x_n, y_n \in U_i \cap H_{m+1} = U_i \cap (H_m \cup Q_m) .$$

If $x_n \in Q_m$ and $y_n \in H_m$ then $x_n*y_n = p_m*y_n \in U_i \cap H_m \subset U_i \cap H_{m+1}$ by the induction hypothesis. By symmetry the statement is true if $x_n \in H_m$ and $y_n \in Q_m$. If $x_n, y_n \in Q_m$ then $x_n*y_n \in Q_m \subset U_i \cap H_{m+1}$, and if $x_n, y_n \in H_m$ the statement follows from the induction hypothesis.

Since $\{x_n\}$ and $\{y_n\}$ are eventually in each U_i, and since for each n and each i we can find $N(n, i)$ such that $x_n, y_n \in U_i \cap H_{N(n,i)}$, we conclude that $\{x_n*y_n\}$ is eventually in each U_i. Thus $\{x_n*y_n\} \rightarrow x = y$.

(b) $x = y \in H_N$, some N. Let $\varepsilon > 0$. There exists $L > N$ so that the diameter of any component of $X\backslash H_L$ is less that $\varepsilon/2$, and so that $B(x, \varepsilon/2) \cap P_L = \varnothing$. We may assume $d(x_n, x) < \varepsilon/2$ and $d(y_n, y) < \varepsilon/2$ for each n. Divide $\{x_n*y_n\}$ into two (perhaps finite) sequences: If $x_n*y_n \in H_L$ then

$$\begin{aligned} x_n*y_n &= P(H_L, x_n*y_n) \\ &= P(H_L, x_n)*P(H_L, y_n) \rightarrow P(H_L, x)*P(H_L, x) = xy = x = y , \end{aligned}$$

by Lemma 3.3 and the continuity of multiplication on H_L. If $x_n*y_n \notin H_L$, then $x_n \notin H_L$ and $y_n \notin H_L$ because $B(x, \varepsilon/2) \cap P_L = \varnothing$ and using the definition of multiplication on H. Also, using the definition of multiplication $x_n*y_n \in \mathrm{Com}\,(x_n, X\backslash H_L)$ or $x_n*y_n \in \mathrm{Com}\,(y_n, X\backslash H_L)$. Thus

$$d(x, x_n*y_n) \leqq d(x, x_n) + d(x_n, x_n*y_n) < \varepsilon$$

or

$$d(y, x_n*y_n) \leqq d(y, y_n) + d(y_n, x_n*y_n) < \varepsilon\ .$$

In either case $d(x, x_n*y) = d(y, x_n*y_n) < \varepsilon$. We conclude that $\{x_n*y_n\} \to x = y$.

Case IV. $y \neq x = 1$. We first establish two facts.

(A) If $a, b \in H$ so that $P(H_0, a) \in \mathrm{Com}\,(1, H_0\backslash\{c_n\})$ and $P(H_0, b) \in \mathrm{Com}\,(0, H_0\backslash\{c_n\})$ for some n, then $a^*b = P(H_0, a)^*b$.

The proof is by the induction on the H_i containing a. It is clear for $a \in H_0$. Suppose the statement is true for $a \in H_m$, $m \geqq 0$, and let $a \in H_{m+1} = H_m \cup Q_{m+1}$. Suppose $a \in Q_{m+1}$, for the induction hypothesis assures the statement is true if $a \in H_m$. Then since a and b are separated by c_m, $b \notin Q_{m+1}$. Hence $a*b = p_{m+1}*b$. But $p_{m+1}*b = P(H_0, p_{m+1})*b$ by the induction hypothesis, and

$$P(H_0, p_{m+1}) = P(H_0, a)\ ,$$

so

$$a*b = P(H_0, a)*b\ .$$

Thus (A) is established.

(B) If $a, b \in H$ so that $a \in \mathrm{Com}\,(1, H_0\backslash\{c_n\})$ and $b \in \mathrm{Com}\,(0, H_0\backslash\{c_n\})$ for some n, then either $a^*b = a^*P(H_n, b)$ or $a^*b \in \mathrm{Com}\,(b, X\backslash H_n)^*$.

The proof is by induction on the H_i containing b. If $b \in H_n$ then $P(H_n, b) = b$ and the statement is true. Suppose the statement is true when $b \in K_m$ for some $m \geqq n$, and let $b \in Q_{m+1}$. If $a \in \mathrm{Com}\,(1, H_0\backslash\{c_m\})$ then $a * b \in Q_{m+1} \subset \mathrm{Com}\,(b, X\backslash H_n)^*$. If $a \in [\mathrm{Com}\,(0, H_0\backslash\{c_m\})]^*$ then $a*b = a*p_m$. But $a*p_m = a*P(H_n, p_m)$ by the induction hypothesis, and $P(H_n, p_m) = P(H_n, b)$. Thus $a*b = a*P(H_n, b)$ and (B) is established.

We now distinguish two subcases of Case IV.

Subcase 1. $y \in H_M$, some M. Let $\varepsilon > 0$. Choose M so large that c_M does not separate y from 0 and the diameter of any component of $X\backslash H_M$ is less than $\varepsilon/2$. We may assume that for each n, $P(H_0, y_n) \in \mathrm{Com}\,(0, H_0\backslash\{c_M\})$ and $P(H_0, x_n) \in \mathrm{Com}\,(1, H_0\backslash\{c_M\})$. Then by (A), $x_n*y_n = P(H_0, x_n)*y_n$, and by (B), $P(H_0, x_n)*y_n = P(H_0, x_n)*P(H_M, y_n)$ or $P(H_0, x_n)*y_n \in \mathrm{Com}\,(b, X\backslash H_n)^*$. If the former then

$$x_n*y_n = P(H_0, x_n)*P(H_M, y_n) \longrightarrow 1*P(H_M, y) = y$$

by the continuity of the multiplication on H_M and Lemma 3.3. In the latter case $d(P(H_0, x_n)*y_n, y_n) < \varepsilon/2$. We may assume $d(y_n, y) < \varepsilon/2$, so $d(y, P(H_0, x_n)*y_n) < \varepsilon$. Thus we conclude that $\{x_n*y_n\} \to y$.

Subcase 2. $y \notin H$. If $V_k = [\mathrm{Com}\,(y, X\backslash H_k)]^*$ then $\{V_k\}$ is a neighborhood basis, so we need only show $\{x_n*y_n\}$ is eventually in each V_k. Fix a V_k. We may assume again that for each n, $P(H_0, y_n) \in \mathrm{Com}\,(0, H_0\backslash\{c_M\})$, $P(H_0, x_n) \in \mathrm{Com}\,(1, H_0\backslash\{c_M\})$, and $y_n \in V_k$ for some $M \geqq k$. By (A) and (B), $x_n*y_n = P(H_0, x_n)*P(H_M, y_n)$ or $x_n*y_n \in \mathrm{Com}\,(y_n, X\backslash H_M)^* \subset V_k$. However $P(H_M, y_n) \in V_k$, and $P(H_0, x_n) \in H_0$, so $P(H_0, x_n)*P(H_M, y_n) \in V_k$. This completes the proof of the lemma.

THEOREM 3.5. *Let X be a finite dimensional cell-cyclic Peano continnum without a nodal element. Then X admits a semilattice with identity.*

Proof. By the above, the dense set H admits a semilattice with identity. For each $x, y \in X$ let $\{x_n\} \to x$, $\{y_n\} \to y$ where $\{x_n\}$, $\{y_n\}$ are sequences in H. Define $xy = \lim\{x_n*y_n\}$. By 3.4 this limit exists and is independent of the choice of the sequences. It follows that this operation is a semilattice with identity on X. Combining this with Theorem 2.3 we have

COROLLARY 3.6. *Let X be a finite dimensional cell-cyclic Peano continuum. Then X admits a commutative semigroup with identity and zero.*

COROLLARY 3.6. *Any retract of a two-cell admits a commutative semigroup with identity.*

Proof. Borsuk [1] has shown that a subset X of a two-cell A is a retract of A if and only if A is a locally connected continuum which does not separate the plane. Whyburn [11] has shown that for locally connected continua in the plane, not separating the plane is equivalent to every cyclic element being a simple closed curve with interior, i.e., a two-cell. Thus a retract of a two-cell is a cell-cyclic Peano continuum, and the result follows from Corollary 3.6.

DEFINITION 3.8. A space X is homogeneous if for each pair of points x and y in X there is a homeomorphism of X onto itself carrying x to y.

THEOREM 3.9. *Any finite dimensional homogeneous cell-cyclic*

Peano continuum (in particular, any homogeneous retract of a two-cell) *is a point.*

Proof. By a result of Hudson and Mostert [5], any homogeneous compact connected semigroup with identity is a group. Combining this with Corollaries 3.6 and 3.7, unless X is a point X admits the structure of a group with two idempotents, a contradiction.

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