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CLASSES OF RINGS TORSION-FREE OVER THEIR CENTERS

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Let $J(\ )$ denote the intersection of the maximals ideals of a ring. The following properties are studied, for a ring R torsion-free over its center C: (i)$J(R) \cap C = J(C)$; (ii) "Going up" from prime ideals of C to prime ideals of R; (iii) If M is a maximal ideal of R then $M \cap C$ is a maximal ideal of C; (iv) if M is a maximal (resp. prime) ideal of C, then $M = MR \cap C$. Properties (i)-(iv) are known to hold for many classes of rings, including rings integral over their centers or finite modules over their centers. However, using an idea of Cauchon, we show that each of (i)-(iv) has a counterexample in the class of prime Noetherian PI-rings.

Let R be a ring with center C. Throughout this note, we assume that R *is torsion-free as C-module*, i.e., $rc \neq 0$ for all nonzero r in R, c in C. (In particular, this is the case if R is prime.) Let $J(R) = \cap\{\text{maximal ideals of } R\}$.

R is a *PI-ring* if there exists a noncommutative polynomial $f(X_1, \cdots, X_m)$ with coefficients ± 1, such that $f(r_1, \cdots, r_m) = 0$ for all r_i in R. The basic facts about PI-rings are in [6, Chapter X], as well as in [10]. Kaplansky's theorem implies that if R is a PI-ring, then $J(R)$ is the Jacobson radical of R, so clearly $J(R) \cap C \subseteq J(C)$. A natural question is, "Under what conditions does $J(R) \cap C = J(C)$?," or, more generally, "Is there any general correspondence between $J(R)$ and $J(C)$?" An answer for PI-rings given in [12, Theorem 5.9], is that $J(R) = 0$ implies $J(C) = 0$. The object of this note is to tie this question in with other notions which often arise (especially in PI-theory). Then we give some pathological examples, which show that many interesting negative properties (including $J(R) \cap C \neq J(C)$) occur in such natural classes as the class of prime Noetherian PI-rings. Some easy theory is developed to cast some light on the sharpness of these counterexamples. (Although the counterexamples are associative, one may note that associativity is not needed in the positive results.)

Call an ideal A of C *contracted* if $A = A' \cap C$ for some ideal A' of R. (By [11, Theorem 2], semiprime PI-rings have a wealth of contracted ideals of the center.)

LEMMA 1. *An ideal A of C is contracted, iff $AR \cap C \subseteq A$.*

Proof. Suppose A is contracted, i.e. $A = A' \cap C$. Then $AR \subseteq A'$, so $AR \cap C \subseteq A' \cap C = A$. Conversely, if $AR \cap C \subseteq A$, then $AR \cap C = A$, so A is contracted.

Lemma 1 gives us a useful way of characterizing contracted ideals of C and shows that any chain condition on the lattice of ideals of R induces the corresponding condition on the lattice of contracted ideals of C. However, it is often hard to apply lemma 1 to determine the precise make-up of {contracted ideals of C}. Some specific information can be obtained.

REMARK 2. Every principal ideal of C is contracted.

Proof. We wish to show $cR \cap C \subseteq cC$ for every nonzero c in C. But if $cr \in C$ then $0 = [cr, x] = c[r, x]$ for all x in R, implying $r \in C$.

REMARK 3. If C is a valuation domain, then every ideal of C is contracted.

Proof. Recall that, given x and y in a valuation domain C, either x divides y or y divides x. Hence, if A is an ideal of C and if $c = \sum_{i=1}^{t} a_i r_i \in AR \cap C$, then (by induction on t) some a_j divides every a_i, $1 \leqq i \leqq t$. Write $a_i = a_j a_{i1}$. Then

$$c = a_j \sum a_{i1} r_i \in a_j R \cap C \subseteq a_j CA$$

(cf. Remark 2). Thus, $AR \cap C \subseteq A$, so A is contracted.

To examine contracted ideals further, we use *central localization* (cf. [12]), which is briefly described as follows: Given a multiplicatively closed set $S \subseteq C$ containing 1, let R_S be the classical localization (as C-module) of R respect to S; $R_S \approx R \otimes_C C_S$. If $T \subseteq R$, we write T_S for $\{xs^{-1} \mid x \in T\}$. If P is a prime ideal of C, then we write R_P for R_{C-P}; note that C_P has a unique maximal ideal P_P. There is a canonical injection $\psi_S \colon R \rightarrow R_S$, given by $r \rightarrow r1^{-1}$, and $C_S = \mathrm{Cent}\,(R_S)$. Moreover, R_S is always torsion free over C_S. If P is a prime ideal of C, write ψ_P for ψ_{C-P} and note that ψ_P^{-1} is a lattice injection of {prime ideals of R_P} into {prime ideals of R}. For $S = C - \{0\}$, call R_S the *ring of central quotients* of R.

LEMMA 4. (i) *If A is a contracted ideal of C, then A_S is a contracted ideal of C_S.* (ii) *If B is a contracted ideal of C_S, then $\psi_S^{-1}(B)$ is a contracted ideal of C.*

Proof. (i) If $cs^{-1} \in C_S \cap A_S R_S$, then, for some s_1 in S, $cs_1 \in$

$AR \cap C \subseteq A$, implying $cs^{-1} = (cs_1)(ss_1)^{-1} \in A_S$.

(ii) Suppose $c \in \psi_S^{-1}(B)R \cap C$. Then $c1^{-1} \in BR_S \cap C_S \subseteq B$, so $c \in \psi_S^{-1}(B)$.

PROPOSITION 5. *If C is Prufer, then every prime ideal of C is contracted.*

Proof. Let P be a prime ideal of C. Then C_P is a valuation domain, so P_P is contracted (by Remark 3). But P is then contracted, by Lemma 4 (ii).

Of course, if every prime ideal of a ring is contracted, then every semiprime ideal of the ring is contracted. Another property of interest is "going up". We say that *R satisfies $GU(P, P_1)$* if, for every prime ideal P' of R with $P = P' \cap C$, there exists a prime ideal $P_1' \supseteq P'$, with $P_1 = P_1' \cap C$. $GU(P, P_1)$ occurs to some extent in every prime PI-ring (cf. [12, Theorem 4.16]); letting GU denote $GU(P, P_1)$ for all prime ideals $P \subseteq P_1$ of C, it is natural to ask under what conditions R satisfies GU.

All the ideas discussed so far can be related through central localization, as follows:

PROPOSITION 6. *Let $\mathscr{R}$ be a class of rings, such that, if $R \in \mathscr{R}$ and P is any prime ideal of R, then $R_P \in \mathscr{R}$. Consider the following sentences:*

(i) *$J(C) = J(R) \cap C$ for all R in $\mathscr{R}$.*

(ii) *$J(C) \subseteq J(R)$ for all R in $\mathscr{R}$.*

(iii) *GU for all R in $\mathscr{R}$.*

(iv) *For every R in $\mathscr{R}$, if P' is a maximal ideal of R, then $P' \cap C$ is maximal in C.*

(v) *For every R in $\mathscr{R}$, each maximal ideal of C is contracted.*

(vi) *For every R in $\mathscr{R}$, each prime ideal of C is contracted.*

We have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v) $\Leftrightarrow$ (vi).

Proof. (i) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (iii). Let $P_1 \subseteq P$ be prime ideals C, with $P_1 = P_1' \cap C$. Take a maximal ideal B of R_P containing $(P_1')_P$. Then

$$P_P = J(C_P) \subseteq J(R_P) \subseteq B ,$$

so $P = \psi_P^{-1}(B) \cap C$; letting $P' = \psi_P^{-1}(B) \supseteq P_1'$, shows that $GU(P_1, P)$ holds.

(iii) $\Rightarrow$ (iv) Clear.

(iv) $\Rightarrow$ (v). Let P be a prime ideal of C. Then P_P is the only maximal ideal of C_P. Thus, for any maximal ideal B of R_P, $P_P = B \cap C_P$, by (iv), implying $P = \psi_P^{-1}(B) \cap C$.

(v) $\Rightarrow$ (vi). Immediate; localize at the given prime.

(vi) $\Rightarrow$ (v). Trivial.

(iv) and (v) $\Rightarrow$ (i). $J(C) = \cap$ {maximal ideals of C} $= C \cap (\cap$ {maximal ideals of R}) $= C \cap J(R)$.

For the rest of this note, (i)-(vi) *refer to the sentences given in Proposition* 6. Sentences (v) and (vi) do *not* imply (i)-(iv), as evidenced by an example (Bergman-Small [1, §1]) of a prime *PI*-ring whose center is a valuation domain, but which does not satisfy *GU*. Hence, by Remark 3, we have (vi), but (iii) fails (and thus (i)-(iv) fail in various central localizations). The following remarks are easy and well known.

REMARK 7. The usual proof of the Cohen-Seidenberg theorem can be modified to show that any integral extension of an integral domain satisfies *GU*. (This fact was observed in [2] and extended in [13].) Since "torsion-free over C" implies C is a domain, we see that {R integral over C} satisfies (i)-(vi).

REMARK 8. If R is finitely spanned as a C-module then R is integral over C, of bounded degree. This is is seen via [8, p. 238 and p. 335]. Hence, any ring of this form satisfies (i)-(iv). (R. Snider showed me a proof of (ii) even in the non-torsion-free case.)

REMARK 9. If R has a unique maximal ideal, then C is local and (i), (ii), (iv), and (v) hold. Indeed, let M be the maximal ideal of R. For any noninvertible element c in C, clearly $cC \subseteqq M$. Thus, {nonunits of C} is the unique maximal ideal of C, equal to $M \cap C$, so (i), (ii), (iv), and (v) follow easily. (Of course this class of rings is not closed under central localization.)

There is also the following general situation where (v) holds:

PROPOSITION 10. (i) *Every prime ideal P of C, minimal over a contracted ideal A of C, is contracted.* (ii) *Every minimal prime ideal of C is contracted.*

Proof. (i) {ideals $\tilde{B} \supseteqq AR \mid \tilde{B} \cap C \subseteqq P$} is nonempty, and this has a maximal element $\tilde{P}$, which is clearly prime. Since $\tilde{P} \cap C$ is prime in C and $A \subseteqq \tilde{P} \cap C \subseteqq P$, we have $P = \tilde{P} \cap C$.

(ii) Every minimal prime ideal of C is minimal over a suitable principal ideal, which is contracted (by Remark 2).

Hence, any prime ring whose center has Krull Dimension 1 (no two nonzero primes are comparable) satisfies GU, so (i)-(vi) hold in this instance. An example of such a ring is the free noncommutative algebra over a commutative domain of Krull Dimension 1.

Having seen some situations in which some all of the sentences in Proposition 6 hold, we shall now look for counterexamples to (v). Example 11(b) will be "generic" in flavor, whereas Example 13 will be Noetherian. Incidentally, in view of Remark 9, this will indicate one of the complications of noncommutative localization of Noetherian PI-rings.

EXAMPLE 11. (a) Let $\xi_{ij}^{(k)}$, $1 \leqq i, j \leqq n$, $k = 1, 2$, be commutative indeterminates over a field F, and let $F(\xi)$ be the field generated by all $\xi_{ij}^{(k)}$ over F. Let T be the $n \times n$ matrix ring $M_n(F(\xi))$, with matric units $\{e_{ij} \mid 1 \leqq i, j \leqq n\}$, and let X_k be the "generic" matrix $\sum_{ij} \xi_{ij}^{(k)} e_{ij}$. The ring R_0 generated by F, X_1, and X_2, is the famous "ring of generic matrices," and, by a theorem of Small, R_0 satisfies GU. Moreover, every central localization of R_0 satisfies GU (and thus (i)-(vi)), by [12, Theorem 4.24]. In fact, this class can be expanded to {rings whose central kernel is a maximal ideal of the center}, cf. [12, Theorem 4.24]. This example makes the following example quite surprising:

(b) Notation as in (a), let $X = X_1$, and let $\mu_1, \cdots, \mu_n$ be the characteristic values of X^{-1}. Define $\alpha_1 = \sum_{i=1}^n \mu_i$, $\alpha_2 = \sum_{i<j} \mu_i \mu_j, \cdots$, $\alpha_n = \mu_1 \mu_2 \cdots \mu_n$. We claim that R, the subring of T generated by R_0 and $\alpha_1, \cdots, \alpha_n$, is a counterexample to (v).

Let $C = \mathrm{Cent}\,(R)$ and let $A = \sum \alpha_i C$. Clearly $AR = R$ (since $\sum_{i=1}^n (-1)^{i-1} \alpha_i X^i = 1$). We will prove the claim by showing $A \neq C$. The starting point is Procesi's observation that the characteristic values of X are algebraically independent (seen by specializing all $\xi_{ij}^{(1)}$ to 0 for $i \neq j$). Hence the μ_i are algebraically independent, and the theory of symmetric polynomials in commutative indeterminates (cf. [8, pp. 133-4]) will be applied to $\alpha_1, \cdots, \alpha_n$.

Let $C_1 = F[\alpha_1, \cdots, \alpha_n]$ and let D be the subring of R generated by X and C_1. Note that $X^{-1} = \sum_{i=1}^n (-1)^{i-1} \alpha_i X^{i-1} \in D$. Suppose there are c_i in C such that $\sum_u \alpha_u c_u = 1$. Specializing all $\xi_{ij}^{(2)}$ to 0, we may assume that each $c_u \in C \cap D$. Since $\alpha_1, \cdots, \alpha_n$ are algebraically independent, we will have reached a contradiction once we prove that $C \cap D = C_1$.

So suppose $c = \sum_{k=q}^t f_k(\alpha) X^k \in C \cap D$, where each $f_k(\alpha) \in C_1$. Write c in this form, with t minimal. First we show that $t \leqq 0$. Otherwise, assume $t > 0$. Write $r_1 = \sum_{k=q}^0 f_k(\alpha) X^k$. Diagonalizing, we may assume $X^{-1} = \sum_{i=1}^n \mu_i e_{ii}$. Let $g(X^{-1}) = \sum_{i=1}^n (-1)^{i-n} \alpha_{n-i-1} X^{i-n}$, where $\alpha_0 = 1$. Clearly $g(X^{-1}) = \alpha_n X$, so we can write

$$\begin{aligned}\alpha_n^t c &= \alpha_n^t r_1 + \sum_{k=1}^{t} \alpha_n^{t-k} f_k(\alpha) g(X^{-1})^k \\ &= \alpha_n(\alpha_n^{t-1} r_1 + \sum_{k=1}^{t-1} \alpha_n^{t-k-1} f_k(\alpha) g(X^{-1})^k) + f_t(\alpha) g(X^{-1})^t ,\end{aligned}$$

a matrix with entries in $F[\mu_1, \cdots, \mu_n]$, a polynomial ring. Now $g(X^{-1})^t e_{jj} = (\mu_1 \cdots \mu_{j-1}\mu_{j+1} \cdots \mu_n)^t e_{jj}$. Examining the entry in the j, j position, for $i \neq j$, we see that μ_i divides both α_n and $f_t(\alpha)g(X^{-1})^t$, implying μ_i divides $\alpha_n^t c$. By symmetry, $\mu_1 \cdots \mu_n \mid \alpha_n^t c$; reversing steps shows that $\mu_j \mid f_t(\alpha)(\mu_1 \cdots \mu_{j-1}\mu_{j+1} \cdots \mu_n)^t$. Hence $\mu_j \mid f_t(\alpha)$ for each j; By symmetry, $f_t(\alpha) = \alpha_n h$ for some element h in $F[\mu_1, \cdots, \mu_n]$.

Since h is symmetric in $\mu_1, \cdots, \mu_n$, h is in D_1; hence, we can write $c = \sum_{k=q}^{t-2} f_k(\alpha)X^k + (f_{t-1}(\alpha) + hg(X^{-1}))X^{t-1}$, contrary to the choice of t minimal. Thus, $t \leqq 0$, after all.

In other words, c is a polynomial in X^{-1} and the α_i. Write $c = \sum_{i=1}^{n} f(\mu_1, \cdots, \mu_n)e_{ii}$. Switching μ_i and μ_j merely interchanges the (equal) coefficients of e_{ii} and e_{jj}, so we see that f is symmetric in the μ_i. Therefore $c \in C_1$, as desired.

Examples 11a and 11b show, in particular, that any of the sentences (i) through (vi) may hold in some prime *PI*-ring, but fail in a finitely generated central extension. Also, 11b is in fact *affine*, that is, finitely generated (as a ring) over a field. However, {affine prime *PI*-rings} is not closed under central localization at prime ideals of the center; in fact, Amitsur proved that all affine prime *PI*-rings are semiprimitive (cf. [10, p. 102]), so (i) holds in this class.

In view of Remarks 7 and 8, and [5], clearly (i)-(vi) hold for large classes of Noetherian *PI*-rings, and it is natural to ask whether (vi) holds for all prime Noetherian *PI*-rings. First let us examine the idea of example 11b. It is well-known that a prime *PI*-ring can be embedded in a matrix ring over a field. Example 11b "works" because there is a suitably general matrix (X) which is not integral over the center, but for which we have the coefficients of the characteristic polynomial of its inverse. But for Noetherian rings, Schelter proved [13, Theorem 2]: If R is a prime Noetherian *PI*-ring then, for any r in R, every characteristic value α of r satisfies an equation of the form $\alpha^t = \sum_{i=0}^{t-1} \alpha^i r_i$, for suitable r_i in R.

Thus, if $\alpha^{-1} \in R$ then, multiplying by α^{1-t}, we conclude that $\alpha \in R$. In particular, for an element r in an arbitrary prime Noetherian *PI*-ring, if $\det(r^{-1}) \in R$ then $\det(r^{-1})$ is a unit in R. Hence, the idea of example 11b fails for prime Noetherian *PI*-rings.

Now we give in an example of a prime, affine Noetherian *PI*-ring which does not satisfy (v). Of course, such an example cannot be integral over its center, by Remark 7, and until recently, all known prime Noetherian *PI*-rings were integral (over their centers). Cauchon [3] and Schelter [13] have discovered non-integral, prime Noetherian *PI*-rings. Although, as can be seen, both examples satisfy (vi), Cauchon's example is representative of a wide class including counterexamples to (v). (Small informed me that, using an approach similar to that of Schelter [13], he has also obtained a counterexample to (v).) Let us start by considering Cauchon's example in its general setting. Recall that a *derivation* of a ring R is an additive map $D: R \to R$ satisfying $(xy)D = (xD)y + x(yD)$ for all x, y in R.

EXAMPLE 12. Let L be a commutative domain with derivation D, and let e_{11}, e_{12}, e_{21}, e_{22} be matric units of $M_2(L)$. For any element a in L, let $a' = a(e_{11} + e_{22}) + (aD)e_{12}$. $H = \{a' \mid a \in L\}$ is a commutative ring isomorphic to L (via the map $a \mapsto a'$). Choose x in L, and let R be the subring of $M_2(L)$ generated by H and $xM_2(L)$. As shown in [3], R is a finitely spanned left (and right) module over H, with generators xe_{ij}, $1 \leqq i, j \leqq 2$. Since the ring of central quotients of R is the (simple) ring of matrices over the field of fractions of L, R is prime. Clearly $H \cap \mathrm{Cent}(R) = \{a' \mid aD = 0\}$.

EXAMPLE 13. A prime, affine Noetherian *PI*-ring R which does not satisfy (v).

Let L_0 be the field generated over $\boldsymbol{Q}$ by the indeterminates x, y_1, y_2, z_1, and z_2, and let L be the $\boldsymbol{Q}$-subalgebra of L_0 generated by x, y_1, y_2, z_1, z_2, and $(1 - y_1z_1)z_2^{-1}$. Let $L_1 = \boldsymbol{Q}[x, z_1](z_2)$, and we extend the zero derivation on L_1 to a derivation D on $L_1[y_1, y_2]$ via the conditions $y_1D = y_2z_2$ and $y_2D = y_2^2$. By restriction, D is also a derivation on L.

We claim $L \cap L_1 = \{g \in L \mid gD = 0\}$. Indeed, suppose $gD = 0$ and $g = \sum_{i=0}^{t} f_i(y_2)y_1^i$ for suitable $f_i(y_2)$ in $L_1[y_2]$, chosen such that t is minimal. The coefficient of y_1^t in gD is $(f_t(y_2))D$, which is thus 0; it follows easily that $f_t(y_2)$ equals some element μ in L_1. If $t > 0$, then the coefficient of y_1^{t-1} in gD is $(f_{t-1}(y_2)D + t\mu y_2z_2) = 0$; hence $\mu = 0$, contrary to the minimality of t. Therefore $t = 0$, and $g = \mu \in L_1$, proving the claim.

Now let R be built from L, using the construction and notation of Example 12. Since L is Noetherian and R is a finite L-module, R is left and right Noetherian. Also, R is clearly affine, as well as prime (cf. Example 12). We claim that R does not satisfy (v). Indeed, with $C = \mathrm{Cent}\, R$, let $A = z_1'C + z_2'C$. Since

$$1 = z_1'y_1' + z_2'((1 - y_1z_1)z_2^{-1})' \in AR ,$$

it suffices to show that $A \neq C$. Suppose to the contrary that

$z_1'c_1 + z_2'c_2 = 1$, for suitable c_i in C. Taking the parts of degree 0 in x, we may assume c_1, $c_2 \in H$. Then we can write $c_i = d_i'$ for suitable d_i in L. By Example 12, $d_iD = 0$, so $d_i \in L \cap L_1$, implying $d_i \in \boldsymbol{Q}[z_1](z_2)$. Now $z_1d_1 + z_2d_2 = 1$, which we assert is an impossibility. Well, taking homogeneous components in terms of z_2, we may assume that $d_1 = h_1(z_1)$ and $d_2 = h_2(z_1)z_2^{-1}$ for suitable $h_i(z_1)$ in $\boldsymbol{Q}[z_1]$. Since $d_2 \in L$, it follows that $d_2 = ((1 - y_1z_1)z_2^{-1})d$ for some element d in L. Viewing d_2 as a polynomial in y_1, with coefficients in L_1, we see that d_2 must have degree $\geqq 1$. But this contradicts the fact that $d_2 \in L_1$. We conclude that $A \neq C$, as wanted.

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