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SUMMABILITY R_r FOR DOUBLE SERIES

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Let r be a positive integer. A trigonometric series T of a single variable is said to be summable R_r at θ_0 if the series obtained by r times formally integrating T has an rth symmetric derivative at θ_0. For even values of r, summability R_r has been applied to double trigonometric series. We study here summability R_r, for odd values of r, for double trigonometric series. We obtain a connection between Bochner-Riesz summable series and series which are summable R_r.

1. Let

$$\sum_{-\infty}^{\infty} c_n e^{in\theta} \tag{1.1}$$

be a trigonometric series of a single variable. Let r be a positive integer. Suppose the series obtained by formally integrating (1.1) r times

$$c_0 \frac{\theta^r}{r!} + \sum_{n \neq 0} \frac{c_n}{(in)^r} e^{in\theta} \tag{1.2}$$

converges to a function $F(\theta)$ in a neighborhood of $\theta_0 \in (0, 2\pi)$. We will say that the series (1.1) is at θ_0 *summable by the method* R_r to sum s if $F(\theta)$ has at θ_0 an rth symmetric derivative with value s. That is, if r is even,

$$\frac{1}{2}\{F(\theta_0 + t) + F(\theta_0 - t)\} = a_0 + \frac{a_2}{2!}t^2 + \cdots + \frac{s}{r!}t^r + o(t^r) \tag{1.3}$$

as $t \to 0$, and if r is odd,

$$\frac{1}{2}\{F(\theta_0 + t) - F(\theta_0 - t)\} = a_1 t + \frac{a_3}{3!}t^3 + \cdots + \frac{s}{r!}t^r + o(t^r)\,, \tag{1.4}$$

as $t \to 0$.

The following result, see [8], p. 66, establishes a connection between summability (C, α) and summability R_r for trigonometric series.

THEOREM A. *Let $\alpha > -1$ and assume the series* (1.1) *is summable (C, α) at θ_0 to sum s. Let r be an integer with $r > \alpha + 1$, and suppose the series* (1.2) *converges in a neighborhood of θ_0. Then the series* (1.1) *is summable R_r to s.*

2. In two variables we will denote points $x \in E_2$ by $x = (x_1, x_2) =$

$te^{i\theta}$ and integral lattice points by $n = (n_1, n_2)$. We write

$$|x| = \sqrt{x_1^2 + x_2^2}\,.$$

We will say a double trigonometric series

$$T: \sum_{n \in Z_2} c_n e^{in\cdot x} \tag{2.1}$$

is *Bochner-Riesz summable* of order α at x_o to sum s_o if

$$\lim_{R\to\infty} \sum_{|n|<R} \left(1 - \left(\frac{|n|}{R}\right)^2\right)^{\alpha} c_n e^{in\cdot x_o} = s_o\,.$$

Suppose r is an even number, $r = 2s$. A two dimensional analogue of summability R_r is given as follows, see [7], [4].

DEFINITION. Let $F(x)$ be defined in a neighborhood of $x_o \in E_2$. F has at x_o a sth *generalized Laplacian* equal to s_o if F is integrable on each circle $|x - x_o| = t$ and

$$\frac{1}{2\pi}\int_0^{2\pi} F(x_o + te^{i\theta})d\theta = a_o + \frac{a_2 t^2}{(2!)^2} + \cdots + \frac{s_o t^{2s}}{(2^s s!)^2} + o(t^{2s}) \tag{2.2}$$

as $t \to 0$.

THEOREM B. *Let the series T of* (2.1) *be Bochner-Riesz-m summable at x_o to sum s_o, where m is a nonnegative integer, and suppose the coefficients of T satisfy*

$$\sum_{n \in Z_2} |n|^{-3+\varepsilon} |c_n|^2 < \infty$$

for some $\varepsilon > 0$. Let $r = 2s$ be an even integer with $r \geqq m + 2$. Set

$$F(x) = \frac{c_o(x_1 + x_2)^{2s}}{2^s(2s)!} + (-1)^s \sum_{n\neq 0} \frac{c_n}{|n|^{2s}} e^{in\cdot x}\,. \tag{2.3}$$

Then the generalized sth *Laplacian of $F(x)$ exists at x_o and is equal to s_o.*

That is, if the series (2.1) is Bochner-Riesz-m summable to s_o and r is an even number with $r \geqq m + 2$, then the series is also summable R_r to sum s_o.

3. The purpose of this paper is to derive a connection between Bochner-Riesz summability and summability R_r, for *odd* values of r. We use the following definition, from [5]. This definition extends the formula of (1.4) to two dimensions in a manner analogous to the extension of (1.3) to two variables by (2.2).

DEFINITION. Let $r = 2s + 1$ be an *odd* positive integer. Let $L(x)$ be a function defined in a neighborhood of $x_o \in E_2$. We will say $L(x)$ has at x_o a *generalized symmetric derivative* of order r with value s_o if L is integrable on each circle $|x - x_o| = t$, for t small, and if

$$(3.1) \qquad \frac{1}{2\pi}\int_0^{2\pi} L(x_o + te^{i\theta})(\cos\theta + \sin\theta)d\theta = a_1t + a_3t^3 + \cdots + \frac{s_o}{2^{2s+1}s!(s+1)!}t^{2s+1} + o(t^{2s+1})$$

as $t \to 0$.

We are able to obtain the following results which, for odd values of r, form a two dimensional version of Theorem A. We begin with the case of double trigonometric series which are Bochner-Riesz summable of integral order, since the statement and proof of our results are much simpler in this case.

THEOREM 1. *Let m be a nonnegative integer. Suppose*

$$(3.2) \qquad T\colon \sum_{n \in Z_2} c_n e^{in\cdot x}$$

is Bochner-Riesz-m summable at x_o to finite sum s_o. Let $r = 2s + 1$ be an odd integer such that $r \geqq m + 1$. Suppose the coefficients of T satisfy

$$(3.3) \qquad \sum_{n_1+n_2=0} |n|^{-2r+3+\varepsilon}|c_n|^2 + \sum_{n_1+n_2\neq 0} (n_1 + n_2)^{-2}|n|^{-2r+3+\varepsilon}|c_n|^2 < \infty$$

for some $\varepsilon > 0$. Then the series

$$(3.4) \qquad \frac{c_o(x_1 + x_2)^r}{(r)!(2r)!2^{s+1}} + \frac{1}{2}(x_1 + x_2)\sum_{n_1+n_2=0}' \frac{c_n}{|n|^{2s}}e^{in\cdot x} + \sum_{n_1+n_2\neq 0} \frac{-ic_n}{(n_1 + n_2)|n|^{2s}}e^{in\cdot x}$$

converges spherically to a function $L(x)$ which has at x_o a generalized symmetric derivative of order r with value s_o.

We are able to extend Theorem 1 to include some, but not all, fractional orders of Bochner-Riesz summability. Let β be a nonnegative real number. We denote by $[\beta]$ the largest integer $\leqq \beta$ and by $\langle\beta\rangle$ the fractional part of β, $\langle\beta\rangle = \beta - [\beta]$.

THEOREM 2. *Let β be a nonnegative real number with $\langle\beta\rangle <$ 1/2. Suppose the series* (3.2) *is summable Bochner-Riesz-β to finite sum s_o. Let $r = 2s + 1$ be an odd integer with $r \geqq [\beta] + 1$. Suppose the coefficients of the series* (3.2) *satisfy formula* (3.3) *for some $\varepsilon > 0$.*

Then the conclusion of Theorem 1 *still holds.*

In particular, in the two dimensional case, Bochner-Riesz summability of order β, for $\beta < 1/2$, is enough to imply summability R_1 (which is Lebesgue summability).

4. Although Theorem 1 is a special case of Theorem 2, we give its proof separately, since its proof is much easier than that of Theorem 2. We will assume, as we may, that $c_0 = 0$, $x_0 = 0$, and $s_0 = 0$. We set

$$S_R = S_R(0) = \sum_{|n|<R} c_n ,$$

and for $\eta > 0$

$$S_R^\eta = \frac{1}{\Gamma(\eta)}\int_0^R (R-u)^{\eta-1} S_u du . \tag{4.1}$$

Note that S_R^η, as a function of R, is the fractional integral of order η of $f(R) = S_R$, see [6].

Hardy, see [2], has shown that a series $\sum c_n$ is Bochner-Riesz-η summable to 0 if and only if

$$\sum_{|n|<R} c_n \left(1 - \frac{|n|}{R}\right)^\eta \longrightarrow 0$$

as $R \longrightarrow \infty$. Thus, for the proof of Theorem 1 we may assume

$$S_R^m = o(R^m) \tag{4.2}$$

as $R \longrightarrow \infty$.

We will need the following lemmas. The first lemma has been adapted from [7].

LEMMA 1. *Suppose* $\sum_{n \in Z_2} c_n e^{in \cdot x}$ *is Bochner-Riesz-*$(m+1)$ *summable to* 0 *at* $x = 0$, *and suppose the coefficients* c_n *satisfy condition* (3.3) *of Theorem* 1, *with* $r \geqq m + 1$. *Then*

$$S_R^k = o(R^{r+1/2}) , \tag{4.3}$$

as $R \longrightarrow \infty$, *for* $k = 0, 1, \cdots, m+1$.

Proof. We first note that for $n_1 + n_2 \neq 0$,

$$\sum_{n_1+n_2\neq 0} (n_1 + n_2)^{-2} |n|^{-2r+3+\varepsilon} |c_n|^2$$
$$\geqq \frac{1}{4} \sum_{n_1+n_2\neq 0} |n|^{-2} |n|^{-2r+3+\varepsilon} |c_n|^2$$
$$= \frac{1}{4} \sum_{n_1+n_2\neq 0} |n|^{-2r+1+\varepsilon} |c_n|^2 .$$

Thus, from (3.3),

$$\sum_{n_1+n_2\neq 0} |n|^{-2r+1+\varepsilon}|c_n|^2 < \infty ,$$

and therefore

$$\sum_{n\in Z_2} |n|^{-2r+1+\varepsilon}|c_n|^2 < \infty .$$

Using Schwartz's inequality,

$$\begin{aligned}\sum_{|n|<R}|c_n| &= \sum_{|n|<R}(|n|^{1/2(-2r+1+\varepsilon)}|c_n|)(|n|^{-1/2(-2r+1+\varepsilon)}) \\ &\leqq (\sum_{n\in Z_2}|n|^{-2r+1+\varepsilon}|c_n|^2)^{1/2}(\sum_{|n|<R}|n|^{2r-1-\varepsilon})^{1/2} \\ &= C\cdot(R^{2r+1-\varepsilon})^{1/2} \\ &= o(R^{r+1/2})\end{aligned} \tag{4.4}$$

as $R\to\infty$.

Now fix an integer j.

$$\sum_{|i|<R} c_i(R-|i|+j)^{m+1} = \sum_{|i|<R+j} c_i(R-|i|+j)^{m+1} - \sum_{R\leqq|i|<R+j} c_i(R-|i|+j)^{m+1} .$$

Since $\sum c_n e^{in\cdot x}$ is Bochner-Riesz-$(m+1)$ summable to 0 at 0,

$$\sum_{|i|<R+j} c_i(R-|i|+j)^{m+1} = o(R^{m+1})$$

as $R\to\infty$.

$$\sum_{R\leqq|i|<R+j} c_i(R-|i|+j)^{m+1} = o(R^{r+1/2}) ,$$

because of (4.4). Thus,

$$\begin{aligned}\sum_{|i|<R} c_i(R-|i|+j)^{m+1} &= o(R^{m+1}) + o(R^{r+1/2}) \\ &= o(R^{r+1/2}) ,\end{aligned} \tag{4.5}$$

as $R\to\infty$.

We next use the fact, see [7], that there are number C_{jk}, for $j=1,\cdots,m+2$, $k=0,\cdots,m+1$ such that for all complex numbers z,

$$\sum_{j=1}^{m+2} C_{jk}(z+j)^{m+1} = z^k .$$

Thus, for $0\leqq k\leqq m+1$,

$$\begin{aligned}S_R^k &= \frac{1}{k!}\sum_{|i|<R} c_i(R-|i|)^k \\ &= \frac{1}{k!}\sum_{|i|<R} c_i \sum_{j=1}^{m+2} C_{jk}(R-|i|+j)^{m+1}\end{aligned}$$

$$= \sum_{j=1}^{m+2} \frac{1}{k!} C_{jk} \sum_{|i|<R} c_i(R - |i| + j)^{m+1}$$
$$= \sum_{j=1}^{m+2} \frac{1}{k!} C_{jk} o(R^{r+1/2})$$
$$= o(R^{r+1/2}) ,$$

by (4.5). This proves Lemma 1.

LEMMA 2. *Let* $x = (x_1, x_2) = te^{i\theta} \in E_2$ *and* $n = (n_1, n_2) \in \boldsymbol{Z}_2$, *with* $|n| \neq 0$. *Define*

$$g_n(x) = \begin{cases} \dfrac{1}{2}(x_1 + x_2)e^{in\cdot x} & \text{if} \quad n_1 + n_2 = 0 \\ \dfrac{-ie^{in\cdot x}}{n_1 + n_2} & \text{if} \quad n_1 + n_2 \neq 0 . \end{cases} \tag{4.6}$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})(\cos\theta + \sin\theta)d\theta = \frac{J_1(|n|t)}{|n|} ,$$

where $J_1(z)$ *is the Bessel function of the first kind of order* 1.

Proof. This is the lemma from [5].

5. Proof of Theorem 1. Let

$$T_R(x) = \sum_{\substack{|n|<R \\ n_1+n_2=0}} \frac{1}{2}(x_1 + x_2)\frac{c_n}{|n|^{2s}} e^{in\cdot x} + \sum_{\substack{|n|<R \\ n_1+n_2\neq 0}} \frac{-ic_n}{(n_1 + n_2)|n|^{2s}} e^{in\cdot x} .$$

The hypothesis (3.3) insures that

$$L(x) = \lim_{R\to\infty} T_R(x)$$

exists a.e. on each circle $|x| = t$, see [3], Theorem 1. Also, by Theorem 2 of [3],

$$\int_0^{2\pi} \sup_R |T_R(te^{i\theta})| d\theta < \infty ,$$

so, using Lebesgue's Dominated Convergence Theorem,

$$\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta$$
$$= \lim_{R\to\infty} \frac{1}{2\pi}\int_0^{2\pi} T_R(te^{i\theta})(\cos\theta + \sin\theta)d\theta$$
$$= \lim_{R\to\infty} \sum_{|n|<R} \frac{c_n}{|n|^{2s}} \frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})(\cos\theta + \sin\theta)d\theta$$

where $g_n(x)$ is defined by (4.6). Using Lemma 2 we get

$$\begin{aligned}(5.1)\qquad &\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta \\ &= \lim_{R\to\infty} \sum_{|n|<R} \frac{c_n}{|n|^{2s}} \frac{J_1(|n|t)}{|n|} \\ &= \lim_{R\to\infty} \sum_{|n|<R} c_n \frac{J_1(|n|t)}{|n|^r} \\ &= t^r \lim_{R\to\infty} \sum_{|n|<R} c_n \gamma(|n|t) ,\end{aligned}$$

where $\gamma(t) = z^{-r}J_1(z)$.

We express the last sum as an integral and integrate by parts $m+1$ times.

$$\begin{aligned}(5.2)\qquad \sum_{|n|<R} c_n\gamma(|n|t) &= S_R\gamma(Rt) - \int_0^R S_u \frac{d}{du}\gamma(ut)du \\ &= S_R\gamma(Rt) - S_R^1\frac{d}{dR}\gamma(Rt) + \int_0^R S_u^1 \frac{d^2}{du^2}\gamma(ut)du \\ &\quad \vdots \\ &= S_R\gamma(Rt) - S_R^1\frac{d}{dR}\gamma(Rt) + \cdots + (-1)^m S_R^m \frac{d^m}{dR^m}\gamma(Rt) \\ &\quad + (-1)^{m+1}\int_0^R S_u^m \frac{d^{m+1}}{du^{m+1}}\gamma(ut)du .\end{aligned}$$

From Lemma 1,

$$S_R^k = o(R^{r+1/2}) \quad \text{for} \quad k = 0, \cdots, m .$$

Repeatedly using the relations from [1],

$$(5.3)\qquad \frac{d}{dz}(z^{-n}J_n(z)) = z^{-n}J_{n+1}(z)$$

and

$$J_\nu(z) = o(z^{-1/2}) ,$$

as $z \to \infty$, we get

$$(5.4)\qquad \frac{d^k}{dz_k}\gamma(z) = o(z^{-r-1/2})$$

as $z \to \infty$. So, for $k = 0, \cdots, m$

$$\begin{aligned}(5.5)\qquad S_R^k \frac{d^k}{dR^k}\gamma(Rt) &= o(R^{r+1/2})o(R^{-r-1/2}) \\ &= o(1) ,\end{aligned}$$

as $R \to \infty$. Thus, returning to (5.2),

$$\lim_{R\to\infty} \sum_{|n|<R} c_n \gamma(|n|t) = (-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut)du \,,$$

and (5.1) becomes,

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta \\ &= t^r \lim_{R\to\infty} \sum_{|n|<R} c_n\gamma(|n|t) \qquad (5.6) \\ &= t^r(-1)^{m+1}\int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}}\gamma(ut)du \,. \end{aligned}$$

Now we make use of the series expansion for $J_1(z)$, [1], p. 4.

$$\begin{aligned} J_1(z) &= \sum_{k=0}^\infty \frac{(-1)^k(\frac{1}{2}z)^{1+2k}}{k!(k+1)!} \qquad (5.7) \\ &= a_1 z + a_3 z^3 + \cdots \,. \end{aligned}$$

Then,

$$\begin{aligned} \gamma(z) &= z^{-r}J_1(z) \\ &= z^{-r}(a_1 z + a_3 z^3 + \cdots + a_{r-2}z^{r-2} + a_r z^r + \cdots) \,. \end{aligned}$$

We define a polynomial $P(z)$ as follows. If $r = 1$, let $P(z) \equiv 0$. Otherwise, let

$$P(z) = a_1 z + a_3 z^3 + \cdots + a_{r-2}z^{r-2}$$

where the a_i's are given by (5.7). Now we let

$$\lambda(z) = \gamma(z) - z^{-r}P(z) \,. \qquad (5.8)$$

Then $\lambda(z)$ is an entire function in the plane and

$$\gamma(z) = z^{-r}P(z) + \lambda(z) \,.$$

Returning to (5.6),

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta \\ &= t^r(-1)^{m+1}\int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}}\gamma(ut)du \\ &= t^r(-1)^{m+1}\int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}}\{(ut)^{-r}P(ut) + \lambda(ut)\}du \qquad (5.9) \\ &= t^r(-1)^{m+1}\int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}}\{(ut)^{-r}P(ut)\}du \\ &\quad + t^r(-1)^{m+1}\int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}}\lambda(ut)du \\ &= A + t^r B(t) \,. \end{aligned}$$

Since $c_0 = 0$, therefore $S_u^m = 0$ for $0 \leqq u < 1$. Thus we may replace the interval of integration of the integral involving A by the interval $(1/2, \infty)$.

$$\begin{aligned} A &= t^r(-1)^{m+1}\int_{1/2}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}}\{(ut)^{-r}P(ut)\}du \\ &= t^r(-1)^{m+1}\int_{1/2}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}}\Big(\sum_{\substack{k=1\\ k \text{ odd}}}^{r-2} a_k(ut)^{k-r}\Big)du \\ &= \sum_{\substack{k=1\\ k \text{ odd}}}^{r-2} t^{r+k-r}a_k(-1)^{m+1}\int_{1/2}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}}u^{k-r}du \\ &= \sum_{\substack{k=1\\ k \text{ odd}}}^{r-2} t^k a_k(-1)^{m+1}\int_{1/2}^{\infty} o(u^m)O(u^{k-r-m-1})du \\ &= \sum_{\substack{k=1\\ k \text{ odd}}}^{r-2} t^k a_k(-1)^{m+1}\int_{1/2}^{\infty} o(u^{k-r-1})du \\ &= \sum_{\substack{k=1\\ k \text{ odd}}}^{r-2} b_k t^k . \end{aligned}$$

Returning to (5.9),

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta \\ &\quad = A + t^r B(t) \\ &\quad = b_1 t + b_3 t^3 + \cdots + b_{r-2}t^{r-2} + 0\cdot t^r + t^r B(t) . \end{aligned}$$

The proof of Theorem 1 will be complete when we establish $B(t) \to 0$ as $t \to 0$.

$$\begin{aligned} B(t) &= (-1)^{m+1}\int_0^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}}\lambda(ut)du \\ &= \int_0^{1/t} + \int_{1/t}^{\infty} \qquad (5.10) \\ &= B_1(t) + B_2(t) . \end{aligned}$$

To estimate $B_1(t)$ we use the fact that $\lambda(z)$ is entire, so for $|z| \leqq 1$,

$$\left|\frac{d^k}{dz^k}\lambda(z)\right| < K .$$

Since $|ut| \leqq 1$ in the interval of integration involving $B_1(t)$,

$$\left|\frac{d^{m+1}}{du^{m+1}}\lambda(ut)\right| \leqq t^{m+1}K$$

in this interval.

$$\begin{aligned} B_1(t) &= (-1)^{m+1}\int_0^{1/t} o(u^m)t^{m+1}K du \\ &= o(t^{m+1})\int_0^{1/t} u^m du \\ &= o(t^{m+1})\left(\frac{1}{t}\right)^{m+1} \\ &= o(1) \end{aligned}$$

as $t \to 0$.

For the estimate of $B_2(t)$ we use the decomposition

$$\lambda(z) = \gamma(z) - z^{-r}P(z) .$$

Clearly, as $z \to \infty$

$$\frac{d^{m+1}}{dz^{m+1}} z^{-r}P(z) = O(z^{-m-3}) ,$$

and by (5.4),

$$\frac{d^{m+1}}{dz^{m+1}}\gamma(z) = O(z^{-r-1/2}) .$$

Thus, for $z \to \infty$

$$\frac{d^{m+1}}{dz^{m+1}}\lambda(z) = O(z^{-r-1/2}) , \tag{5.11}$$

and

$$\begin{aligned} B_2(t) &= (-1)^{m+1}\int_{1/t}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}}\lambda(ut)du \\ &= (-1)^{m+1}\int_{1/t}^{\infty} o(u^m)t^{m+1}O(ut)^{-r-1/2}du \\ &= o(t^{m+1-r-1/2})\int_{1/t}^{\infty} o(u)^{m-r-1/2}du \\ &= o(t^{m-r+1/2})\, o\left(\frac{1}{t}\right)^{m-r+1/2} \\ &= o(1) . \end{aligned}$$

(Note we needed $m - r - 1/2 < -1$ to perform the last integration.) Thus $B_2(t) \to 0$ as $t \to 0$, and returning to (5.10), the proof of Theorem 1 is complete.

6. Proof of Theorem 2. We may assume that the fractional part of β is not zero. Otherwise Theorem 2 reduces to Theorem 1. Write $\beta = m + \alpha$, where m is an integer and $0 < \alpha < 1/2$.

We again assume $c_o = 0$, $x_o = 0$, $s_o = 0$. We proceed as in the beginning of the proof of Theorem 1.

$$\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta$$
$$= t^r \lim_{R\to\infty} \sum_{|n|<R} c_n \gamma(|n|t) ,$$

with $\gamma(z) = z^{-r}J_1(z)$.

As in the proof of Theorem 1 we integrate the last sum by parts. We now integrate by parts $m + 2$ times.

$$\sum_{|n|<R} c_n\gamma(|n|t) = S_R\gamma(Rt) - \int_0^R S_u \frac{d}{du}\gamma(ut)du$$
$$\vdots$$
$$= S_R\gamma(Rt) - S_R^1\frac{d}{dR}\gamma(Rt) + \cdots + (-1)^{m+1}S_R^{m+1}\frac{d^{m+1}}{dR^{m+1}}\gamma(Rt) \tag{6.1}$$
$$+ (-1)^{m+2}\int_0^R S_u^{m+1}\frac{d^{m+2}}{du^{m+2}}\gamma(ut)du .$$

We are now assuming the series (3.1) is summable Bochner-Riesz-β to 0 at $x_0 = 0$, so it is also summable Bochner-Riesz-$(m + 1)$ to 0 at $x_0 = 0$. Therefore we may again apply Lemma 1. For $0 \leqq k \leqq m + 1$,

$$S_R^k\frac{d^k}{dR^k}\gamma(Rt) = o(R^{r+1/2})O(R^{-r-1/2})$$
$$= o(1) ,$$

as $R \to \infty$, so

$$\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta$$
$$= t^r \lim_{R\to\infty} \sum_{|n|<R} c_n\gamma(|n|t) \tag{6.2}$$
$$= t^r(-1)^m\int_0^\infty S_u^{m+1}\frac{d^{m+2}}{du^{m+2}}\gamma(ut)du .$$

We define $P(z)$ and $\lambda(z)$ as in the proof of Theorem 1:

$$P(z) = \begin{cases} 0 & \text{if } r = 1 \\ a_1z + a_3z^3 + \cdots + a_{r-2}z^{r-2} & \text{if } r \neq 1 \end{cases}$$

and

$$\lambda(z) = \gamma(z) - z^{-r}P(z) .$$

Then (6.2) becomes,

$$\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta$$
$$= t^r(-1)^m\int_0^\infty S_u^{m+1}\frac{d^{m+2}}{du^{m+2}}[(ut)^{-r}P(ut) + \lambda(ut)]du$$

$$= t^r(-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}}[(ut)^{-r}P(ut)]du$$

$$+ t^r(-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}}\lambda(ut)du$$

$$= A(t) + t^r B(t) .$$

$$A = t^r(-1)^m \int_{1/2}^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}}\Big(\sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} a_k(ut)^{k-r}\Big)du$$

$$= \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} t^{r+k-r} a_k (-1)^m \int_{1/2}^\infty o(u)^{m+1} \frac{d^{m+2}}{du^{m+2}} u^{k-r} du$$

$$= \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} b_k t^k .$$

Hence,

$$\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta = \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} b_k t^k + t^r B(t) \tag{6.3}$$

where

$$B(t) = (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}}\lambda(ut)du . \tag{6.4}$$

To complete the proof of Theorem 2 we must show $B(t) \to 0$ as $t \to 0$.

If $f(u)$ is a function defined for $u > 0$ and η is a positive real number, denote by

$$I^\eta f(z) = \frac{1}{\Gamma(\eta)}\int_0^z (z-u)^{\eta-1} f(u)du ,$$

the fractional integral of order η, see [6]. Now if we set

$$f(u) = S_u = \sum_{|n|<u} c_n ,$$

then by (4.1),

$$S_u^\eta = I^\eta S_u ,$$

so

$$\begin{aligned} S_u^{m+1} &= I^{m+1} S_u \\ &= I^{1-\alpha} I^{m+\alpha} S_u \\ &= I^{1-\alpha} S_u^{m+\alpha} . \end{aligned}$$

Thus,

$$\begin{aligned} S_u^{m+1} &= \frac{1}{\Gamma(1-\alpha)}\int_0^u (u-z)^{1-\alpha-1} S_z^{m+\alpha} dz \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^u (u-z)^{-\alpha} S_z^{m+\alpha} dz . \end{aligned}$$

Returning to (6.4)

$$\begin{aligned}
B(t) &= (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du \\
&= \lim_{R\to\infty} (-1)^m \int_0^R \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{-\alpha} S_z^{m+\alpha} dz \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du \\
&= \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^R S_z^{m+\alpha} \int_z^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)dudz \\
&= \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^R S_z^{m+\alpha} H(z, t, R)dz ,
\end{aligned}$$

where

$$H(z, t, R) = \int_z^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du .$$

$$\begin{aligned}
B(t) &= \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^{1/t} S_z^{m+\alpha} H(z, t, R)dz \\
&\quad + \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_{1/t}^R S_z^{m+\alpha} H(z, t, R)dz \\
&= B_1(t) + B_2(t) .
\end{aligned}$$

We will make separate estimates of $H(z, t, R)$ for $B_1(t)$ and for $B_2(t)$.

First, in the interval of integration involving $B_1(t)$, $0 \leqq z \leqq 1/t$.

$$\begin{aligned}
H(z, t, R) &= \int_z^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du \\
&= \int_z^{1/t} + \int_{1/t}^R \qquad (6.5) \\
&= H_1 + H_2 .
\end{aligned}$$

Using the fact that λ is entire,

$$\begin{aligned}
|H_1| &\leqq \int_z^{1/t} (z-u)^{-\alpha} t^{m+2} \cdot K du \\
&\leqq K t^{m+2} \int_z^{1/t} (z-u)^{-\alpha} du \\
&= O(t^{m+2}) \left(\frac{1}{t} - z\right)^{1-\alpha} .
\end{aligned}$$

We estimate H_2 by employing (5.11)

$$\begin{aligned}
H_2 &= \int_{1/t}^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du \\
&= \int_{1/t}^\infty (u-z)^{-\alpha} t^{m+2} O(ut)^{-r-1/2} du
\end{aligned}$$

$$= O(t^{m-r+3/2})\left(\frac{1}{t} - z\right)^{-\alpha}\int_{1/t}^{\infty} u^{-r-1/2}du$$
$$= O(t^{m-r+3/2})\left(\frac{1}{t} - z\right)^{-\alpha}\left(\frac{1}{t}\right)^{-r+1/2}$$
$$= O(t^{m+1})\left(\frac{1}{t} - z\right)^{-\alpha}.$$

Returning to (6.5),

$$H(z, t, R) = O(t^{m+2})\left(\frac{1}{t} - z\right)^{1-\alpha} + O(t^{m+1})\left(\frac{1}{t} - z\right)^{-\alpha}.$$

and

$$B_1(t) = \frac{(-1)^m}{\Gamma(1-\alpha)}\int_0^{1/t} S_z^{m+\alpha}H(z, t, R)dz$$
$$= \int_0^{1/t} o(z^{m+\alpha})\left\{O(t^{m+2})\left(\frac{1}{t} - z\right)^{1-\alpha} + O(t^{m+1})\left(\frac{1}{t} - z\right)^{-\alpha}\right\}dz$$
$$= o\left(\frac{1}{t}\right)^{m+\alpha}\left\{O(t^{m+2})\int_0^{1/t}\left(\frac{1}{t} - z\right)^{1-\alpha}dz + O(t^{m+1})\int_0^{1/t}\left(\frac{1}{t} - z\right)^{-\alpha}dz\right\}$$
$$= o\left(\frac{1}{t}\right)^{m+\alpha}\left\{O(t^{m+2})\left(\frac{1}{t}\right)^{2-\alpha} + O(t^{m+1})\left(\frac{1}{t}\right)^{1-\alpha}\right\}$$
$$= o(1),$$

as $t \to 0$.

It remains to be shown that $B_2(t) \to 0$. In the interval of integration for B_2, $1/t \leqq z \leqq R$, and

$$H(z, t, R) = \int_z^R (u - z)^{-\alpha}\frac{d^{m+2}}{du^{m+2}}\lambda(ut)du$$
$$= \int_z^R (u - z)^{-\alpha}\frac{d^{m+2}}{du^{m+2}}\left(\frac{-P(ut)}{(ut)^r}\right)du$$
$$+ \int_z^R (u - z)^{-\alpha}\frac{d^{m+2}}{du^{m+2}}\gamma(ut)du$$
$$= H_a + H_b.$$

$$H_a = -\int_z^R (u - z)^{-\alpha}\frac{d^{m+2}}{du^{m+2}}\Big(\sum_{\substack{k=1\\ k \text{ odd}}}^{r-2} a_k(ut)^{k-r}\Big)du$$
$$= \int_z^R (u - z)^{-\alpha}t^{m+2}O(ut)^{-m-4}du$$
$$= t^{-2}\left\{\int_z^{2z}(u - z)^{-\alpha}O(u)^{-m-4}du + \int_{2z}^{\infty}(u - z)^{-\alpha}O(u)^{-m-4}du\right\}$$
$$= t^{-2}\{O(z)^{1-\alpha}z^{-m-4} + O(z^{-\alpha})z^{-m-3}\}$$
$$= t^{-2}O(z^{-m-\alpha-3}).$$

We change variables in the interval for H_b by letting $x = ut$.

$$\begin{aligned}
H_b(z, t, R) &= \int_z^R (u - z)^{-a} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du \\
&= \int_{tz}^{tR} \left(\frac{x}{t} - z\right)^{-a} t^{m+2} \frac{d^{m+2}}{du^{m+2}} \gamma(x) \frac{dx}{t} \\
&= t^{m+1+\alpha} \int_{tz}^{tR} (x - tz)^{-\alpha} \gamma^{(m+2)}(x) dx \\
&= t^{m+1+\alpha} \left\{ \int_{tz}^{tz+1} + \int_{tz+1}^{tR} \right\} \\
&= H_b' + H_b'' .
\end{aligned}$$

Recall that $1/t \leqq z$, so in the interval of integration for H_b, $x > tz \geqq 1$. Thus, by (5.11)

$$|\gamma^{(m+2)}(x)| \leqq Cx^{-r-1/2} ,$$

and

$$\begin{aligned}
H_b' &= t^{m+1+\alpha} \int_{tz}^{tz+1} (x - tz)^{-\alpha} \gamma^{(m+2)}(x) dx \\
&= t^{m+1+\alpha} O(tz)^{-r-1/2} \int_{tz}^{tz+1} (x - tz)^{-\alpha} dx \\
&= t^{m+1+\alpha} O(tz)^{-r-1/2} .
\end{aligned}$$

We estimate H_b'' by integrating by parts.

$$\begin{aligned}
H_b'' &= t^{m+1+\alpha} \int_{tz+1}^{tR} (x - tz)^{-\alpha} \gamma^{(m+2)}(x) dx \\
&= t^{m+1+\alpha} (x - tz)^{-\alpha} \gamma^{)m+1)}(x) \Big|_{tz+1}^{tR} \\
&\quad + t^{m+1+\alpha} \alpha \int_{tz+1}^{tR} (x - tz)^{-\alpha-1} \gamma^{(m+1)}(x) dx \\
&= t^{m+1+\alpha} (x - tz)^{-\alpha} \gamma^{(m+1)}(x) \Big|_{tz+1}^{tR} \\
&\quad + t^{m+1+\alpha} O(tz)^{-r-1/2} \int_{tz+1}^{tR} (x - tz)^{-\alpha-1} dx \\
&= t^{m+1+\alpha} (tR - tz)^{-\alpha} \gamma^{(m+1)}(tR) - t^{m+1+\alpha} \gamma^{(m+1)}(tz + 1) \\
&\quad + t^{m+1+\alpha} O(tz)^{-r-1/2} \left(\frac{1}{-\alpha}\right) \{(tR - tz)^{-\alpha} - 1\} \\
&= t^{m+1+\alpha} (tR - tz)^{-\alpha} O(tz)^{-r-1/2} + t^{m+1+\alpha} O(tz)^{-r-1/2} \\
&= t^{m+1+\alpha} O(tz)^{-r-1/2} .
\end{aligned}$$

Hence, in the interval of integration for B_2,

$$\begin{aligned}
H_b(z, t, R) &= H_b' + H_b'' \\
&= t^{m+1+\alpha} O(tz)^{-r-1/2} ,
\end{aligned}$$

and

$$H(z, t, R) = H_a + H_b$$
$$= t^{-2}O(z^{-m-\alpha-3}) + t^{m+1+\alpha}O(tz)^{-r-1/2} .$$

So,

$$B_2(t) = \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_{1/t}^{R} S_z^{m+\alpha} H(z, t, R)dz$$
$$= \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_{1/t}^{R} o(z)^{m+\alpha}\{t^{-2}O(z^{-m-\alpha-3}) + t^{m+1+\alpha}O(tz)^{-r-1/2}\}dz$$
$$= t^{-2}\int_{1/t}^{\infty} o(z^{m+\alpha-m-\alpha-3})dz + t^{m+1+\alpha-r-1/2}\int_{1/t}^{\infty} o(z^{m+\alpha-r-1/2})dz$$
$$= t^{-2}o(z^{-2})\Big|_{1/t}^{\infty} + t^{m+1/2+\alpha-r}o(z^{m+\alpha-r+1/2})\Big|_{1/t}^{\infty}$$
$$= o(1) .$$

(Note that the hypothesis $\alpha < 1/2$ is necessary here to insure that the last integral converge.) This completes the proof of Theorem 2.

REFERENCES

1. Erdelyi et al, *Higher Transcendental Functions*, vol. II, McGraw-Hill, New York, 1953.
2. G. H. Hardy, *The second theorem of consistency for summable series*, Proc. London Math. Soc., (2), **15** (1916), 72-88.
3. M. Kohn, *Spherical convergence and integrability of multiple trigonometric series on hypersurfaces*, Studia Math., **44** (1972), 345-354.
4. M. Kohn, *Riemann summability of multiple trigonometric series*, Indiana Univ. Math. J., **24** (1975), 813-823.
5. M. Kohn, *Lebesgue summability of double trigonometric series*, Trans. Amer. Math. Soc., to appear.
6. M. Riesz, *L'integrale de Riemann-Liouville et le problem de Cauchy*, Acta Math., **81** (1949), 1-233.
7. V. Shapiro, *Circular summability C of double trigonometric series*, Trans. Amer. Math. Soc., **76** (1954), 223-233.
8. A. Zygmund, *Trigonometric Series*, Vol. II, Cambridge University Press, Cambridge, 1968.

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