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POSITIVE DEFINITE FUNCTIONS WHICH VANISH AT INFINITY

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Let G be a separable noncompact locally compact group. Let $A(G)$ and $B(G)$, respectively, be the Fourier algebra and the Fourier-Stieltjes algebra of G as defined by P. Eymard. We prove that if G is unimodular and satisfies an additional hypothesis, which implies noncompactness, there exists an element of $B(G)$, indeed a positive definite function, which vanishes at infinity but is not in $A(G)$. This function actually belongs to $B^\rho(G)$, that is, it defines a unitary representation of G which is weakly contained in the regular representation.

We refer to [3] for the definitions and properties of $A(G)$, $B(G)$, $B_\rho(G)$ and of the related spaces $VN(G)$, $C^*(G)$ and $C^*_\rho(G)$.

We recall only that $C^*_\rho(G)$ and $VN(G)$ are, respectively, the C^*-algebra and the von Neumann algebra, generated by $L^1(G)$ acting by left convolution on $L^2(G)$. While $C^*(G)$ is the C^*-algebra of the group obtained by completing the algebra $L^1(G)$ with respect to the norm $\|f\|C^*(G) = \sup_{\pi\in\Sigma}\|\pi(f)\|$, where Σ is the space of all $*$-representations of $L^1(G)$ as an algebra of operators on a Hilbert space. We also recall that $B(G)$ is, in a natural fashion, the dual of $C^*(G)$, while $B_\rho(G)$ is the dual of $C^*_\rho(G)$ which is a quotient algebra of $C^*(G)$. Finally $VN(G)$ is the dual of $A(G)$;

When G is commutative and $\hat{G}$ is the character group of G, $A(G)$ and $B(G)$, respectively, coincide with the algebra of Fourier transforms of elements of $L^1(\hat{G})$ and the algebra of Fourier-Stieltjes transforms of bounded regular measures on $\hat{G}$.

Thus for G commutative our result reduces to the classical theorem which asserts that on any nondiscrete locally compact abelian group $\hat{G}$, one can construct a singular measure with Fourier-Stieltjes transform vanishing at infinity. This classical result was proved for the first time, for the case $\hat{G} = T$ and $G = Z$, by M. D. Menchoff [10].

When G is noncommutative the situation may be quite different: I. Khalil proved in [6] that if G is the affine group of the line, i.e., the group of transformations $x \to ax + b$ of $\boldsymbol{R}$ into $\boldsymbol{R}$, then $B(G) \cap C_0(G) = A(G)$.

Therefore some other hypothesis, in addition to noncompactness of G, is needed for our result to be true. In this paper we show that $A(G) \neq B(G) \cap C_0(G)$ provided that G is unimodular and in addition satisfies the following condition:

(H) *The von Neumann algebra $VN(G)$ is not purely atomic.*

This same hypothesis, which implies noncompactness, was used in [2] for another construction. A list of groups satisfying (H) is given in [2]. An example of a noncompact unimodular group which does not satisfy (H) is given in [1] and [7]. This example, as shown in [9] and in [13] does not satisfy the conclusion of the theorem. It seems possible that, for unimodular groups, (H) is necessary as well as sufficient for the existence of functions of B_ρ which vanish at infinity and are not in A. We also do not know whether, for nonunimodular groups condition (H) is sufficient and/or necessary to yield the conclusion of the theorem.

The techniques used in this paper are related to those developed in [2] which in turn are related to the techniques of [4].

2. Preliminaries. The main device in proof of the theorem consists in transferring certain techniques of the theory of Fourier series on commutative compact groups to the present context. The simplest commutative infinite compact group is the Cantor group D which we define as the product $D = \prod_{j=1}^{\infty}\{-1, 1\}_j$, of countably many multiplicative groups of two elements. The character group $\hat{D}$ of D can be simply described: first we have the Rademacher functions $r_j(t) = \varepsilon_j$ for $t = \{\varepsilon_j\} \in D$, which are simply the projections of an element into its j-th coordinate. The functions r_j are of course characters of D and also independent random variables with zero mean. All characters w of D may be obtained then as finite products of Rademacher functions. The characters of D are called the Walsh functions and are numbered (according to R.E.A.C. Paley) as follows: $w_0 = 1$, $w_n = r_{j_1} \cdots r_{j_s}$ if $n = 2^{j_1-1} + \cdots + 2^{j_s-1}$, with $j_1 < j_2 < \cdots < j_s$. We shall indicate by w a generic element of the character group $\hat{D}$ of D. We shall summarize in the following lemma a few well known technical results concerning Fourier series on D.

LEMMA. *Let $-1 \leqq \beta_j \leqq 1$ be a sequence of real numbers. Let $\phi_n(t) = \prod_{j=1}^{n}(1 + \beta_j r_j(t))$, for $t \in D$. Then there exists a probability measure μ defined on the Borel subsets of D, such that*

(1) $\phi_n(t) = \sum_{k=0}^{2^n} \hat{\mu}(w_k)w_k(t)$

(2) $\hat{\mu}(w_n) = \beta_{j_1} \cdots \beta_{j_s}$ *if* $n = 2^{j_1-1} + \cdots + 2^{j_s-1}$, *so that* $\lim_n \hat{\mu}(w_n) = 0$ *if and only if* $\lim_j \beta_j = 0$

(3) *μ is absolutely continuous if and only if $\Sigma\beta_j^2 < \infty$. Thus if $\Sigma\beta_j^2 = \infty$, no subsequence of Φ_n can converge in the norm of $L^1(D)$.*

Proof. The measure μ is obtained as a weak* limit of the positive functions ϕ_n which have all integral one. Statements (1) and

(2) are verified immediately. For (3) one proves that if $\Sigma\beta_j^2 < \infty$, $\phi_n \in L^2(D)$, with bounded norms, and that if $\Sigma\beta_j^2 = +\infty$, $\phi_n(t)$ converges to zero almost everywhere. For a more detailed proof cfr. [5, Th. 3.2. and Th. 4.1.].

We shall describe now the machinery which will allow us to bring to bear upon our problem ideas, results and techniques pertaining to Fourier analysis on the Cantor group. An analogous machinery was used in [2]. First of all, since we assume that G is unimodular, we can make use, as in [2], of the canonical gace space $\Gamma = (L^2(G), VN(G), \mathrm{tr})$, defined for a unimodular group G, by I. E. Segal [11], and extensively studied by R. A. Kunze [8] and W. F. Stinespring [12]. We refer to the cited papers and to [2] for the statements of the results needed here. We recall only that $A(G)$ is the image under an "inverse Fourier transform" of the space $L^1(\Gamma)$ of the operators which are measurable with respect to VN and "absolutely integrable" with respect to the trace tr. The trace is defined as $\mathrm{tr}(P) = \|f\|^2_{L^2(G)}$, when P is a projection and $P = L_f$ (left convolution by f), with $f \in L^2(G)$, and $\mathrm{tr}(P) = \infty$, otherwise. The fact that G satisfies property (H), which will be always assumed from now on, implies that *there exists a projection P, of finite positive trace, which contains no minimal projections of VN.* By choosing an appropriate multiple of the Haar measure on G, we may assume that $\mathrm{tr}(P) = 1$. (If G is discrete we may take P to be the identity and thus keep the usual normalization for the Haar measure of G). Since P contains no minimal projections, for each θ, $0 < \theta < 1$, there exists a subprojection $Q_\theta \subseteqq P$ such that $\mathrm{tr}(Q_\theta) = \theta$. This is proved just as in the commutative case when instead of projections one deals with measurable sets containing no atom for a given measure (cfr. [2]). Using this fact we construct a sequence of partitions of P as indicated in [2]. That is, we let $\pi_n = \{P_{n1}, \cdots, P_{n2^n}\}$, $P = P_{n1} + \cdots + P_{n2^n}$, $P_{nk} \in VN(G)$, $\mathrm{tr}(P_{nk}) = 1/2^n$. We also require that π_{n+1} partitions each of the projections of π_n into two projections of equal trace.

We define now the Rademacher operators $\{R_n\}_{n=1}^\infty$ and the Walsh operators $\{W_n\}_{n=0}^\infty$ as in [2], starting from the partitions $\{\pi_n\}$. Recall that $R_n = \sum_{k=1}^{2^n}(-1)^{k+1}P_{nk}$, where $\pi_n = \{P_{n1}, \cdots, P_{n2^n}\}$, and the projections of π_n are indexed so that $P_{n-1,k} = P_{n,2k-1} + P_{n,2k}$ for $k = 1, \cdots, 2^{n-1}$. Recall also that $W_0 = P$ and the Walsh operators W_n are finite products of the Rademacher operators R_n, in such a way that $n = 2^{j_1-1} + \cdots + 2^{j_s-1}$, if $j_1 < j_2 < \cdots j_s$ and $W_n = R_{j_1} \cdots R_{j_s}$. The sequence $\{W_n\}_{n=0}^\infty$ is an orthonormal system in $L^2(\Gamma)$ which will be referred to as the Walsh system.

We define then the operators $\mathscr{E}_n$ mapping $L^1(\Gamma)$ into $L^1(\Gamma)$ as follows:

$$\mathscr{E}_n(T) = \sum_{k=0}^{2^n-1} \operatorname{tr}(W_k T) W_k \,, \quad \text{for} \quad T \in L^1(\Gamma) \,.$$

We notice that if $\pi_n = \{P_{n1}, \cdots, P_{n2^n}\}$, then

$$\mathscr{E}_n(T) = \sum_{j=1}^{2^n} m_{jn}(T) P_{nj} \,,$$

where

$$m_{jn}(T) = 2^n \operatorname{tr}(P_{nj} T) \,.$$

Therefore, since $\sum_{j=1}^{2^n} |\operatorname{tr}(P_{nj}T)| \leqq \|T\|_{L^1(\Gamma)}$, we also have

$$\|\mathscr{E}_n(T)\|_{L^1(\Gamma)} \leqq \|T\|_{L^1(\Gamma)} \,.$$

We shall define now an operator $\mathscr{E}$ of $L^1(\Gamma)$ into $L^1(\Gamma)$ which is the limit in the strong operator topology of the sequence $\mathscr{E}_n$. We start by defining $\mathscr{E}(VN)$ to be the commutative v. Neumann subalgebra of VN generated by the projections $\{P_{nj}\}$. Since $\mathscr{E}(VN)$ is commutative, it is isomorphic to $L^\infty(Y, \mathscr{M}, m)$ where Y is a set, $\mathscr{M}$ a σ-algebra of subsets of Y and m a positive measure of total mass one.[1] Let E_{nj} be the elements of $\mathscr{M}$ corresponding to P_{nj}. Then $m(E_{nj}) = \operatorname{tr}(P_{nj})$ and the sets $\{E_{nj}\}$ generate $\mathscr{M}$. Now if $T \in L^1(\Gamma)$ we can define a measure μ on $\mathscr{M}$ by letting $\mu(E_{nj}) = \operatorname{tr}(P_{nj}T)$ and $\mu(A) = \operatorname{tr}(QT)$ where $A \in \mathscr{M}$ and $Q \in \mathscr{E}(VN)$ is the projection corresponding to A. The Radon-Nikodym derivative of μ with respect to m is a function in $L^1(X, \mathscr{M}, m)$ which will be denoted by $\mathscr{E}(T)$. It is clear that $\mathscr{E}(T)$ can be identified with an element of $L^1(\Gamma)$. Furthermore $\mathscr{E}_n(\mathscr{E}(T)) = \mathscr{E}(\mathscr{E}_n(T)) = \mathscr{E}_n(T)$. Since $\mathscr{E}(T)$ is a limit in $L^1(\Gamma)$ of linear combinations of the projections P_{nj}, we also have

$$\|\mathscr{E}_n(T) - \mathscr{E}(T)\|_{L^1(\Gamma)} \longrightarrow 0 \,.$$

3. The main theorem. We now have all the ingredients for the proof of our result.

THEOREM. *There exists a positive definite function $f \in B(G)$ which is not in $A(G)$ and such that* $\lim_{x\to\infty} f(x) = 0$

Proof. Let $\{\alpha_j\}$ be a decreasing sequence of positive real numbers, bounded by one, and satisfying $\lim \alpha_j = 0$ and $\sum \alpha_j^2 = \infty$. Let D be the Cantor group. For each $t = \{\varepsilon_j\} \in D$ we shall construct inductively a sequence f_n^t of positive definite elements of $A(G)$, which converge uniformly to an element $f^t \in B_\rho$. We shall prove that for

[1] Y can be chosen to be D, the Cantor group with its Haar measure as shown in footnote 2).

some $t \in D$, $f^t \notin A(G)$. We construct f^t_n as the Fourier transform $(F^t_n)^\wedge(x) = \operatorname{tr}(L^*_x F^t_n)$ of operators $F^t_n \in \mathscr{C}(VN) \subseteqq L^1(\Gamma)$. We refer to [11, p. 47] for properties of the transform $T^\wedge$ of an element T of $L^1(\Gamma)$. We let $F_1 = F^t_1 = P$ for each $t \in D$. The operators F^t_n will have the form

$$F^t_n = \prod_{k=1}^{n-1}\left(P + r_{j_k}(t)\frac{\alpha_k}{2}R_{j_k}\right), \quad \text{for} \quad t = \{r_j(t)\} \in D ,$$

where the sequence j_k will be determined inductively. Together with the sequence F^t_n we shall construct a sequence of compact sets K_n, which will be used in the induction procedure. We let $K_1 = \{x \in G: |F^\wedge_1(x)| \geqq \alpha_1/2\}$. Suppose that, for each t, $F^t, \cdots, F^t_n$ and $K_1, \cdots, K_{n-1}$, have been defined. We let $K_n = \{x \in G: |(F^t_n)^\wedge(x)| \geqq \alpha_n/2$, for some $t \in D\} \cup K_{n-1}$.

Notice that because of the form of F^t_n, as t varies in D, we only have finitely many, in fact 2^{n-1}, different operators F^t_n. Thus K_n is the union of finitely many compact sets and it is compact. Let $\nu(x) = \sum_{j=0}^\infty |\operatorname{tr}(L^*_x W_j)|^2$. Then, as proved in [2], ν is a continuous function of x. Therefore, by Dini's theorem, for some positive integer h_n.

$$\sum_{j \geqq h_n} |\operatorname{tr}(L^*_x W_j)|^2 < 2^{-4n} , \quad \text{for} \quad x \in K_n .$$

Let j_n be such that $j_n > j_{n-1}$, and $2^{j_n - 1} > h_n$, and define

$$F^t_{n+1} = F^t_n + r_{j_n}(t)\frac{\alpha_n}{2}R_{j_n}F^t_n , \quad \text{where} \quad t = \{r_j(t)\} \in D .$$

We notice that $R_{j_n} = W_{2^{j_n - 1}}$. Therefore all the Walsh operators appearing in the expansion of $R_{j_n}F^t_n$ have indices greater than h_n. As a consequence, if W is any such Walsh operator,

$$|\operatorname{tr}(L^*_x W)| < 2^{-2n} , \quad \text{for} \quad x \in K_n .$$

Since the number of terms appearing in the expansion of $R_{j_n}F^t_n$ is exactly 2^{n-1}, we deduce that

$$|\operatorname{tr}(L^*_x R_{j_n} F^t_n)| < \left(\frac{1}{2}\right)^{n+1} , \quad \text{for} \quad x \in K_n .$$

Now,

$$(F^t_{n+1})^\wedge(x) = \operatorname{tr}(L^*_x F^t_n) + r_{j_n}(t)\frac{\alpha_n}{2}\operatorname{tr}(L^*_x R_{j_n} F^t_n) .$$

Therefore, for $x \notin K_n$,

$$|(F^t_{n+1})^\wedge(x)| \leqq \frac{\alpha_n}{2} + \frac{\alpha_n}{2} = \alpha_n .$$

On the other hand, if $x \in K_m \backslash K_{m-1}$, with $m \leqq n$, then

$$|(R_{j_k} F_k^t)\hat{}(x)| < \frac{1}{2^{k+1}}, \quad \text{for} \quad m \leqq k \leqq n .$$

Therefore, if $x \in K_m \backslash K_{m-1}$,

$$\begin{aligned} |(F_{n+1}^t)\hat{}(x)| &= \Big| \frac{\alpha_n}{2} r_{j_n}(t)(R_{j_n} F_n^t)\hat{}(x) \\ &\quad + \cdots + \frac{\alpha_{m-1}}{2} r_{j_{m-1}}(t)(R_{j_{m-1}} F_{m-1}^t)\hat{}(x) + (F_{m-1}^t)\hat{}(x) \Big| \\ &\leqq \frac{\alpha_{m-1}}{2}\Big(\frac{1}{2^{n+1}} + \cdots + \frac{1}{2^m}\Big) + \frac{\alpha_{m-1}}{2} \leqq \alpha_{m-1} , \end{aligned}$$

(we have used the fact that the sequence α_j is decreasing). We have proved therefore that

$$|(F_{n+1}^t)\hat{}(x)| \leqq \alpha_j \quad \text{for} \quad x \notin K_j, \ j \leqq n .$$

Since $K_{n+1} \supseteqq \{x : |(F_{n+1}^t)\hat{}(x)| \geqq \alpha_{n+1}/2\}$, we can say that for all n and $j \leqq n$,

$$|f_n^t(x)| = |(F_n^t)\hat{}(x)| \leqq \alpha_j \quad \text{for} \quad x \notin K_j .$$

Since for $t = \{r_j(t)\} \in D$,

$$F_n^t = \prod_{k=1}^{n-1} \Big(P + r_{j_k}(t)\frac{\alpha_k}{2} R_{j_k}\Big) ,$$

it follows that F_n^t is a positive element of the commutative algebra $\mathscr{E}(VN)$ and $\mathrm{tr}(F_n^t) = \mathrm{tr}(P) = 1$. We show now that f_n^t converges uniformly (and hence weak*) to a positive definite function $f^t \in B_\rho$. Let $\varepsilon > 0$ and n be such that $2\alpha_n < \varepsilon$. Then for all positive integers p,

$$|f_n^t(x) - f_{n+p}^t(x)| \leqq 2\alpha_n < \varepsilon \quad \text{for} \quad x \notin K_n .$$

On the other hand if $x \in K_n$ and $i \geqq 0$, then $x \in K_{n+i}$, and hence

$$|f_{n+i}^t(x) - f_{n+i+1}^t(x)| = \Big| \frac{\alpha_{n+i}}{2} (R_{j_{n+i}} F_{n+i}^t)\hat{}(x) \Big| < \frac{\alpha_{n+i}}{2} \frac{1}{2^{n+i}} .$$

Thus for $x \in K_n$

$$|f_n^t(x) - f_{n+p}^t(x)| \leqq \sum_{i=0}^{p-1} |f_{n+i}^t(x) - f_{n+i+1}^t(x)| \leqq \frac{\alpha_n}{2^n} .$$

We conclude that f_n^t is uniformly Cauchy and converges, uniformly and weak*, to a positive definite function $f^t \in B_\rho$. Since $1 = \mathrm{tr}(F_n^t) = f_n^t(1)$ converges to $f^t(1)$, we have that $\|f^t\|_{B_\rho} = 1$.

It remains to prove that for at least some $t \in D$, $f^t \notin A(G)$. Suppose that for all t, $f^t \in A(G)$. We let F^t be the element of $L^1(\Gamma)$ such that $\operatorname{tr}(L_x^* F^t) = f^t(z)$. We consider now an application Θ mapping $\mathscr{E}(VN)$ onto $L^\infty(D)$ and $\mathscr{E}(L^1)$ onto $L^1(D)$. We simply define $\Theta W_n = w_n(t)$, where the $w_n(t)$ are the Walsh functions and extend Θ linearly and by continuity (weak* continuity for VN).[2] We denote by X the subspace of $L^\infty(D)$ which is the image under Θ of $\{\mathscr{E}(T)\colon T \in C_\rho^*(G)\}$. Let $\beta_{j_k} = \alpha_k/2$ and $\beta_h = 0$ if $h = j_k$, and construct the functions ϕ_n as in the Lemma. Then

$$\phi_{j_{n-1}}(s+t) = \prod_{k=1}^{n-1}\left(1 + \frac{\alpha_k}{2} r_{j_k}(t) r_{j_k}(s)\right) = (\Theta F_n^t)(s)\,,$$

and $\phi_k = \phi_{j_n}$ for $j_n \leqq k < j_{n+1}$.

Let T be an arbitrary element of $C_\rho^*(G)$, then

$$\begin{aligned}\operatorname{tr}(F^t T) &= \lim_n \operatorname{tr}(F_n^t T) = \lim_n \operatorname{tr}(F_n^t \mathscr{E}(T)) \\ &= \lim_n \int_D \phi_n(t+s) u(s) ds = \lim_n \phi_n * u(t)\,,\end{aligned}$$

where $u \in X$ is the element $u = \Theta\mathscr{E}(T)$. On the other hand (cfr. the Lemma) the functions ϕ_n converge weak* (with respect to $C(D)$) to the measure μ. Thus μ is a singular measure, because $\sum (\alpha_k/2)^2 = +\infty$, and $\lim_n \mu(w_n) = 0$. Since ϕ_n is a partial sum of the series of μ, $\lim_n \|\hat{\phi}_n - \hat{\mu}\|_\infty = 0$. This means that the operator on $L^2(D)$ of convolution by ϕ_n approaches in norm the operator of convolution by μ. This implies that

$$\lim_n \|\phi_n * u - \mu * u\|_{L^2(D)} = 0$$

It follows that

$$\operatorname{tr}(F^t T) = \lim \phi_n * u(t) = \mu * u(t)\,,$$

the last equality holding for almost every $t \in D$. Since G is separable the unit sphere of VN is metrisable in the strong operator topology.[3] Therefore given one of the Walsh operators W, there exists a sequence T_n of elements of C_ρ^*, of norm less than or equal to one, which converges to W in the strong operator topology. Let $u_n = \Theta\mathscr{E}(T_n)$, then u_n converges to w in the weak* topology of $L^\infty(D)$,

[2] Consider the gage space $\Gamma = (L^2(G), VN(G), \operatorname{tr})$ mentioned after the proof of the Lemma, and let $\mathscr{H}$ be the closed linear span of the Walsh operators $(W_n)_{n=0}^\infty$ in $L^2(G)$. Then the gage space $(\mathscr{H}, \mathscr{E}(VN), \operatorname{tr})$, where $\mathscr{E}(VN)$ acts on $\mathscr{H}$ by multiplication is unitarily equivalent to the commutative gage space $(L^2(D), L^\infty(D), m)$, where D is the Cantor group and m is its Haar measure. It follows from this that Θ has the asserted properties.

[3] This is the only place where separability of G is used.

because T_n converges to W weak*, and $\mathscr{E}$ is weak* continuous from VN to $\mathscr{E}(VN)$ (it is the adjoint of a continuous map between their respective pre-duals). For each $t \in D$

$$\operatorname{tr}(F^t W) = \lim_n \operatorname{tr}(F^t T_n) .$$

But $\operatorname{tr}(F^t T_n) = \mu * u_n(t)$ a.e., and convolution by a bounded measure is a weak* continuous map of $L^\infty(D)$ into $L^\infty(D)$. Thus $\lim_n \mu * u_n = \mu * w$ in the weak* topology of L^∞. On the other hand $\mu * u_n(t)$ converges pointwise to $\operatorname{tr}(F^t W)$, except for those t which belong to the set $\{t \in D: \operatorname{tr}(F^t T_n) \neq \mu * u_n(t)$ for some $n\}$. Since this set is the countable union of sets of zero measure, $\mu * u_n(t) \to \operatorname{tr}(F^t T)$ a.e., and hence $\mu * u_n \to \operatorname{tr}(F^t T)$ in the weak* topology of L^∞, from which it follows that $\operatorname{tr}(F^t W) = \mu * w(t)$, a.e., and, since the Walsh functions are denumerable,

$$\operatorname{tr}(F^t W_n) = \mu * w_n(t) = \hat{\mu}(w_n) w_n(t) , \quad \text{for all } n,$$

except for a negligible set of $t \in D$. We fix $t \in D$ so that the above identity holds. It follows that $\operatorname{tr}(F^t W_n) = \hat{\phi}^t_m(w_n) w_n(t)$ if $n < 2^{j_m}$ and $\mathscr{E}_{j_n}(F^t) = F^t_n$. This means that $\lim_n F^t_n = \mathscr{E}(F^t)$ in $L^1(\Gamma)$, but if this were the case the functions ϕ^t_n would converge in $L^1(D)$ which is impossible because they are the partial sums of the Fourier-Stieltjes series of the singular measure μ^t, defined by $\mu^t(E) = \mu(E + t)$. We conclude that for at least one t the function f^t cannot be in $A(G)$ and the proof is complete.

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