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A SOBOLEV SPACE AND A DARBOUX PROBLEM

M. B. SURYANARAYANA

This paper deals with a class of functions which are defined in an n-dimensional rectangle and which possess there, only the generalized partial derivatives of mixed type. It is shown that (i) this class contains as a proper subset the usual Sobolev class of order n, the dimension of the domain and (ii) this class can be imbedded in the space of continuous functions. In addition to the compactness of the imbedding operator, the closedness of certain nonlinear partial integro differential operators is also studied. Finally, a system of partial integro differential equations with Darboux type boundary data in a rectangle, is shown to have solutions in this class. The results of this paper are used in certain existence theorems of optimal control theory.

1. Introduction. In recent studies on existence theorems for optimization problems involving Darboux type side conditions, it was found useful and necessary to introduce a special class of functions, (see for example [11] and [7]). This special class which we shall denote as $W_p^*(G)$, $G \subset E^n$, consists of all functions $z(t)$, $t \in G$, with $z \in L_p(G)$, and such that z has all (and only) the mixed partial generalized derivatives $D_\alpha z$ of orders upto and including n (the number of independent variables) with $D_\alpha z \in L_p(G)$; thus, derivatives of z taken more than once with respect to any of the variables $t_1, \cdots, t_n$, may not even exist. In the case of $n = 2$, for example, with independent variables x and y, this would mean that for $z \in W_p^*(G)$, the generalized partials z_x, z_y and z_{xy} exist and belong to $L_p(G)$ while the pure partials z_{xx} and z_{yy} need not even exist. This class is thus analogous to the classical $C^*(G)$ where only the derivatives z_x, z_y and z_{xy} exist and are continuous.

Clearly, $W_p^*(G)$ contains $W_p^n(G)$, the usual Sobolev class of functions for which all the generalized partial derivatives of order upto and including n exist and belong to $L_p(G)$. We shall show in no. 2 below that there exist functions in $W_p^*(G)$ which are not in $W_p^n(G)$. One of the purposes of this paper is to analyze this special class of functions $W_p^*(G)$ for $G = [a, a + h] \subset E^n$, $n \geqq 1$ and in particular to show that it can be imbedded in $C(G)$, the space of continuous functions, for all p, $1 \leqq p \leqq \infty$. In particular, it follows that $W_p^n(G) \subset C(G)$ even for $p = 1$. We shall also study criteria for the compactness of the imbedding operator. In the same context, the closedness of some nonlinear integro differential operators is investigated, a result used in the closure theorems relevant to the existence theory of optimal control problems (see [11]).

The last section of this paper is devoted to establish an existence and uniqueness theorem in the special class mentioned above, for solutions of a system of partial integro differential equations with Darboux type boundary data in a rectangle in E^n. This theorem is derived as a consequence of the existence theorems for multi-dimensional integral equations of Volterra type given by the author in [9]. Since the Darboux problem considered here involves a control function also, the existence of solutions for each measurable control under the specified hypothesis), therefore proves the controllability of the system. These results are used in [11].

Examples are provided through out the paper to illustrate the statements.

2. Notations. Let E^n, $n \geqq 1$ denote the n-dimensional Euclidean space. Let

$$\begin{aligned} G &= [a, a+h] \\ &= \{t \in E^n \,|\, t = (t_1, \cdots, t_n),\ a_i \leqq t_i \leqq a_i + h_i,\ i = 1, \cdots, n\} . \end{aligned}$$

Let $C(G)$ denote the space of continuous functions on G, as usual with the supremum norm and let $L_p(G)$ be the space of all measurable functions whose p-th powers are integrable over G, for $1 \leqq p < \infty$. Let $L_\infty(G)$ be the space of all essentially bounded functions measurable on G.

Let T_i denote the operator defined on X^m, $m \geqq 1$ (with $X = C(G)$, $L_p(G)$ or $L_\infty(G)$) as follows:

$$T_i z(t) = \int_{a_i}^{t_i} z(t'_i, s)ds , \quad z \in X^m .$$

Here $t'_i = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n)$ so that $t = (t_i, t'_i)$. We define $T_i T_j$ as composition so that $T_i^r z = T_i(T_i^{r-1} z)$, $r \geqq 1$ and $T_i^0 z = z$ for $z \in X^m$. Let $D_i z$ denote as usual the generalized derivative of z with respect to t_i, $i = 1, \cdots, n$.

A mutiindex $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-vector with arbitrary nonnegative integers α_i, $i = 1, \cdots, n$. As usual $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. Let J denote the set of all multiindices α with $|\alpha| < n$ and $\alpha_i = 0$ or 1 for $i = 1, \cdots, n$. Let $\bar{J} = J \cup \{(1, \cdots, 1)\}$. In the sequel, unless specified otherwise, multiindices (denoted by greek letters) are from $\bar{J}$. Also $\alpha = 1$ shall mean $\alpha_i = 1$ for all i while $|\alpha| = 1$ means $\alpha_i = 1$ for one and only one i. For α, $\beta \in \bar{J}$, the inner product $\Sigma \alpha_i \beta_i$ is denoted by $\alpha \cdot \beta$ and α' denotes $1 - \alpha$; that is $\alpha'_i = 1 - \alpha_i$, $i = 1, \cdots, n$. By $\alpha \leqq \beta$ we shall mean $\alpha_i \leqq \beta_i$ for each i while $\alpha < \beta$ means $\alpha \leqq \beta$ and $\alpha \neq \beta$. Thus, for example,

$(1, 0, 1) < (1, 1, 1)$.

Given $\beta \in J$, there are just as many $\pi \in \bar{J}$, $\pi \cdot \beta = 0$ with $|\pi|$ even, as there are with $|\pi|$ odd (the total number being $2^{n-|\beta|}$). So, $\Sigma(-1)^{|\pi|} = -1$ where the summation is taken over all $\pi \in \bar{J} - \{0\}$ with $\pi \cdot \beta = 0$. The following lemma is now an immediate consequence of this.

LEMMA 1. *Given any real valued function $f(\beta)$ on $\bar{J}$ and $\alpha \in \bar{J}$ we have*

$$\sum_{\substack{0 \leqq \beta < \alpha \\ \beta \in J}} f(\beta) = \sum_{i=1}^{|\alpha|} \sum_{\substack{0 < \pi \leqq \alpha \\ |\pi| = i \\ \pi \in J}} \sum_{\substack{\pi \cdot \beta = 0 \\ 0 \leqq \beta < \alpha \\ \beta \in J}} (-1)^{i-1} f(\beta) .$$

REMARK. The above formula is that of the cardinality of union of a collection of sets.

Illustration. If $\alpha = (1, 0, 1, 0)$ then $0 \leqq \beta < \alpha$ is satisfied by $\beta_1 = (0, 0, 0, 0)$, $\beta_2 = (1, 0, 0, 0)$ and $\beta_3 = (0, 0, 1, 0)$. Also, $0 < \pi \leqq \alpha$ is satisfied by $\pi_1 = (1, 0, 0, 0)$, $\pi_2 = (0, 0, 1, 0)$ and $\pi_3(1, 0, 1, 0)$. In this case, $\pi_1 \cdot \beta_1 = \pi_1 \cdot \beta_3 = 0$; $\pi_2 \cdot \beta_1 = \pi_2 \cdot \beta_2 = 0$ and $\pi_3 \cdot \beta_1 = 0$ so that the right hand side of the equation in the lemma is

$$f(\beta_1) + f(\beta_3) + f(\beta_1) + f(\beta_2) - f(\beta_1) = \sum_{i=1}^{3} f(\beta_i) = \sum_{0 \leqq \beta < \alpha} f(\beta) .$$

In the sequel, multiindices from $\bar{J}$ (that is with $\alpha_i = 0$ or 1 for all i) appear as subscripts in three different ways: (1) As subscripts of the set G or a point in G, they mean projection into the corresponding coordinates. Thus t_α means the $|\alpha|$-vector obtained omitting those t_i in $(t_1, \cdots, t_n)$ for which $\alpha_i = 0$. Also

$$G_\alpha = \{t_\alpha \in E^{|\alpha|} \colon a_\alpha \leqq t_\alpha \leqq a_\alpha + h_\alpha\} .$$

If $\alpha = 0$, that is $\alpha_i = 0$ for all i then G_α is the empty set. (2) As subscripts of an operator (T or D) we mean the composition $T_1^{\alpha_1} \cdots T_n^{\alpha_n}$ of those T_i for which $\alpha_i \neq 0$. If $\alpha = 0$ then $T_\alpha = T_0$ and $D_\alpha = D_0$ are understood as identity operators; Thus $T_0 z = D_0 z = z$. In either of the two cases (1) and (2), a prime $'$ on the top shall indicate the complementary index. Thus $T'_\alpha = T_{\alpha'} = T_{1-\alpha}$. For $n = 3$, and $\alpha = (1, 0, 1)$ for example, $T_\alpha = T_1 T_3$ while $T'_\alpha = T_2$. (3) As subscripts of functions, no further meaning is attached. Thus, $R_\alpha(t)$ is just the αth function on G.

For $1 \leqq k \leqq n$ if k is the subscript of an operator (T or D) then it is same as having a subscript α with $\alpha_i = 1$ for $1 \leqq i \leqq k$ and $\alpha_i = 0$ for $i > k$. Thus, $D^n z = D_\alpha z$ with $\alpha = (1, \cdots, 1)$.

For $1 \leqq p \leqq \infty$, let $W_p^n(G)$ denote as usual (see Sobolev [8] or

Morrey [5]) the set of all $z \in L_p(G)$ with $D_\alpha z \in L_p(G)$ for all α with $|\alpha| \leqq n$; (α may not be in J) We define the space $W_p^*(G)$ as the set of all $z \in L_p(G)$ for which $D_\alpha z \in L_p(G)$ for $\alpha \in \bar{J}$. The norm on $W_p^*(G)$ is defined by $||z||^* = \sum_{\alpha \in \bar{J}} ||D_\alpha z||_p$ where $||\cdot||_p$ is the usual L_p-norm.

It may be recalled that $W_p^n(G)$ is normed by $\sum_{|\alpha| \leqq n} ||D_\alpha z||_p$. Also, by definition, $W_p^n(G) \subset W_p^*(G)$. However, as following example shows, $W_p^*(G)$ is certainly a larger class than $W_p^n(G)$. Indeed, in the case $n = 2$, and $G = [0, 1] \times [0, 1]$, if $z(x, y) = \int_0^x c(\alpha)d\alpha + \int_0^y c(\beta)d\beta$ where $c(\,.\,)$ is the Cantor function on $[0, 1]$ then $z \in W_p^*(G) - W_p^2(G)$ because z_{xx} does not exist (which follows from the fact that the Cantor function is not absolutely continuous.)

We define the weak convergence in $W_p^*(G)$ analogous to $W_p^n(G)$.

3. Imbedding theorem. It is well known that $W_p^k(G) \subset C(G)$ if $kp > n$. In particular, if $p > 1$, then $W_p^n(G) \subset C(G)$. In this section, we show that $W_p^*(G) \subset C(G)$ for all p, $1 \leqq p \leqq \infty$. In particular then, $W_p^n(G) \subset W_p^*(G) \subset W_1^*(G) \subset C(G)$, where inclusions are understood as imbeddings. It is to be noted that in view of this, $W_p^n(G) \subset C(G)$ even for $p = 1$. This result is analogous to Theorem 2.2.7 of Hormander [4], but of course here for bounded rectangles.

THEOREM 1. *Let* $G = [a, a + h] \subset E^n$. *Then for each* $z \in W_1^*(G)$, *there exists a* $\bar{z} \in C(G)$ *such that* $z = \bar{z}$ a.e. *in* G.

Proof. Since $z \in W_1^*(G)$, $D^n z \in L_1(G)$ where D^n denotes D_α with $\alpha = 1$. Let $w \in L_1(G)$ be defined by

$$w(t) = \int_a^t D^n z(s)ds = T^n D^n z \tag{3.1}$$

By Fubini's theorem $w \in C(G)$. Also w has generalized derivatives $D_i w$ given by

$$D_i w(t) = D_i w(t_i, t_i') = \int_{a_i'}^{t_i'} D^n z(t_i, s)ds \,. \tag{3.2}$$

It is seen similarly that for $\alpha \in \bar{J}$, $D_\alpha w$ exists and is given by $\int_{a_\alpha'}^{t_\alpha'} D^n z(t_\alpha, s)ds$. Furthermore, $D^n z = D^n w$. Now, $D^{n-1}z$ and $D^{n-1}w$ are both absolutely continuous and have the same generalized derivative with respect to t_n. Thus, (see [8, p. 21]) there is a function $c_{1,n}(t_n')$ of t_n' alone such that $D^{n-1}(z - w)(t_n, t_n') = c_{1,n}(t_n')$ for almost all t_n' and almost all t_n. Since $D^{n-1}(z - w)$ is in $L_1(G_n')$ for almost all t_n, same is true for $c_{1,n}(t_n')$. Repeating the process (see illustration below), it is seen that for $\alpha \neq 0$,

$$D'_\alpha(z-w)(t) = \sum \int_{a_\beta}^{t_\beta} c_{|\gamma|,\delta}(a_\theta, s_\beta, t'_\alpha)ds_\beta \tag{3.3}$$

where the summation in taken over all multiindices $\beta, \gamma, \delta, \theta$ with $\delta+\theta=\gamma$, $\gamma+\beta=\alpha$; $|\delta|=1$; $|\gamma|\geqq 1$; $\beta\geqq 0$; $\theta\geqq 0$. If $\beta=0$, it is understood that there is no integration and only one (any of $|\alpha|=|\gamma|$ choices) pair δ, θ is chosen with $\delta+\theta=\gamma=\alpha$. If $|\alpha|=1$, that is $\alpha_i=1$ for one and only one i, then clearly $\beta=0$.

If $\alpha=1$, that is $\alpha_i=1$ for all i, then $D'_\alpha(z-w)=z-w$ and thus (3.3) yields

$$z(t) = w(t) + \sum \int_{a_\beta}^{t_\beta} c_{|\gamma|,\delta}(a_\theta, s_\beta)ds_\beta \tag{3.4}$$

where summation is taken as above with $\alpha=1$. It follows that $z(t)$ is equal to a continuous function a.e. in G.

Illustration. Let $n=3$, $(t_1, t_2, t_3)=(x, y, u)$ and $\alpha=(0,1,1)$; then $\alpha'=(1,0,0)$. Let us consider the set of multiindices $\beta, \gamma, \delta, \theta$ to be used in formula (3.3) for this α. Since $|\delta|=1$ and $\delta\leqq\gamma\leqq\alpha$, γ can be either $\gamma_1=(0,0,1)$, $\gamma_2=(0,1,0)$ or $\gamma_3=(0,1,1)$ while δ has to be $\delta_1=(0,0,1)$ for γ_1, $\delta_2=(0,1,0)$ for γ_2, $\delta_3=(0,1,0)$ for γ_3 or $\delta_4=(0,0,1)$ also for γ_3. Then $\beta_1=\gamma_2$ for γ_1, $\beta_2=\gamma_1$ for γ_2 and $\beta_3=(0,0,0)$ for γ_3. In the last case, that is with γ_3, θ can take only one value: either $\theta=(0,0,1)$ with γ_3, δ_3 or $\theta=(0,1,0)$ with γ_3, δ_4. Thus (3.3) yields

$$(z-w)_x = \int_{a_2}^{y} c_{1,\delta_1}(x, s)ds + \int_{a_3}^{u} c_{1,\delta_2}(x, t)dt + c_{2,\delta_3}(a_3, x) \tag{3.5}$$

or a similar equation with the last term on the right hand side replaced by $c_{2,\delta_4}(a_2, x)$. Here, $w=\int_{a_1}^{x}\int_{a_1}^{y}\int_{a_3}^{u} z_{xyu}$. It is seen that we arrive at (3.5) by observing that since $z_{xyu}=w_{xyu}$, we have on integrating out u, $z_{xy}=w_{xy}+c_{1,\delta_1}(x, y)$ which becomes

$$z_x = w_x + \int_{a_2}^{y} c_{1,\delta_1}(x, s)ds + \tilde{c}(x, u)\,. \tag{3.6}$$

Similarly, starting from $z_{xyu}=w_{xyu}$ and integrating y, we get $z_{xu}=w_{xu}+c_{1,\delta_2}(x, u)$ which becomes

$$z_x = w_x + \int_{a_3}^{u} c_{1,\delta_2}(x, t)dt + \tilde{\tilde{c}}(x, y)\,. \tag{3.7}$$

Equation (3.6) and (3.7) (as well as (3.3) and (3.4)) are valid for $y=a_2$ and $u=a_3$ (as boundary values). Substitution and comparison yields (3.5).

REMARKS.

1. There exist sets $H_i \subset [a_i, a_i + h_i]$ with meas $(H_i) = 0$, $i = 1, 2, \cdots, n$ such that formulae (3.3) and (3.4) are valid for $t \in [G - \bigcup_{i=1}^{n} (G_i' \times H_i)]$. It is to be noted that this is stronger than saying that the equations are valid a.e. in G. However, the equations are valid for (a_π, t_π') for almost all, $t_\pi' \in G_\pi'$, $1 \leqq |\pi| \leqq n$.

2. The function $(D_\alpha' z)(t) = (D_\alpha' z)(t_\pi, t_\pi')$ is equal to a continuous function in t_π for $0 < \pi \leqq \alpha$.

3. Formula (3.3) can be written more explicity as follows:

$$D_\alpha'(z - w)(t) = \sum_{\substack{0 \leqq \beta < \alpha \\ \beta \in J}} \int_{a_\beta}^{t_\beta} c_{|\alpha-\beta|,\delta}(a_{\alpha-\beta-\delta}, s_\beta, t_\beta') ds_\beta \,. \tag{3.8}$$

If $\pi \in \bar{J}$ (and $\alpha \neq 0$ as before)

$$D_\alpha'(z - w)(a_\pi, t_\pi') = \sum \int_{a_\beta}^{t_\beta} c_{|\alpha-\beta|,\delta}(a_\sigma, s_\beta, t_\rho) ds_\beta \tag{3.9}$$

where summation is taken over all β with $\beta \in J$, $0 \leqq \beta < \alpha$ and $\pi \cdot \beta = 0$; and where $\sigma = \alpha - \beta - \delta + (\pi \cap \alpha')$ and $\rho = \alpha' - (\pi \cap \alpha')$ with $(\alpha')_i = 1 - \alpha_i$ for all i and $(\pi \cap \alpha')_i = 1$ if $\pi_i = \alpha_i' = 1$ and $(\pi \cap \alpha')_i = 0$ otherwise. In particular, if $\pi \cap \alpha' = 0$ then the integrands in (3.8) and (3.9) are the same, so that using Lemma 1, no. 2 yields

$$\begin{aligned} D_\alpha'(z - w)(t) &= \sum (-1)^{i-1} D_\alpha'(z - w)(a_\pi, t_\alpha') \\ &= \sum (-1)^{i-1} D_\alpha' z(a_\pi, t_\pi') \end{aligned} \tag{3.10}$$

because $D_\alpha' \; w(a_\pi, t_\pi') = 0$ for $\pi \cap \alpha' = 0$. The summations in (3.10) are taken over all i, π with $1 \leqq i \leqq |\alpha|$ and $\pi \in \bar{J}$, $|\pi| = i$, $\pi \cdot \alpha' = 0$. $\left(\text{Let us recall that } w = \int_a^t D^n z(s) ds.\right)$ In particular, if $\alpha = 1$ (that is $\alpha_i = 1$ for all i) then

$$(z - w)(t) = \sum_{i=1}^{n} \sum_{\substack{|\pi| = i \\ \pi \in \bar{J}}} (-1)^{i-1} z(a_\pi, t_\pi') \tag{3.11}$$

If $\pi = \alpha'$ then $\rho = 0$ and (3.9) reduces to

$$D_\alpha'(z - w)(a_\alpha', t_\alpha) = \sum \int_{a_\beta}^{t_\beta} c_{|\alpha-\beta|,\delta}(a_{1-\beta-\delta}, s_\beta) ds_\beta \tag{3.12}$$

where 1 stands for $(1, \cdots, 1)$ and the summation is over $\beta \in J$ with $0 \leqq \beta < \alpha$ and $|\delta| = 1$, $\delta \in J$. On the other hand from (3.9) with $\alpha = 1$ we get,

$$(z - w)(a_\pi, t_\pi') = \int_{a_\beta}^{t_\beta} c_{|\gamma|,\delta}(a_\theta, s_\beta) ds_\beta \tag{3.13}$$

where the summation is over all $\beta \in J$ with $0 \leqq \beta < 1$ and $\pi \cdot \beta = 0$ and where $\gamma = 1 - \beta$ while $\theta = 1 - \beta - \delta$. (In all these, 1 as before stands for $(1, \cdots, 1)$.) Once again using Lemma 1, no. 2 we get

$$z(t) = w(t) + \sum (-1)^{i-1} D'_\alpha z(a'_\alpha, t_\alpha) \,. \tag{3.14}$$

where the summation is taken over all i, α with $1 \leqq i < n$ and $|\alpha| = i$, $\alpha \in J$.

4. **Compactness of the imbedding operator.** In this section, we shall obtain some results analogous to the following well known fact (see [5], p. 75) in the usual Sobolev spaces regarding weak convergence:

(P) If $\{z_k\}$ is a sequence of functions in $W_p^n(G)$, $p > 1$ converging weakly to some $z \in W_p^n(G)$ then $z_k \to z$ uniformly in G and $D_\alpha z_k \to D_\alpha z$ strongly in $L_p(G)$ for all multiindices α with $0 \leqq |\alpha| < n$ (need not be in $\bar{J}$). The same conclusion holds for $p = 1$ provided in addition the set $\{D_\alpha z_k \,|\, |\alpha| = n, k = 1, 2, \cdots\}$ is equiabsolutely integrable.

It is to be noted that the above statement is not valid in $W_p^*(G)$ as the following examples show:

EXAMPLE 1. Let

$$G = [(x, y) \,|\, 0 \leqq x \leqq 1, 0 \leqq y \leqq 1] \,.$$

For $k = 1, 2, \cdots$ and $i = 0, \cdots, k - 1$, let

$$I_{ik} = [(x, y) \in G \,|\, (i/k) \leqq x \leqq (2i + 1)/2k]$$

and

$$I'_{ik} = [(x, y) \in G \,|\, (2i + 1)/2k \leqq x \leqq (i + 1)/k]$$

and let $z_k(x, y)$ be defined on G by $z_k(x, y) = y(x - i/k)$ in I_{ik} and $= y(-x + (i + 1)/k)$ in I'_{ik}, $i = 0, \cdots, (k - 1)$. Then z_k is continuous and belongs to $W_p^*(G)$, $p \geqq 1$; the derivatives z_{kx}, z_{ky}, z_{kxy} are given by $z_{kx} = y$; $z_{ky} = x - i/k$; $z_{kxy} = +1$ in I_{ik}; $z_{kx} = -y$; $z_{ky} = -x + (i + 1)/k$; $z_{kxy} = -1$ in I'_{ik}, $i = 0, \cdots, (k - 1)$. Clearly, $z_k \to 0$ weakly in $W_p^*(G)$, $p \geqq 1$ and $z_{kxy} \to 0$ weakly in $L_p(G)$, while z_{ky} and z_k converge uniformly to 0. But z_{kx} has norm $(p + 1)^{-1/p}$ for each k and as such does not converge to 0 strongly in $L_p(G)$. It is to be noted that $z_k \notin W_p^2(G)$ since the generalized derivative z_{kxx} does not exist.

EXAMPLE 2. Let

$$G = [(x, y) \,|\, -\pi \leqq x \leqq \pi; -\pi \leqq y \leqq \pi] \,.$$

Let $z_k(x, y) = (\sin kx + \sin ky)/k$. Then $z_{kx} = \cos kx$: $z_{ky} = \cos ky$ and $z_{kxy} = 0$; $k = 1, 2, \cdots$. Clearly, $z_k \in W_2^*(G)$ and $z_{kxy} \to 0$ weakly. But

$z_{kx} \to 0$, $z_{ky} \to 0$ weakly and not strongly. However, $z_k \to 0$ uniformly in G. It is to be observed that z_{kx} and z_{ky} do not converge pointwise.

More generally, if I is any interval on the real line, if $\{\varphi_k(x)\}$ is an orthogonormal sequence in $L_2(I)$ and if $G = I \times I$, then $z_k(x, y) = \varphi_k(y) + \varphi_k(y)$ will be in $W_2^*(G)$ and converge to 0 uniformly as well as weakly in $W_2^*(G)$; but the derivatives z_{kx} and z_{ky} do not converge strongly.

Now, the following theorem valid in $W_p^*(G)$, $p > 1$ is analogous to statement (P):

THEOREM 2. *Let*

$$G = [a, a + h] = \{t \mid t = (t_1, \cdots, t_n), a_i \leqq t_i \leqq a_i + h_i, i = 1, \cdots, n\} .$$

If $\{z_k\}$ *is a sequence in* $W_p^*(G)$, $p > 1$ *weakly convergent to an element* z *in* $W_p^*(G)$, *and if* $z_k(a_i, t_i') \to z(a_i, t_i')$ *uniformly in* G_i' *for* $i = 1, \cdots, n$ *then* $z_k(t) \to z(t)$ *uniformly in* G.

REMARK. Here $G_i' = \{t_i' \mid t_i' = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n) \in [a_i', a_i' + h_i']\}$ as in no. 2.

Proof. Weak convergence of z_k to z in $W_p^*(G)$ implies in particular, weak convergence of the highest order derivatives $D^n z_k$ to $D^n z$ in $L_p(G)$, $p > 1$. This implies in turn the pointwise convergence of $w_k(t) = \int_a^t D^n z_k(s) ds$ to $w(t) = \int_a^t D^n z(s) ds$. In view of the formula (3.11) and the hypothesis, this yields that $z_k(t) \to z(t)$ a.e. in G. But since weak convergence of $D^n z_k$ to $D^n z$ in $L_p(G)$, $p > 1$ implies boundedness of the norms of $\{D^n z_k, k = 1, 2, \cdots\}$, if follows that the sequence $\{w_k(t)\}$ is equicontinuous and equibounded. It follows that the sequence z_k is uniformly convergent to z. Indeed, if $z_k(t_0) \to z(t_0)$ for some fixed $t_0 \in G$, then

$$|z_k(t) - z(t)| \leqq |z_k(t) - z_k(t_0)| + |z_k(t_0) - z(t_0)| + |z(t_0) - z(t)| .$$

Also, since $\{w_k(t)\}$ are equicontinuous and $z_k(a_i, t_i')$ is uniformly convergent, it follows that $\{z_k, z, k = 1, 2, \cdots\}$ is equicontinuous. This fact along with the above inequalities yields uniform convergence of z_k to z in a neighborhood of t_0. But since G is bounded (and closed), the same conclusion is valid through G. Thus, $z_k \to z$ in $C(G)$.

REMARKS. 1. Above theorem is valid for $p = 1$ provided $D^n z_k$, $k = 1, 2, \cdots$ is assumed to be equiabsolutely integrable. The proof remains the same.

2. In view of the remarks in no. 3, the above hypothesis of

uniform convergence of $z_k(a_i, t_i')$ can be replaced by weak convergence of $D_\alpha z_k(a_\alpha, t_\alpha')$ to $D_\alpha z(a_\alpha, t_\alpha')$ in $L_p(G_\alpha')$ for α, $0 \leqq |\alpha| < n$ and $p > 1$. For $p = 1$, we assume in addition that

$$\{D_\alpha z_k(a_\alpha, t_\alpha') \mid k = 1, 2, \cdots; 0 \leqq |\alpha| < n, \alpha \in \bar{J}\}$$

is equiabsolutely integrable.

3. The above hypothesis on the boundary values of z_k and $D_\alpha z_k$ are all automatically satisfied in many applications. For example, in a Darboux problem, the boundary values are prescribed apriori so that $z_k(a_i, t_i') = \varphi_i(t_i')$ and the required convergence is obviously satisfied.

4. The hypothesis of uniform convergence of $z_k(a_i, t_i')$ can be replaced by that of $z_k(\bar{t}_i, t_i')$ where $\bar{t}$ is a fixed element of G.

5. Integro-differential operators. Weak convergence in $W_p^*(G)$ as usual implies weak convergence in $L_p(G)$ of each of the derivatives $D_\alpha z_k$, $\alpha \in \bar{J}$, to $D_\alpha z$. However, weak convergence in $W_p^*(G)$ does not imply the following condition on weak convergence on "lines of boundary":

(B) The sequence $z_k \in W_p^*(G)$ is said to converge in (B) to z if for each $\alpha \in J$ (that is $\alpha_i \neq 1$ for some i), $D_\alpha(z_k - z)(a_\pi, s_\alpha, t'_{\alpha+\pi})$ con vergences weakly in $L_p(G_\alpha)$ to zero, for almost all $t'_{\alpha+\pi}$ in $G'_{\alpha+\pi}$ with $\pi \in J$, $1 \leqq |\pi| \leqq n - |\alpha|$, $\pi \cdot \alpha = 0$; [In particular, $D_\alpha(z_k - z)(a_\pi, t_\pi')$ (with $s = 0$) should converge weakly to zero].

EXAMPLE. Let $G = [0, 1] \times [0, 1] = \{(x, y) \mid 0 \leqq x, y \leqq 1\}$. Let $z_k(x, y)$ be defined on G as Example 1, no. 4. Then, the deriatives of z_k are given by $z_{kx} = y$; $z_{ky} = x - (i/k)$; $z_{kxy} = +1$ for $(i/k) \leqq x \leqq (2i + 1)/2k$; $z_{kx} = -y$; $z_{ky} = -x + (i + 1)/k$; $z_{kxy} = -1$ for $(2i + 1)/2k \leqq x \leqq (i + 1)/k$; for $i = 0, \cdots, (k - 1)$.

In this case $z_{kx}(1, y) = -y$ and this certainly does not converge weakly to zero in $[0, 1]$. Thus, z_k does not coverge to zero in (B). However, $z_k \to 0$ weakly in $W_p^*(G)$.

REMARK. Condition (B) is certainly satisfied if there is a fixed function $\varphi(t)$, $t \in \partial G$ with $z_k(t) = \varphi(t)$ for all k and $t \in \partial G$.

The condition (B) is used in the following theorem which in turn is used in establishing closure properties of certain operators. We recall from no. 2 that for each $\alpha \in \bar{J} - \{0\}$ ($\alpha_i = 0$ or 1 for all i, $\alpha_i \neq 0$ for some i) t_α denotes the $|\alpha|$-vector obtained by omitting those coordinates t_i in $(t_1, \cdots, t_\nu)$ for which $\alpha_i = 0$. Let $G_\alpha = \{t_\alpha \mid a_\alpha \leqq t_\alpha \leqq a_\alpha + h_\alpha\}$.

THEOREM 3. *Let* $z_k, z \in W_p^*(G)$, $1 \leqq p \leqq \infty$, $k = 1, 2, \cdots$, *and let*

$z_k \to z$ weakly in $W_p^*(G)$ as $k \to \infty$. Let condition (B) *hold. Let $R_\alpha(t)$ be a given element of $L_q(G)$ with $\alpha \in J$ and $1/p + 1/q = 1$* (with the usual convention that $q = \infty$ if $p = 1$ and vice versa). *Let there exist functions $K_\alpha(t_\alpha) \in L_q(G_\alpha)$ such that $|R_\alpha(t'_\alpha, t_\alpha)| \leqq K_\alpha(t_\alpha)$ for almost all $t'_\alpha \in G'_\alpha$. Then the sequences $T_\alpha R_\alpha D_\alpha z_k = \int_{a_\alpha}^{t_\alpha} R_\alpha(t'_\alpha, s_\alpha) D_\alpha z_k(t'_\alpha, s_\alpha) ds_\alpha$ belong to $L_p(G)$ and converge in measure to $T_\alpha R_\alpha D_\alpha z$, $\alpha \in J$. If $p > 1$, then this convergence is also in L_r norm, $1 \leqq r < p$.*

Proof. (i) For $z \in W_p^*(G)$ and $\alpha \in J$, $D_\alpha z \in L_p(G)$ and

$$\begin{aligned}\int_G |T_\alpha R_\alpha D_\alpha z|^p dt &= \int_G \left(\int_{a_\alpha}^{t_\alpha} R_\alpha D_\alpha z\right)^p dt \\ &\leqq \int_G \left(\int_{a_\alpha}^{t_\alpha} |R_\alpha(s_\alpha, t'_\alpha)|^q\right)^{p/q} \cdot \left(\int_{a_\alpha}^{t_\alpha} |D_\alpha z(a'_\alpha, s_\alpha)|^p\right) dt \\ &\leqq \int_G \left(\int_{a_\alpha}^{t_\alpha} |K_\alpha(s_\alpha)|^q\right)^{p/q} \left(\int_{a_\alpha}^{t_\alpha} |D_\alpha z(s_\alpha, t'_\alpha)|^p\right) dt \\ &\leqq \int_{a_\alpha}^{a_\alpha + h_\alpha} \int_{a'_\alpha}^{a'_\alpha + h'_\alpha} \left(\int_{a_\alpha}^{a_\alpha + h_\alpha} |K_\alpha(s_\alpha)|^q\right)^{p/q} \cdot \left(\int_{a_\alpha}^{t_\alpha} |D_\alpha z(s'_\alpha, t_\alpha)|^p\right) \\ &\leqq h_\alpha(\|K_\alpha\|_q \cdot \|D_\alpha z\|_p)^p .\end{aligned}$$

If $p = 1$, $q = \infty$, $R_\alpha \in L_\infty$ say $|R_\alpha| \leqq M_\alpha$ then

$$\int_G |T_\alpha R_\alpha D_\alpha z| \leqq M_\alpha \int_G |T_\alpha D_\alpha z| \leqq M_\alpha h_\alpha \|D_\alpha z\| .$$

This proves that $T_\alpha R_\alpha D_\alpha z \in L_p(G)$.

(ii) If $z_k \to z$ weakly in $W_p^*(G)$ then $T_\alpha R_\alpha D_\alpha z_k \to T_\alpha R_\alpha D_2 z$ a.e. in G. Indeed, using (3.10) with z replaced by $z_k - z$, we get

$$\begin{aligned}&\left|\int_{a_\alpha}^{t_\alpha} R_\alpha(s_\alpha, t'_\alpha) D_\alpha(z_k - z)(s_\alpha, t'_\alpha)\right| \\ &\qquad \leqq \left|\int_{a_\alpha}^{t_\alpha} R_\alpha(s, t'_\alpha)\left(\int_{a'_\alpha}^{t'_\alpha} D^n(z_k - z)(s_\alpha, s'_\alpha) ds'_\alpha\right) ds_\alpha\right| \\ &\qquad + \left|\int_{a_\alpha}^{t_\alpha} R_\alpha(s_\alpha, t'_\alpha) \Sigma(-1)^{i-1} D_\alpha(z_k - z)(a_\pi, s_\alpha, t'_{\alpha+\pi}) ds_\alpha)\right|\end{aligned}$$

where the summation in the last term is taken over i, π with $1 \leqq i \leqq n - |\alpha|$; and $\pi \in J$, $|\pi| = i$, $\pi \cdot \alpha = 0$.

The first term tends to zero a.e. in G since $D^n(z_k - z) \to 0$ weakly in $L_p(G)$. The second term tends to zero due to condition (B).

(iii) Since G is of finite measure and $T_\alpha R_\alpha D_\alpha z_k \to T_\alpha R_\alpha D_\alpha z$ a.e. in G, it follows that convergence is also in measure.

(iv) Let us observe that (a) for $p > r \geqq 1$

$$\begin{aligned}\int_E |T_\alpha R_\alpha D_\alpha z_k|^r &\leqq \|T_\alpha R_\alpha D_\alpha z_k\|^r (|\text{meas}\,(E)|)^{(p-r)/p} \\ &\leqq \|K_\alpha\|^r h_\alpha^{r/p} \|D_\alpha z_k\|^r (|\text{meas}\,(E)|)^{(p-r)/p}\end{aligned}$$

and (b) $\{\|D_\alpha z_k\|, k = 1, 2, \cdots\}$ is bounded due to weak convergence of $D_\alpha z_k$. Thus, $\{T_\alpha R_\alpha D_\alpha z_k\}$ is a sequence in L_p converging in measure to $T_\alpha R_\alpha D_\alpha z$ and such that $\int_E |T_\alpha R_\alpha D_\alpha z_k|^r$, $1 \leqq r < p$, $k = 1, 2, \cdots$ is uniformly absolutely continuous. By using a theorem in L_r-convergence (see [6], p. 231, Theorem 32.2, and p. 235, Problem 32.f) it follows that $T_\alpha R_\alpha D_\alpha z_k$ converges to $T_\alpha R_\alpha D_\alpha z$ strongly in L_r norm.

REMARK. Under the conditions of the Theorem 3 above, the operator $Az = (z, T_\alpha R_\alpha D_\alpha z)$ from $W_p^*(G)$ in to $(L_p(G))^{2^n}$, $p > 1$ takes weakly convergent sequences in to sequences with strongly convergent subsequences. In this case, A is compact.

6. Closure properties in $W_p^*(G)$. We have seen that $W_p^*(G) \subset C(G)$ for $1 \leqq p \leqq \infty$, and thus it is a subspace. However, $W_p^*(G)$ is not closed in $C(G)$. Indeed, for $n = 1$, $G = [0, 1]$, let $z_k(t)$ be a sequence of polynomials converging uniformly to the Cantor function $z(t)$ on G. Since $z(t)$ is not absolutely continuous, $z \notin W_p^*(G)$ while $z_k \in W_p^*(G)$, $k = 1, 2, \cdots$. This situation prompts us to consider closable and closed operators on $W_p^*(G)$. First, we shall follow Cesari and Kaiser [2], and make the following definitions:

DEFINITION 1. Let (X, τ) be a topological space and $S \subseteqq X$. Let $A: S \to B$, B a Banach space. We say that A is a weakly (strongly) closed operator if $x_n \in S$, $x_n \to x$(in τ), $x \in X$, and $Ax_n \to y \in B$ weakly (strongly) implies $x \in S$ and $Ax = y$.

DEFINITION 2. Let (X, τ) be a topological space and B a Banach space. Let $S \subset X$ and $A: S \to B$. We say that A has weak (strong) convergence property if x_n, $x \in S$, $x_n \to x$(in τ) implies $Ax_n \to Ax$ weakly (strongly).

EXAMPLES. 1. Obvious from the definition of the norm in $W_p^*(G)$, $G \subset E^n$, $n \geqq 1$, $p \geqq 1$ that for $\alpha \in J$ (that is $\alpha_i = 0$ or 1 for all i) the operator D_α from $W_p^*(G)$ into $L_p(G)$ has the weak (and strong) convergence properties provided the topology τ of $W_p^*(G)$ is the norm topology.

2. $X = C(G)$, the space of contiuous functions on

$$G = [t \in E^n \,|\, a_i \leqq t_i \leqq a_i + h_i]$$

with uniform topology τ. Let $S = W_p^*(G)$ and

$$B = L_p(G) \times \prod_{0 \leqq |\alpha| < n} L_p(G'_\alpha)$$

and

$$Az = (D^n z, D_\alpha z(a_\alpha, t'_\alpha)) .$$

Then A is a weakly (and hence strongly) closed operator from S into B for $p > 1$. In fact, $Az_k \to Az$ weakly implies $z_k \to w$ uniformly (see Theorem 2 and Remark 2 of no. 4) where

$$w = \int_a^t D^n z + \sum (-1)^{i-1} D_\alpha z(a_\alpha, t'_\alpha)$$

and the summation is over all i, α with $1 \leqq i < n$, $|\alpha| = i$, $\alpha \in J$. Also, $w \in W_p^*(G)$. Now $z_k \to z$ in $C(G)$ and $z_k \to w$ also in $C(G)$ implies $z = w$ and hence proof.

3. Let $X = C(G)$, $G = [t \in E^n | a_i \leqq t_i + h_i] = [a, a + h]$, $S = \{z \in W_p^*(G) | z(t) = 0 \text{ for } t \in \partial G\}$; $B = L_p(G)$, $p > 1$. Then $Az = D^n z = \partial^n z / \partial t_1 \cdots \partial t_n$ is a weakly (and hence strongly) closed operator from S in to B.

4. If $X = W_p^1(G)$, $p > 1$, $G = [a, a + h] \subset E^n$ and $S = \{z \in W_p^*(G) | z(t) = 0, t \in \partial G\}$ and τ is the weak topology; $B = L_p(G)$ and $Az = D^n z$; then A is *not* necessarily closed; The lower order derivatives may not converge in the L_p norm.

5. Let $X = W_p^*(G)$, $G = [a, a + h] \subset E^n$, $p > 1$, τ is the weak topology, $S_\varphi = \{z \in X | z(t) = \varphi(t)\}$ for $t \in \partial G$ and let $B = L_r(G)$. Let $Az = f\left(t, z, \int_{a_\alpha}^{t_\alpha} R_\alpha D_\alpha z\right)$ where $R_\alpha(t_\alpha)$ are functions of t_α alone and are elements of $L_q(G_\alpha)$ and where $f = f(t, \theta)$ is a real valued function defined on $G \times E^{2^n}$, f is continuous in t, Lipschitzian in θ, and $f(t, 0) \in L_r(G)$. Then A has strong (and hence weak) covergence properties for $1 \leqq r < p$ while A is weakly closed for $r = p$. Indeed, if $z_k \to z$ weakly in $W_p^*(G)$, $p > 1$ then we have by Theorem 3, no. 5, that $Az_k \to Az$ strongly in L_r, $1 \leqq r < p$. This proves the convergence properties. If $r = p$ then Az_k need not converge to Az. But if $Az_k \to w$ weakly for some $w \in (L_p(G))^{2^n}$ then $Az_k \to w$ weakly also in $(L_s(G))^{2^n}$ for $1 \leqq s < p$. But $Az_k \to Az$ strongly (and hence weakly) in $(L_s(G))^{2^n}$. Thus, $Az = w$ and weak closure of A is proved.

An example of a function f satisfying the requirements above is $f(t, x, y) = e^t \sin x \cos y$, $t \in G = [0, 1]$ and $(x, y) \in E^2$.

7. A Darboux problem. In this section, we shall obtain an existence theorem for solutions of a Darboux problem with integro-differential system of equations. We shall follow notations in no. 5. Thus, $G = [t \in E^n | t = (t_1, \cdots, t_n), a_i \leqq t_i \leqq a_i + h_i, i = 1, \cdots, n]$ and J denotes $\{\alpha = (\alpha_1, \cdots, \alpha_n) | \alpha_i = 0 \text{ or } 1 \text{ for all } i, \alpha_i \neq 1 \text{ for some } i\}$. Also $\bar{J} = J \cup \{(1, \cdots, 1)\}$. For $\alpha \in \bar{J} - \{0\}$, $G_\alpha = \{t_\alpha | a_\alpha \leqq t_\alpha \leqq a_\alpha + h_\alpha\}$ where t_α is obtained from $(t_1, \cdots, t_n)$ by deleting those t_i for which $\alpha_i = 0$. Let $\varphi_\alpha(t_\alpha)$ be a given element of $(W_p^*(G_\alpha))^\nu$, $1 \leqq p \leqq \infty$. Let $\varphi_{\alpha_1}(a_{\alpha_2}) = \varphi_{\alpha_2}(a_{\alpha_1})$ for $\alpha_1, \alpha_2 \in J - \{0\}$. Let U be a fixed closed subset of

E^m. Let $g = g(t, \zeta, u) = (g_1, \cdots, g_\nu)$ be defined on $G \times E^\theta \times U$ where $\theta = (2^n - 1)\cdot 3\nu$. Let g take values in E^ν, $\nu \geqq 1$. Let g be measurable in t, continuous in u and Lipchitzian in ζ. Thus, there exists a constant $K > 0$ such that

$$|g(t, \zeta, u) - g(t, \zeta', u)| \leqq K|\zeta - \zeta'|, \zeta, \zeta' \in E^\theta .$$

For $\alpha \in \bar{J} - \{0\}$ let $R_\alpha(t)$, $R'_\alpha(t)$ be $\nu \times \nu$ matrices with entries in $L_q(G)$, $1/p + 1/q = 1$. Further, let there exist functions $K_\alpha(t_\alpha)$ in $L_q(G_\alpha)$ and $K'_\alpha(t'_\alpha)$ in $L_q(G'_\alpha)$ such that for $\alpha \in J - \{0\}$, $|R_\alpha(t)| \leqq K_\alpha(t_\alpha)$ for almost all $t'_\alpha \in G'_\alpha$ and $|R'_\alpha(t)| \leqq K'_\alpha(t'_\alpha)$ for almost all $t_\alpha \in G_\alpha$; and for $\alpha = (1, \cdots, 1)$, $|R_\alpha(t)| \leqq K_\alpha$, a constant and R'_α is an arbitrary $L_q(G)$ function. With this notation, we may state the Darboux problem as follows: Given $u(t) = (u^1, \cdots, u^m)$ measurable on G, with values in U, find $z(t) \in (W^*_p(G))^\nu$ satisfying

$$\begin{aligned} &D^n z^i(t) = g_i(t, z, A_1 z, A_2 z, u(t)) \\ &(A_1 z^i_\alpha = D_\alpha z^i , \quad \alpha \in J - \{0\} \\ &(A_2 z)^i_\alpha = (T_\alpha R_\alpha D_\alpha z^i, T_\alpha R'_\alpha D'_\alpha z^i) , \quad \alpha \in \bar{J} - \{0\} \\ &i = 1, \cdots, \nu, z(t) = (z^1, \cdots, z^\nu) , \quad u(t) = (u^1, \cdots, u^m) , \quad t \in G \end{aligned} \tag{7.1}$$

with boundary data

$$z(a'_\alpha, t_\alpha) = \varphi_\alpha(t_\alpha) , \quad t_\alpha \in G_\alpha . \tag{7.2}$$

In the equations (7.1) above for each i, $(A_1 z)^i$ and $(A_2 z)^i$ are vectors with components $(A_1 z)^i_\alpha$, $\alpha \in J - \{0\}$ and $(A_2 z)^i_\alpha$, $\alpha \in \bar{J} - \{0\}$ respectively. Also, $A_i z = ((A_i z)^1, \cdots, (A_i z)^\nu)$, $i = 1, 2$.

By using a new set of variables and using the equations of no. 3, the Darboux problem stated above can be written in an equivalent form with integral equations as follows:

$$\begin{aligned} &\sigma_1(t) = T^n g(s, \sigma(s), u(s)) + \Sigma(-1)^{i-1}\varphi_{\pi'}(t'_\pi) \\ &\sigma_{2,\alpha}(t) = T'_\alpha g(t_\alpha, s_\beta, \sigma(s_\beta)u(s_\beta)) + \Sigma(-1)^{i-1}D_\alpha\varphi_{\pi'}(t'_\pi) , \quad \alpha \in J - \{0\} , \\ &\sigma_{3,\alpha}(t) = (T_\alpha R_\alpha \sigma_{2,\alpha}(s), T_\alpha R'_\alpha \sigma_{2,\alpha'}(s)) , \quad \alpha \in J - \{0\} , \\ &\sigma_{3,\alpha}(t) = (T_\alpha R_\alpha g, T_\alpha R'_\alpha \sigma_1) , \quad \alpha = (1, \cdots, 1) \end{aligned} \tag{7.3}$$

where the summations are taken over all i, π with $1 \leqq i \leqq |\alpha'|$ and $\pi \in \bar{J}$, $|\pi| = i$, $\pi\cdot\alpha = 0$. Also, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, σ_2 is a vector with components $\sigma^i_{2,\alpha}$, $i = 1, \cdots, \nu$; $\alpha \in J - \{0\}$, and σ_3 is a vector with components $\sigma^i_{3,\alpha}$, $i = 1, \cdots, \nu$ and $\alpha \in \bar{J} - \{0\}$ while $\sigma_1 = (\sigma^1_1, \cdots, \sigma^\nu_1)$. The solutions of (7.1, 2) and those of (7.3) correspond as follows: $\sigma_1 = z$: $\sigma_{2,\alpha} = D_\alpha z$, $\alpha \in J - \{0\}$ and $\sigma_{3,\alpha} = (T_\alpha R_\alpha D_\alpha z, T_\alpha R'_\alpha D'_\alpha z)$ for $\alpha \in \bar{J} - \{0\}$.

The integral equations (7.3) being of the form studied in [9] by the author, we obtain the existence and uniqueness of the solutions of the integral system and hence of the Darboux problem. More precisely, we state

THEOREM 4. *With the above notation, let* $u(t)$ *be a measurable function on* G *such that* $s_u(t) = g(t, 0, u(t)) \in (L_p(G))^\nu$, $1 \leqq p \leqq \infty$. *Then, there exists corresponding to* u, *a unique* $z \in (W_p^*(G))$, *satisfying the boundary conditions* $z(a'_\alpha, t_\alpha) = \varphi_\alpha(t_\alpha)$ *and whose generalized partial derivatives* $D_\alpha z$, $\alpha \in \bar{J}$, *satisfy the equation* (7.1) a.e. *in* G. *Furthermore, there exist constants* B_1, B_2 *depending only on* $(h_1, \cdots, h_n)$, p, k (the Lipschitz constant of g), $\|K_\alpha\|$, $\|K'_\alpha\|$, $\alpha \in J - \{0\}$; *and* $\|R'_\alpha\|$, K_α, $\alpha = (1, \cdots, 1)$, *such that*

$$\sum_{\alpha \in J} \|D_\alpha z\|_p \leqq B_1[\sum_{\alpha \in J} (h_\alpha)^{1/p}(\|D_\alpha \varphi_\alpha\| + 2^{-1}\|\varphi_\alpha\|_p) + |h| \cdot \|S_u\|_p] \tag{7.4}$$

$$|z(t)| \leqq 2^{-1}\left[\sum_{\alpha \in J} \|\varphi_\alpha\| + BB_2|h| + e^{K|h|}\left(B + \int_G |S_u|\right)\right] \tag{7.5}$$

$$|D_\alpha z(t)| \leqq \theta_\alpha(t_\alpha) + BB_2\,, \quad \alpha \in J - \{0\} \tag{7.6}$$

where

$$B = \sum \|D_\alpha \varphi_\alpha\|_p h_\alpha^{1/p} + Kh_1 \cdots h_n(\|\varphi_\alpha\|_\infty) + \int_G |S_u(s)|\, ds \tag{7.7}$$

and

$$\theta_\alpha(t_\alpha) = e^{K|h'_\alpha|}(|D_\alpha \varphi_\alpha(t_\alpha)| + K|h'_\alpha||\varphi_\alpha(t_\alpha)| + \int_{a'_\alpha}^{a'_\alpha + h'_\alpha} |S_u(t_\alpha, t'_\alpha)|\, dt'_\alpha \qquad \alpha \in J - \{0\} \tag{7.8}$$

REMARK 1. Since g is Lipchitzian, we observe that

$$\begin{aligned} \|D^n z\|_p &= \|g(t, z, A_1 z, A_2 z, u)\|_p \\ &\leqq \|g(t, 0, u)\| + K(\|z\| + \|A_1 z\| + \|A_2 z\|)\,. \end{aligned}$$

Thus, the inequality (7.4) can be written as

$$\|z\|^* = \sum_{\alpha \in J} \|D_\alpha z\|_p \leqq B_3[\|S_u\| + \sum_{\alpha \in J} \|\varphi_\alpha\|\, \|D_\alpha \varphi_\alpha\|] \tag{7.9}$$

where $\|\cdot\|^*$ denotes the norm in $W_p^*(G)$; and where B_3 is a constant depending only on h, K, p and the functions R_α, R'_α.

REMARK 2. Let z', z'' denote solutions of (7.1, 2) corresponding to data φ'_α, φ''_α in $(W_p^*(G_\alpha))^\nu$, $\alpha \in J$ and control functions u', u'' respectively. Let $z = z' - z''$; $\varphi_\alpha = \varphi'_\alpha - \varphi''_\alpha$; $S(t) = |g(t, z', A_1 z', A_2 z', u') - g(t, z'', A_1 z'', A_2 z'', u'')|$. With this notation, the above inequalities (7.4-6) hold with S_u replaced by S. In particular, if $\varphi'_\alpha = \varphi''_\alpha$, for all α the inequality shows the behaviour of z with respect to u.

Illustration. We shall illustrate the above theorem in the case $n = 2$. Let $G = [a, a + h] \times [b, b + k]$. Let $\varphi(x)$ and $\psi(y)$ be absolutely continuous functions for $x \in [a, a + h]$ and $y \in [b, b + k]$. The function $g(x, y, \zeta, u)$ is defined on $G \times E^6 \times U$. Let $R(x, y)$, $S(x, y)$, $T(x, y)$ be given functions in $L_q(G)$, $1/p + 1/q = 1$. Let there exist functions $K_1(x) \in L_q([a, a + h])$, $K_2(y) \in L_q([b, b + k])$, such that $|S(x, y)| \leqq K_1(x)$ for almost all y and $|T(x, y)| \leqq K_2(y)$ for almost all x. Then, the above theorem states that if $u(x, y)$ is a measurable function such that $s_u(x, y) = |g(x, y, 0, u(x, y))| \in L_p(G)$, $1 \leqq p \leqq \infty$, then there exists a uique $z \in W_p^*(G)$, (corresponding to u), satisfying the boundary conditions $z(a, y) = \psi(y)$; $z(x, b) = \varphi(x)$ and whose generalized derivatives z_x, z_y, z_{xy} exist and satisfy the equation

$$z_{xy}(x, y) = g\left(x, y, z, z_x, z_y, \int_a^x \int_b^y Rz, \int_a^x Sz_x, \int_b^y Tz_y, u(x, y)\right) \quad \text{a.e. in } G .$$

REMARK. This type of equations have been studied by R. H. J. Germay [3].

EXAMPLES.

1. Let $G = [0, 1] \times [0, 1]$ and $U = [-1, 1]$. Let $\varphi(x) = \psi(y) = 0$; $R(x, y) = S(x, y) = T(x, y) = 0$; Let $g = \zeta_2 + x \tan u$ so that differential equation is $z_{xy} = z_x + x \tan u$ with boundary conditions $z(x, 0) = z(0, y) = 0$. By integration, the solution of this equation is seen to be $z(x, y) = e^{-y} \int_0^x \int_0^y e^{\beta}\alpha \tan u(\alpha, \beta) d\alpha\, d\beta$. In this case, the constants B_2 and B_3 of the above proposition are both equal to e.

2. Let $G = [0, 1] \times [0, 1]$ as before. Let $C(.)$ denote the Cantor function on $[0, 1]$ (or any other function whose derivative exists almost everywhere but which is not absolutely continuous). Let us consider the equation $z_{xy} = 0$ a.e. in G with side conditions $z(x, 0) = \int_0^x C(\alpha) d\alpha$ for $x \in [0, 1]$ and $z(0, y) = \int_0^y C(\beta) d\beta$ for $y \in [0, 1]$. It is readily seen that the solution of this equation is

$$z(z, y) = z(x, 0) + z(0, y) = \int_0^x C(\alpha) d\alpha + \int_0^y C(\beta) d\beta .$$

It is to be noted that this equation cannot have any solutions in $W_p^2(G)$, even though $C(.)$ possesses ordinary derivatives (equal to zero). Indeed, since $C(.)$ is not absolutely continuous, it does not have a generalized derivative and consequently z_x and z_y do not have the corresponding generalized derivatives z_{xx} and z_{yy}. However, the above solution belongs to $W_p^*(G)$ and the constants B_2 and B_3 are $= 1$ in this case.

3. Let $G = [0, 1] \times [0, 1]$ and $U = E^2$. Let a_i, b_i, $i = 1, \cdots, 7$

be some positive constants and $\varphi_i(x)$, $\psi_i(y)$, $i = 1, 2$ be some absolutely continuous functions. Let us consider the system of equations

$$\begin{aligned} z_{xy}(x, y) = \Big[& a_1 x^{-1/2} + a_2 z + a_3 \sin(xz_y) + a_4 \cos(yz_x) \\ & + a_5 \int_0^x \varphi_1(\alpha) w_x(\alpha, y) d\alpha + a_6 \int_0^y \psi_1(\beta) w_y(x, \beta) d\beta \\ & + a_7 \int_0^x \int_0^y \varphi_1(\alpha)\psi_1(\beta) w(\alpha, \beta) d\alpha d\beta \Big] \sin u(x, y) \\ w_{xy}(x, y) = \Big[& b_1 y^{-1/2} + b_2 w + b_3 \cos(xw_y) + b_4 \sin(yw_x) \\ & + b_5 \int_0^x \varphi_2(\alpha) z_x(\alpha, y) d\alpha + b_6 \int_0^y \psi_2(\beta) z_y(x, \beta) d\beta \\ & + b_7 \int_0^x \int_0^y \varphi_2(\alpha)\psi_2(\beta) z(\alpha, \beta) d\alpha d\beta \Big] \cos v(x, y) \end{aligned}$$

with side conditions $z(x, 0) = z(0, y) = w(x, 0) = w(0, y) = 0$. The solution exists in $W_p^*(G)$, $1 \leqq p < 2$, for any measurable function (u, v); because $g(x, y, 0, u, v) = (a_1 x^{-1/2} \sin u, a_2 y^{-1/2} \cos v)$ is in $L_p(G)$ for $1 \leqq p < 2$. Also there are constants C_1, C_2, C_3 depending only on a_i, b_i, $i = 1, \cdots, 7$ and φ_i, ψ_i, $i = 1, 2$ such that $|w(x, y)|$, $|z(x, y)|$, $\|z\|^*$, $\|w\|^* \leqq C_1$, $(x, y) \in G$, $|w_x(x, y)|$, $|z_x(x, y)| \leqq a_1 x^{-1/2} + C_2$; $|w_y(x, y)|$, $|z_y(x, y)| \leqq b_1 y^{-1/2} + C_3$.

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EASTERN MICHIGAN UNIVERSITY
YPSILANTI, MI 48197