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ASYMPTOTIC PROPERTIES OF NONOSCILLATORY SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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Recently, there is an increasing interest in studying the nth order differential equations involving the so called nth order r-derivative of x

$$(r_{n-1}(t)(r_{n-2}(t)(\cdots(r_1(t)(r_0(t)x(t))')'\cdots)')')'$$

which causes damped terms. Here, the asymptotic behavior of nonoscillatory solutions of such general differential equations with deviating argument is studied and, more precisely, sufficient conditions which guarantee that

$$\lim_{t\to\infty} x(t) = 0$$

for the bounded nonoscillatory solutions $x(t)$ are established. A basic theorem is obtained for the general case and then it is specialized into four corollaries concerning the particular case

$$r_j = 1 \quad \text{for} \quad j \neq n-N \quad \text{and} \quad r_{n-N} = r \qquad (1 \leqq N \leqq n-1)$$

which is of special interest. Finally, some examples are given to illustrate the significance of the results.

In this paper we consider the nth order $(n > 1)$ differential equation with deviating argument of the form

$$\text{(E)} \qquad \begin{aligned} &(r_{n-1}(t)(r_{n-2}(t)(\cdots(r_1(t)(r_0(t)x(t))')'\cdots)')')' \\ &\qquad + a(t)F(x[\sigma(t)]) = b(t), \qquad t \geqq t_0 \end{aligned}$$

where the functions r_i $(i = 0, 1, \cdots, n-1)$ are supposed positive at least on the interval $[t_0, \infty)$. The continuity of the functions involved in the above equation (E) as well as sufficient smoothness to guarantee the existence of solutions of (E) on an infinite subinterval of $[t_0, \infty)$ will be assumed without mention. In what follows the term "solution" is always used only for such solutions $x(t)$ of (E) which are defined for all large t. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined on an interval of the form $[T, \infty)$

is called *oscillatory* if it has no last zero, and otherwise it is called *nonoscillatory*.

Furthermore, the conditions (i) and (ii) below are assumed to hold throughout the paper:

(i) $\lim_{t\to\infty} \sigma(t) = \infty$

(ii) $y \neq 0 \Rightarrow yF(y) > 0$.

The results of this paper are included in a general theorem given in §2, which establishes sufficient conditions in order that all bounded nonoscillatory solutions $x(t)$ of the differential equation (E) tend to zero as $t \to \infty$. This theorem extends a recent result given by the authors in [**9**, Theorem 3] concerning also the equation (E). Also, it unifies and extends two results by Kusano and Onose [**5**, Theorems 3 and 4] concerning, in particular, the differential equation

$$(\mathrm{D}_N) \qquad [r(t)x^{(n-N)}(t)]^{(N)} + a(t)F(x[\sigma(t)]) = b(t), \qquad t \geqq t_0,$$

where $1 \leqq N \leqq n-1$ and the function r is continuous and positive at least on the interval $[t_0, \infty)$.

The technique used in the proof of our theorem is based on three lemmas which are given in §1. Lemma 1 is fundamental and has been proved by the authors in [**9**], Lemma 2 is proved here and it is an extension of a result due to Hardy and Littlewood [**4**], while Lemma 3 is new and it is rather technical.

In §3 our main theorem is specialized for the differential equation (D_N), which is obtained from the equation (E) by setting

$$r_j = 1 \quad \text{for} \quad j \neq n - N \quad \text{and} \quad r_{n-N} = r$$

and which is of special interest. Thus, four corollaries are obtained, from which Corollary 1 is the main result of a recent paper by Kusano and Onose [**5**] while Corollaries 2, 3 and 4 are new and are illustrated by examples.

1. Preliminaries. Let q_i $(i = 0, 1, \cdots, m)$ be positive continuous functions on an interval I of the real line. For a real-valued function h on I and any $\mu = 0, 1, \cdots, m$ we define the μth *q-derivative* of h by the formula

$$D_q^{(\mu)}h = q_\mu(q_{\mu-1}(q_{\mu-2}(\cdots(q_1(q_0h)')'\cdots)')')'$$

when obviously we have

$$D_q^{(0)}h = q_0h$$

and

$$D_q^{(j)}h = q_j(D_q^{(j-1)}h)' \qquad (j = 1, 2, \cdots, m).$$

Moreover, if $D_q^{(m)}h$ is defined as a continuous function on I, then h is said to be m-times *continuously q-differentiable.* We note that in the case where $q_0 = q_1 = \cdots = q_m = 1$ the above notion of q-differentiability specializes to the usual one.

By using this shorthand notation, the differential equation (E) can be written

$$\text{(E)} \qquad (D_r^{(n)}x)(t) + a(t)F(x[\sigma(t)]) = b(t), \qquad t \geqq t_0$$

where $r_n = 1$.

Now, let ρ be a real-valued function which is defined and positive at least on the interval (t_0, ∞) and let R_i $(i = 0, 1, \cdots, n)$ be the functions defined as follows:

$$R_n = \rho$$

and for every $j = n - 1, n - 2, \cdots, 0$

$$R_j = r_j R'_{j+1}.$$

The function ρ is said to be *of the type* $r[k]$, $0 \leqq k \leqq n - 1$, if:

(α) the functions R_j $(j = k + 1, \cdots, n)$ are defined at least on (t_0, ∞),

(β) R_{k+1} is a constant nonzero function on (t_0, ∞),

(γ) if $k < n - 1$, then for every $j = k + 2, \cdots, n$

$$\lim_{t \to \infty} R_j(t) \text{ exists in } \{-\infty, 0, +\infty\}$$

and

(δ) if $k < n - 2$, then for every $j = k + 2, \cdots, n - 1$

$$R_j(t) \neq 0 \text{ for all } t > t_0.$$

For some interesting examples of functions of the above type we refer to **[9]**.

The technique used here is based on the following lemma, which has been proved by the authors in **[9]**.

LEMMA 1. *Let ρ be a function of the type $r[k]$, $0 \leqq k \leqq n - 1$, and h an n-times continuously r-differentiable function of $[T, \infty)$, $T > t_0$.*

If the improper integral

$$\int_T^\infty \rho(t)(D_r^{(n)}h)(t)dt$$

exists in the extended real line $\mathbf{R}^*$, *then so does the* $\lim_{t\to\infty}(D_r^{(k)}h)(t)$. *Moreover,*

$$\int_T^\infty \rho(t)(D_r^{(n)}h)(t)dt = \pm\infty \quad \textit{implies} \quad \lim_{t\to\infty} |(D_r^{(k)}h)(t)| = \infty.$$

In order to obtain our results we need further the following lemma which is an extension of a result due to Hardy and Littlewood **[4]**.

LEMMA 2. *Let* q_i $(i = 0, 1, \cdots, m)$, *where* $m > 1$, *be positive continuous functions on an interval* $[T, \infty)$ *such that*

$$\liminf_{t\to\infty} q_i(t) > 0 \qquad (i = 1, 2, \cdots, m)$$

and

$$\limsup_{t\to\infty} q_i(t) < \infty \qquad (i = 1, 2, \cdots, m-1).$$

Moreover, let h *be an* m*-times continuously* q*-differentiable function on* $[T, \infty)$.

If $D_q^{(0)}h$ *is bounded on* $[T, \infty)$ *and*

$$\lim_{t\to\infty} (D_q^{(m)}h)(t) = 0,$$

then

$$\lim_{t\to\infty} (D_q^{(j)}h)(t) = 0 \qquad (j = 1, 2, \cdots, m-1).$$

The above lemma follows immediately from the following proposition, which in the particular case $q_0 = q_1 = \cdots = q_m = 1$ is an improved version of a result due to Landau **[6]** (cf. also Coppel **[2**, p. 140**]**).

PROPOSITION. *Let* q_i $(i = 0, 1, \cdots, m)$, *where* $m > 1$, *be positive continuous functions on an interval* I *such that*

$$A \equiv \min_{i=1,\cdots,m} \inf_{t\in I} q_i(t) > 0 \quad \textit{and} \quad B \equiv \max_{i=1,\cdots,m-1} \sup_{t\in I} q_i(t) < \infty.$$

Moreover, let h be an m-times continuously q-differentiable function on I with

$$|(D_q^{(0)}h)(t)| \leqq K \quad \text{and} \quad |(D_q^{(m)}h)(t)| \leqq M \quad \text{for every } t \in I.$$

If L is the length of the interval I and

(a) *I is closed and* $L \geqq 2A\left(\frac{K}{M}\right)^{1/m}$

or

(b) $L > 2A\left(\frac{K}{M}\right)^{1/m}$,

then for every $t \in I$

$$|(D_q^{(j)}h)(t)| \leqq c_m K^{1-j/m} M^{j/m} \qquad (j = 1, 2, \cdots, m-1),$$

where

$$c_m \equiv \left(\frac{2B}{A}\right)^{2^m-2}.$$

Proof. It suffices to prove the proposition in the case where (a) is satisfied. Indeed, if (b) holds, then for any $t \in I$ we can choose a closed subinterval J of I with length $L' \geqq 2A(K/M)^{1/m}$ and $t \in J$. So, applying the proposition for the closed interval J we obviously obtain

$$|(D_q^{(j)}h)(t)| \leqq c_m K^{1-j/m} M^{j/m} \qquad (j = 1, 2, \cdots, m-1).$$

Now, we suppose that (a) is satisfied and define

$$S = \max\left\{c_m, \max_{0<j<m} \max_{t \in I} \frac{|(D_q^{(j)}h)(t)|}{K^{1-j/m}M^{j/m}}\right\}.$$

Obviously, for every $j = 0, 1, \cdots, m-1$

$$1 \leqq \left(\frac{2B}{A}\right)^{2^{j+1}-2} \leqq \left(\frac{2B}{A}\right)^{2^m-2} = c_m \leqq S,$$

by which, after some manipulations, we derive that

$$\left(\frac{4B^2}{A^2}S\right)^{\gamma_j} \leqq S, \tag{1}$$

where $\gamma_j = 1 - 2^{-j}$.

We shall prove next that for $j = 0, 1, \cdots, m-1$

(2) $$|(D_q^{(j)}h)(t)| \leqq \left(\frac{4B^2}{A^2}S\right)^{\gamma_j} K^{1-j/m}M^{j/m} \equiv K_j \text{ for every } t \in I.$$

Indeed, (2) is valid for $j = 0$, since

$$|(D_q^{(0)}h)(t)| \leqq K = K_0 \text{ for every } t \in I.$$

We suppose that (2) is satisfied for $j = l$, $0 \leqq l < m - 1$, i.e.

(3) $$|(D_q^{(l)}h)(t)| \leqq K_l \text{ for every } t \in I.$$

For this l we have that

(4) $$|(D_q^{(l+2)}h)(t)| \leqq SK^{1-(l+2)/m}M^{(l+2)/m} \equiv M_l \text{ for every } t \in I$$

which, by the definition of S, is obvious for $l < m - 2$ and follows from the inequality

$$|(D_q^{(m)}h)(t)| \leqq M \leqq MS = M_{m-2} \text{ for every } t \in I,$$

when $l = m - 2$. Since, by (1),

$$\left(\frac{K_l}{M_l}\right)^{1/2} \leqq \left(\frac{K}{M}\right)^{1/m}$$

and because of (a), there exist t_1, t_2 in I with $t_2 - t_1 = 2A(K_l/M_l)^{1/2}$ and $t_1 \leqq T_0 \leqq t_2$, where T_0 is such that

$$|(D_q^{(l+1)}h)(T_0)| = \max_{t \in I} |(D_q^{(l+1)}h)(t)|.$$

It is easy to verify that

$$(D_q^{(l)}h)(t_2) - (D_q^{(l)}h)(T_0) = (D_q^{(l+1)}h)(T_0)\int_{T_0}^{t_2} \frac{ds}{q_{l+1}(s)} + \int_{T_0}^{t_2} \frac{1}{q_{l+1}(s)} \int_{T_0}^{s} \frac{1}{q_{l+2}(w)} (D_q^{(l+2)}h)(w)dwds$$

and

$$(D_q^{(l)}h)(t_1) - (D_q^{(l)}h)(T_0) = (D_q^{(l+1)}h)(T_0)\int_{T_0}^{t_1} \frac{ds}{q_{l+1}(s)} + \int_{T_0}^{t_1} \frac{1}{q_{l+1}(s)} \int_{T_0}^{s} \frac{1}{q_{l+2}(w)} (D_q^{(l+2)}h)(w)dwds,$$

when, by subtraction, we get

$$(D_q^{(l)}h)(t_2)-(D_q^{(l)}h)(t_1)=(D_q^{(l+1)}h)(T_0)\int_{t_1}^{t_2}\frac{ds}{q_{l+1}(s)}$$
$$+\int_{T_0}^{t_2}\frac{1}{q_{l+1}(s)}\int_{T_0}^{s}\frac{1}{q_{l+2}(w)}(D_q^{(l+2)}h)\ (w)dwds$$
$$-\int_{t_1}^{T_0}\frac{1}{q_{l+1}(s)}\int_{s}^{T_0}\frac{1}{q_{l+2}(w)}(D_q^{(l+2)}h)\ (w)dwds.$$

Thus, we obtain

$$|(D_q^{(l+1)}h)(T_0)|\int_{t_1}^{t_2}\frac{ds}{q_{l+1}(s)}\leqq|(D_q^{(l)}h)(t_1)|+|(D_q^{(l)}h)(t_2)|$$
$$+\int_{t_1}^{T_0}\frac{1}{q_{l+1}(s)}\int_{s}^{T_0}\frac{1}{q_{l+2}(w)}|(D_q^{(l+2)}h)\ (w)|\,dwds$$
$$+\int_{T_0}^{t_2}\frac{1}{q_{l+1}(s)}\int_{T_0}^{s}\frac{1}{q_{l+2}(w)}|(D_q^{(l+2)}h)\ (w)|\,dwds$$

and consequently, by (3) and (4),

$$|(D_q^{(l+1)}h)(T_0)|\leqq\frac{1}{\displaystyle\int_{t_1}^{t_2}\frac{ds}{q_{l+1}(s)}}\ \left\{2K_l+M_l\left[\left[\int_{t_1}^{T_0}\frac{1}{q_{l+1}(s)}\int_{s}^{T_0}\frac{1}{q_{l+2}(w)}\,dwds\right.\right.\right.$$
$$\left.\left.+\int_{T_0}^{t_2}\frac{1}{q_{l+1}(s)}\int_{T_0}^{s}\frac{1}{q_{l+2}(w)}\,dwds\right]\right\}.$$

Hence,

$$|(D_q^{(l+1)}h)(T_0)|\leqq\frac{B}{t_2-t_1}\left\{2K_l+\frac{M_l}{2A^2}\left[(T_0-t_1)^2+(t_2-T_0)^2\right]\right\}$$
$$\leqq\frac{B}{t_2-t_1}\left[2K_l+\frac{M_l}{2A^2}(t_2-t_1)^2\right]$$
$$=B\left[\frac{2K_l}{t_2-t_1}+\frac{M_l}{2A^2}(t_2-t_1)\right]$$
$$=\frac{2B}{A}(K_lM_l)^{1/2}.$$

But, it is a matter of calculations to verify that

$$\frac{2B}{A}(K_lM_l)^{1/2}=K_{l+1},$$

when

$$|(D_q^{(l+1)}h)(T_0)| \leqq K_{l+1},$$

which proves (2) for $j = l+1$ and consequently for all $j = 0, 1, \cdots, m-1$.

From (2) it follows that

$$\max_{0<j<m} \max_{t\in I} \frac{|(D_q^{(j)}h)(t)|}{K^{1-j/m}M^{j/m}} \leqq \left(\frac{4B^2}{A^2} S\right)^{\gamma_{m-1}}.$$

Also,

$$c_m = \left(\frac{2B}{A}\right)^{2^m-2} = \left(\frac{4B^2}{A^2} c_m\right)^{\gamma_{m-1}} \leqq \left(\frac{4B^2}{A^2} S\right)^{\gamma_{m-1}}.$$

Therefore, by the definition of S,

$$S \leqq \left(\frac{4B^2}{A^2} S\right)^{\gamma_{m-1}},$$

which gives

$$S \leqq \left(\frac{2B}{A}\right)^{2^m-2} = c_m, \quad \text{i.e.} \quad S = c_m.$$

Hence, by the definition of S, we have that for every $t \in I$

$$|(D_q^{(j)}h)(t)| \leqq c_m K^{1-j/m} M^{j/m} \qquad (j = 1, 2, \cdots, m-1).$$

Finally, for our purpose we need one more lemma.

LEMMA 3. *Let q_i $(i = 0, 1, \cdots, \lambda)$ be positive continuous functions on an interval $[T, \infty)$ such that for $\lambda > 0$,*

$$\text{(I)} \qquad \int^{\infty} \frac{dt}{q_i(t)} = \infty \qquad (i = 1, 2, \cdots, \lambda)$$

and let

$$Q_{0\lambda}(t; T) = \begin{cases} 1, & \text{if } \lambda = 0 \\ \displaystyle\int_T^t \frac{1}{q_1(s_1)} \int_T^{s_1} \frac{1}{q_2(s_2)} \cdots \int_T^{s_{\lambda-1}} \frac{1}{q_\lambda(s_\lambda)} ds_\lambda \cdots ds_2 ds_1, & \text{if } \lambda > 0. \end{cases}$$

Moreover, let h be a λ-times continuously q-differentiable function on the interval $[T,\infty)$. *Then we have*:

(α) *If*

$$\liminf_{t\to\infty} (D_q^{(\lambda)}h)(t)>0 \quad \textit{or} \quad \limsup_{t\to\infty} (D_q^{(\lambda)}h)(t)<0$$

and

$$\int^{\infty} \frac{Q_{0\lambda}(t;T)}{q_0(t)}\,dt=\infty, \tag{II}$$

then

$$\int^{\infty} h(t)dt=\pm\infty.$$

(β) *If*

$$\lim_{t\to\infty} (D_q^{(\lambda)}h)(t)=0$$

and

$$\limsup_{t\to\infty} \frac{Q_{0\lambda}(t;T)}{q_0(t)}<\infty, \tag{III}$$

then

$$\lim_{t\to\infty} h(t)=0.$$

Proof. Since the lemma is obvious for $\lambda=0$, we suppose that $\lambda>0$. For any integers i and j, $0\leqq i\leqq j\leqq\lambda$, and for every u and v with $v\geqq u\geqq T$, we define

$$Q_{ij}(v;u)=\begin{cases}1, \text{ if } \quad i=j\\ \displaystyle\int_u^v \frac{1}{q_{i+1}(s_{i+1})}\int_u^{s_{i+1}} \frac{1}{q_{i+2}(s_{i+2})}\cdots\int_u^{s_{j-1}} \frac{1}{q_j(s_j)}\,ds_j\cdots ds_{i+2}ds_{i+1}, \\ \hfill \text{if } i<j.\end{cases}$$

Then, taking into account condition (I), we have that for every $T^* \geqq T$,

$$\lim_{t\to\infty} Q_{ij}(t; T^*) = \infty \qquad (0 \leqq i < j \leqq \lambda)$$

and consequently, by L'Hospital's rule,

$$\lim_{t\to\infty} \frac{Q_{0j}(t; T^*)}{Q_{0\lambda}(t; T^*)} = 0 \qquad (j = 0, 1, \cdots, \lambda - 1) \tag{5}$$

and

$$\lim_{t\to\infty} \frac{Q_{0\lambda}(t; T^*)}{Q_{0\lambda}(t; T)} = 1. \tag{6}$$

Moreover, for every t and T^* with $t \geqq T^* \geqq T$, it is easy to derive the following generalized Taylor's formula

$$(D_q^{(0)}h)(t) = \sum_{j=0}^{\lambda-1} (D_q^{(j)}h)(T^*)Q_{0j}(t; T^*) \\ + \int_{T^*}^{t} \frac{1}{q_1(s_1)} \int_{T^*}^{s_1} \frac{1}{q_2(s_2)} \cdots \int_{T^*}^{s_{\lambda-1}} \frac{1}{q_\lambda(s_\lambda)} (D_q^{(\lambda)}h)(s_\lambda)ds_\lambda \cdots ds_2 ds_1. \tag{7}$$

(α) Let $d > 0$ and $T^* > T$ be chosen so that

$$(D_q^{(\lambda)}h)(t) \geqq d \text{ for every } t \geqq T^*$$

or

$$(D_q^{(\lambda)}h)(t) \leqq -d \text{ for every } t \geqq T^*.$$

Then from the formula (7) we have respectively

$$\frac{(D_q^{(0)}h)(t)}{Q_{0\lambda}(t; T)} \geqq \sum_{j=0}^{\lambda-1} (D_q^{(j)}h)(T^*) \frac{Q_{0j}(t; T^*)}{Q_{0\lambda}(t; T)} + d\frac{Q_{0\lambda}(t; T^*)}{Q_{0\lambda}(t; T)}, \qquad t \geqq T^*$$

or

$$\frac{(D_q^{(0)}h)(t)}{Q_{0\lambda}(t; T)} \leqq \sum_{j=0}^{\lambda-1} (D_q^{(j)}h)(T^*) \frac{Q_{0j}(t; T^*)}{Q_{0\lambda}(t; T)} - d\frac{Q_{0\lambda}(t; T^*)}{Q_{0\lambda}(t; T)}, \qquad t \geqq T^*$$

and hence, by virtue of (5) and (6), we obtain

$$\liminf_{t\to\infty} \frac{(D_q^{(0)}h)(t)}{Q_{0\lambda}(t; T)} > 0 \quad \text{or} \quad \limsup_{t\to\infty} \frac{(D_q^{(0)}h)(t)}{Q_{0\lambda}(t; T)} < 0.$$

But this, by condition (II), gives

$$\int^{\infty} h(t)dt = \pm\infty.$$

(β) For any arbitrary $\epsilon > 0$, we consider a $T^* > T$ so that

$$|(D_q^{(\lambda)}h)(t)| < \epsilon \text{ for every } t \geqq T^*,$$

when from the formula (7) we obtain

$$\frac{|(D_q^{(0)}h)(t)|}{Q_{0\lambda}(t;T)} \leqq \sum_{j=0}^{\lambda-1} |(D_q^{(j)}h)(T^*)| \frac{Q_{0j}(t;T^*)}{Q_{0\lambda}(t;T)} + \epsilon \frac{Q_{0\lambda}(t;T^*)}{Q_{0\lambda}(t;T)}$$

for every $t \geqq T^*$. Thus, by (5) and (6), we have

$$\limsup_{t\to\infty} \frac{|(D_q^{(0)}h)(t)|}{Q_{0\lambda}(t;T)} \leqq \epsilon$$

and, since ϵ is arbitrary,

$$\lim_{t\to\infty} \frac{(D_q^{(0)}h)(t)}{Q_{0\lambda}(t;T)} = 0,$$

which, by condition (III), gives

$$\lim_{t\to\infty} h(t) = 0.$$

2. The main result.

Our main theorem establishes conditions which essentially guarantee that

$$\lim_{t\to\infty} x(t) = 0$$

for the bounded nonoscillatory solutions x of the equation (E).

THEOREM. *Consider the differential equation* (E) *subject to the conditions* (i), (ii) *and*

(iii) $\limsup_{t\to\infty} r_0(t) < \infty$.

Let m and k be integers with $1 \leqq m \leqq k \leqq n-1$ so that the conditions (iv), (v) *and* (vi) *below are satisfied*:

(iv) *If $m < k$, then for every $j = m+1, \cdots, k$*

$$\int^{\infty} \frac{dt}{r_j(t)} = \infty.$$

(v) $\int^{\infty} \frac{R_{mk}(t)}{r_m(t)}\,dt = \infty$ *and* $\limsup_{t\to\infty} \frac{R_{mk}(t)}{r_m(t)} < \infty$,

where

$$R_{mk}(t) = \begin{cases} 1, & \text{if} \quad m = k \\ \int_{t_0}^{t} \frac{1}{r_{m+1}(s_{m+1})} \int_{t_0}^{s_{m+1}} \frac{1}{r_{m+2}(s_{m+2})} \cdots \int_{t_0}^{s_{k-1}} \frac{1}{r_k(s_k)}\, ds_k \cdots ds_{m+2} ds_{m+1}, & \text{if} \quad m < k. \end{cases}$$

(vi) *If* $m > 1$, *then for every* $i = 1, 2, \cdots, m-1$

$$0 < \liminf_{t\to\infty} r_i(t) \leqq \limsup_{t\to\infty} r_i(t) < \infty.$$

Moreover, let there exist a function ρ *of the type* $r[k]$ *such that*:

$$(C_1) \qquad \int^{\infty} \rho(t)|b(t)|\,dt < \infty$$

and

(C_2) *For some* $\delta > 0$,

$$\liminf_{t\to\infty} \int_t^{t+\delta} \rho(s)a^{+}(s)ds > 0 \quad \text{and} \quad \int^{\infty} \rho(t)a^{-}(t)dt < \infty$$

or

$$\liminf_{t\to\infty} \int_t^{t+\delta} \rho(s)a^{-}(s)ds > 0 \quad \text{and} \quad \int^{\infty} \rho(t)a^{+}(t)dt < \infty,$$

where

$$a^{+}(t) = \max\{a(t), 0\} \quad \text{and} \quad a^{-}(t) = \max\{-a(t), 0\}.$$

If the function σ *is differentiable with bounded derivative on* $[t_0, \infty)$, *then for all bounded nonoscillatory solutions* x *of the equation* (E),

$$\lim_{t\to\infty} (D_r^{(k)}x)(t) = 0 = \lim_{t\to\infty} (D_r^{(m-1)}x)'(t)$$

and

$$\lim_{t\to\infty} (D_r^{(l)}x)(t) = 0 \qquad (l = 0, 1, \cdots, m-1).$$

Proof. Let x be a bounded nonoscillatory solution of (E). Without loss of generality, we assume that x is a solution on the whole interval $[t_0, \infty)$. Moreover, this solution is supposed positive on $[t_0, \infty)$, since the substitution $u = -x$ transforms (E) into an equation of the same form satisfying the assumptions of the theorem.

If, by (i), $T > t_0$ is chosen so that for every $t \geqq T$

$$\sigma(t) \geqq t_0,$$

then from equation (E) we obtain

$$\begin{aligned}\int_T^t \rho(s)(D_r^{(n)}x)(s)ds &= \int_T^t \rho(s)b(s)ds - \int_T^t \rho(s)a^+(s)F(x[\sigma(s)])ds \\ &\quad + \int_T^t \rho(s)a^-(s)F(x[\sigma(s)])ds\end{aligned} \tag{8}$$

for every $t \geqq T$. Because of (ii) and the boundedness of x, the function $F \circ x \circ \sigma$ is positive and bounded on $[T, \infty)$. Thus, by conditions (C_1) and (C_2), we conclude from (8) that the improper integral

$$\int_T^\infty \rho(t)(D_r^{(n)}x)(t)dt$$

exists in $\mathbf{R}^*$ and consequently, by virtue of Lemma 1, the $\lim_{t\to\infty} (D_r^{(k)}x)(t)$ also exists in $\mathbf{R}^*$. Moreover,

$$\lim_{t\to\infty} (D_r^{(k)}x)(t) = 0. \tag{9}$$

Indeed, in the opposite case we have

$$\liminf_{t\to\infty} (D_r^{(k)}x)(t) > 0 \quad \text{or} \quad \limsup_{t\to\infty} (D_r^{(k)}x)(t) < 0.$$

Thus, because of conditions (iv) and (v), we can apply Lemma 3(a) for $h = (D_r^{(m-1)}x)'$, $\lambda = k - m$ and $q_j = r_{m+j}$ $(j = 0, 1, \cdots, \lambda)$, when we obtain

$$\int^\infty (D_r^{(m-1)}x)'(t)dt = \pm\infty, \quad \text{i.e.} \quad \lim_{t\to\infty} (D_r^{(m-1)}x)(t) = \pm\infty.$$

From this, by condition (vi), it is easy to derive

$$\lim_{t\to\infty} (D_r^{(0)}x)(t) = \pm\infty$$

a contradiction, since, by condition (iii), $D_r^{(0)}x = r_0 x$ is bounded.

Now, taking into account (9) and conditions (iv) and (v), we apply again Lemma 3(β) for $h = (D_r^{(m-1)}x)'$, $\lambda = k - m$ and $q_j = r_{m+j}$ ($j = 0, 1, \cdots, \lambda$), when we get

$$\lim_{t\to\infty} (D_r^{(m-1)}x)'(t) = 0. \tag{10}$$

So, if $m > 1$, then, by virtue of condition (vi), we can apply Lemma 2 for $q_j = r_j$ ($j = 0, 1, \cdots, m-1$) and $q_m = 1$, to obtain

$$\lim_{t\to\infty} (D_r^{(j)}x)(t) = 0 \qquad (j = 1, 2, \cdots, m-1). \tag{11}$$

Thus, it remains to prove that

$$\lim_{t\to\infty} (D_r^{(0)}x)(t) = 0. \tag{12}$$

To do this, we first observe that

$$\liminf_{t\to\infty} (D_r^{(0)}x)(t) = 0. \tag{13}$$

Indeed, in the opposite case for some positive constant d_1 and for every $t \geqq t_0$ we have

$$(D_r^{(0)}x)(t) \geqq d_1$$

and consequently, by (iii),

$$x(t) \geqq d_2 \text{ for every } t \geqq t_0$$

where $d_2 = d_1/\sup_{t\geqq t_0} r_0(t)$. From this and the boundedness of x we conclude that

$$F(x[\sigma(t)]) \geqq d \text{ for every } t \geqq T,$$

where d is a positive constant. Hence, from (8) and on account of conditions (C_1) and (C_2), it follows that

$$\int_T^\infty \rho(t)(D_r^{(n)}x)(t)dt = \pm\infty$$

and consequently, by applying Lemma 1,

$$\lim_{t\to\infty}(D_r^{(k)}x)(t) = \pm\infty,$$

which contradicts (9).

To complete the proof of (12), we have verify that

$$\limsup_{t\to\infty}(D_r^{(0)}x)(t) = 0.$$

Indeed, in the opposite case we have

$$\limsup_{t\to\infty}(D_r^{(0)}x)[\sigma(t)] \geqq K$$

for some positive constant K. Hence, on account of (13) and based on the arguments of Hammett **[3]** (cf. also Singh **[7, 8]** and Staikos and Philos **[9]**), we derive that there exist three sequences (α_ν), (β_ν) and (γ_ν) with $\lim \alpha_\nu = \infty$ and such that for every $\nu = 1, 2, \cdots$

$$T \leqq \alpha_\nu < \gamma_\nu < \beta_\nu \leqq \alpha_{\nu+1}$$
$$(D_r^{(0)}x)[\sigma(\alpha_\nu)] = \frac{K}{2} = (D_r^{(0)}x)[\sigma(\beta_\nu)]$$
$$(D_r^{(0)}x)[\sigma(\gamma_\nu)] > K$$
$$(D_r^{(0)}x)[\sigma(t)] > \frac{K}{2} \text{ for every } t \in (\alpha_\nu, \beta_\nu).$$

By mean-value theorem, we have

$$\frac{(D_r^{(0)}x)[\sigma(\gamma_\nu)] - (D_r^{(0)}x)[\sigma(\alpha_\nu)]}{\gamma_\nu - \alpha_\nu} = \sigma'(\xi_\nu)(D_r^{(0)}x)'[\sigma(\xi_\nu)]$$

and consequently

$$\frac{K}{2(\beta_\nu - \alpha_\nu)} < \sigma'(\xi_\nu)(D_r^{(0)}x)'[\sigma(\xi_\nu)],$$

where obviously $\lim \xi_\nu = \infty$. But, because of (10), (11) and (vi), it is easy to see that

$$\lim_{t\to\infty}(D_r^{(0)}x)'(t) = 0.$$

Thus, we obtain

$$\lim(\beta_\nu - \alpha_\nu) = \infty. \tag{14}$$

Next, we observe that for every $\nu = 1, 2, \cdots$

$$(D_r^{(0)}x)[\sigma(t)] \geqq \frac{K}{2} \text{ for every } t \in [\alpha_\nu, \beta_\nu]$$

and consequently, by (iii),

$$x[\sigma(t)] \geqq \frac{K}{2r_0[\sigma(t)]} \geqq \frac{K}{2\sup_{t \geqq t_0} r_0(t)} > 0 \text{ for every } t \in [\alpha_\nu, \beta_\nu].$$

We have thus proved that the bounded function $x \circ \sigma$ has a positive lower bound on the set $\bigcup_{\nu=1}^{\infty}[\alpha_\nu, \beta_\nu]$. Hence, because of (ii), we have

$$F(x[\sigma(t)]) \geqq M \text{ for every } t \in \bigcup_{\nu=1}^{\infty} [\alpha_\nu, \beta_\nu],$$

where the constant M is positive.

Obviously,

$$\int_T^\infty \rho(t)a^{\pm}(t)F(x[\sigma(t)])dt \geqq \sum_{\nu=1}^{\infty} \int_{\alpha_\nu}^{\beta_\nu} \rho(t)a^{\pm}(t)F(x[\sigma(t)])dt$$
$$\geqq M\sum_{\nu=1}^{\infty} \int_{\alpha_\nu}^{\beta_\nu} \rho(t)a^{\pm}(t)dt.$$

But, by virtue of (14) and condition (C_2), we have

$$\sum_{\nu=1}^{\infty} \int_{\alpha_\nu}^{\beta_\nu} \rho(t)a^{+}(t)dt = \infty \quad \text{or} \quad \sum_{\nu=1}^{\infty} \int_{\alpha_\nu}^{\beta_\nu} \rho(t)a^{-}(t)dt = \infty$$

and consequently

$$\int_T^\infty \rho(t)a^{+}(t)F(x[\sigma(t)])dt = \infty \quad \text{or} \quad \int_T^\infty \rho(t)a^{-}(t)F(x[\sigma(t)])dt = \infty.$$

Thus, from (8) it follows that

$$\int_T^\infty \rho(t)(D_r^{(n)}x)(t)dt = \pm\infty.$$

Finally, by Lemma 1, we obtain

$$\lim_{t\to\infty}(D_r^{(k)}x)(t) = \pm\infty,$$

which contradicts (9).

3. Applications. We shall give now some interesting applications of our main result for the particular case

$$r_j = 1 \quad \text{for} \quad j \neq n-N \quad \text{and} \quad r_{n-N} = r,$$

where N is an integer with $1 \leqq N \leqq n-1$. More precisely, we shall derive some corollaries concerning the differential equation

$$(\mathrm{D}_N) \qquad [r(t)x^{(n-N)}(t)]^{(N)} + a(t)F(x[\sigma(t)]) = b(t), \qquad t \geqq t_0.$$

All corollaries are new except Corollary 1, which is the main result of a recent paper by Kusano and Onose **[5]**.

COROLLARY 1. *Consider the differential equation* (D_N) *subject to the conditions* (i) *and* (ii). *Moreover, let* k, $0 \leqq k \leqq N-1$, *be an integer such that*:

$$(\alpha) \qquad \int^{\infty} \frac{t^{N-1-k}}{r(t)}\,dt = \infty \quad \textit{and} \quad \limsup_{t\to\infty} \frac{t^{N-1-k}}{r(t)} < \infty,$$

$$(\beta) \qquad \int^{\infty} t^k\,|b(t)|\,dt < \infty$$

and

(γ) *For some* $\delta > 0$,

$$\liminf_{t\to\infty} \int_t^{t+\delta} s^k a^+(s)ds > 0 \quad \textit{and} \quad \int^{\infty} t^k a^-(t)dt < \infty$$

or

$$\liminf_{t\to\infty} \int_t^{t+\delta} s^k a^-(s)ds > 0 \quad \textit{and} \quad \int^{\infty} t^k a^+(t)dt < \infty.$$

If the function σ *is differentiable with bounded derivative on* $[t_0, \infty)$, *then for all bounded nonoscillatory solutions* x *of the differential equation* (D_N),

$$\lim_{t\to\infty} [r(t)x^{(n-N)}(t)]^{(N-1-k)} = 0$$

and

$$\lim_{t\to\infty} x^{(i)}(t) = 0 \qquad (i = 0, 1, \cdots, n - N).$$

Proof. It is easy to see that the function ρ,

$$\rho(t) = (t - t_0)^k$$

is of the type $r[n-1-k]$. Thus, the corollary follows by applying our theorem for $m = n - N$ and with $n - 1 - k$ in place of k.

COROLLARY 2. *Consider the differential equation* (D_N) *with* $N > 1$, *subject to the conditions* (i), (ii) *and*

$$0 < \liminf_{t\to\infty} r(t) \leqq \limsup_{t\to\infty} r(t) < \infty.$$

Moreover, let $k, 0 \leqq k < N - 1$, *be an integer such that* (β) *and* (γ) *are satisfied.*

If the function σ *is differentiable with bounded derivative on* $[t_0, \infty)$, *then for all bounded nonoscillatory solutions* x *of the differential equation* (D_N),

$$\lim_{t\to\infty} [r(t)x^{(n-N)}(t)]^{(j)} = 0 \qquad (j = 0, 1, \cdots, N - 1 - k)$$

and

$$\lim_{t\to\infty} x^{(i)}(t) = 0 \qquad (i = 0, 1, \cdots, n - N).$$

Proof. It follows from our theorem, by taking the same function ρ as in the proof of Corollary 1, $m = n - 1 - k > n - N$ and $n - 1 - k$ in place of k.

COROLLARY 3. *Consider the differential equation* (D_N) *with* $N < n - 1$, *subject to the conditions* (i) *and* (ii). *Moreover, let* k, $1 \leqq k \leqq n - N - 1$, *be an integer such that*:

$$(\delta) \qquad \int^{\infty} \frac{t^{n-N-1-k}}{r(t)} dt = \infty,$$

(ϵ) $$\int^{\infty} \rho_k(t)|b(t)|\,dt < \infty$$

and for some $\delta > 0$ *either*

$$\liminf_{t\to\infty} \int_t^{t+\delta} \rho_k(s)a^+(s)ds > 0 \quad \textit{and} \quad \int^{\infty} \rho_k(t)a^-(t)dt < \infty$$

or

$$\liminf_{t\to\infty} \int_t^{t+\delta} \rho_k(s)a^-(s)ds > 0 \quad \textit{and} \quad \int^{\infty} \rho_k(t)a^+(t)dt < \infty,$$

where

$$\rho_k(t) = \int_{t_0}^{t} \frac{(t-s)^{N-1}(s-t_0)^{n-N-1-k}}{r(s)}\,ds.$$

If the function σ *is differentiable with bounded derivative on* $[t_0, \infty)$, *then for all bounded nonoscillatory solutions* x *of the differential equation* (D_N),

$$\lim_{t\to\infty} x^{(i)}(t) = 0 \qquad (i = 0, 1, \cdots, k).$$

Proof. Here, we have to apply our theorem for $m = k$ and $\rho = \rho_k$, since, as it is easy to see, the function ρ_k is of the type $r[k]$.

COROLLARY 4. *Consider the differential equation* (D_N) *with* $N < n-1$, *subject to the conditions* (i) *and* (ii). *Moreover, let* k, $1 \leqq k \leqq n-N-1$ *be an integer such that*:

(ζ) $$\int^{\infty} \frac{t^{n-N-1-k}}{r(t)}\,dt < \infty,$$

(η) *If* $N > 1$, *then for every* $j = 1, 2, \cdots, N-1$

$$\lim_{t\to\infty} \int_{t_0}^{t} (t-s)^{j-1} \int_s^{\infty} \frac{(u-t_0)^{n-N-1-k}}{r(u)}\,du\,ds \textit{ exists in } \{0, \infty\}$$

(ϑ) $$\int^{\infty} \rho_k(t)|b(t)|\,dt < \infty$$

and for some $\delta > 0$ *either*

$$\liminf_{t\to\infty} \int_t^{t+\delta} \rho_k(s)a^+(s)ds > 0 \quad \textit{and} \quad \int^{\infty} \rho_k(t)a^-(t)dt < \infty$$

or

$$\liminf_{t\to\infty}\int_t^{t+\delta}\rho_k(s)a^-(s)ds>0 \quad and \quad \int^\infty \rho_k(t)a^+(t)dt<\infty,$$

where

$$\rho_k(t)=\begin{cases}\displaystyle\int_t^\infty \frac{(s-t_0)^{n-2-k}}{r(s)}\,ds, & if \quad N=1\\[2ex] \displaystyle\int_{t_0}^t (t-s)^{N-2}\int_s^\infty \frac{(u-t_0)^{n-N-1-k}}{r(u)}\,duds, & if \quad N>1.\end{cases}$$

If the function σ is differentiable with bounded derivative on $[t_0,\infty)$, then for all bounded nonoscillatory solutions x of the differential equation (D_N),

$$\lim_{t\to\infty} x^{(i)}(t)=0 \qquad (i=0,1,\cdots,k).$$

Proof. It is easy to verify that the function ρ_k is of the type $r[k]$, when the corollary follows immediately by applying our theorem for $m=k$.

The significance of Corollaries 2, 3 and 4 is illustrated by the three examples below, where in each case exactly one of these corollaries can be applied and in addition Corollary 1 fails.

EXAMPLE 1. The retarded differential equation

$$[(1+e^{-t})x'(t)]''+\frac{1}{t}x^2(\log t)\operatorname{sgn} x(\log t)=\frac{1}{t^3}+e^{-t}(1+8e^{-t}), \qquad t\geqq 1$$

has the bounded nonoscillatory solution $x(t)=e^{-t}$ with the property

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}x'(t)=\lim_{t\to\infty}[(1+e^{-t})x'(t)]'=0.$$

Further, as it follows from Corollary 2, every bounded nonoscillatory solution x of the above equation has this property.

EXAMPLE 2. By Corollary 3, all bounded nonoscillatory solutions of the differential equation

$$[t^{1/2}x''(t)]'+t^{-1/2}x^5(\gamma t)=(\gamma^{-5/2}-3/2)t^{-3}, \qquad t\geqq 1,$$

where γ is a positive constant, tend to zero as $t \to \infty$ together with their derivatives. For example, $x(t) = t^{-1/2}$ is a bounded nonoscillatory solution of the above equation with $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0$.

EXAMPLE 3. Consider the retarded differential equation

$$[t^2 x''(t)]' + tx^6(\sqrt{t})\operatorname{sgn} x(\sqrt{t}) = -\frac{1}{t^2}, \qquad t \geqq 1.$$

This equation has the bounded nonoscillatory solution $x(t) = 1/t$ with $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0$. Further, from Corollary 4 it follows that all bounded nonoscillatory solutions of the considered equation tend to zero as $t \to \infty$ together with their derivatives.

REMARK. The results of this paper can be formulated in the case of more general differential equations of the form

$$\text{(E*)} \quad \begin{aligned} &(D_r^{(n)}x)(t) + a(t)F(x[\sigma(t)]) \\ &+ G(t; x\langle\tau_0(t)\rangle; (D_r^{(1)}x)\langle\tau_1(t)\rangle, \cdots, (D_r^{(n-1)}x)\langle\tau_{n-1}(t)\rangle) = b(t), \qquad t \geqq t_0 \end{aligned}$$

where

$$g\langle\tau_i(t)\rangle = (g[\tau_{i1}(t)], g[\tau_{i2}(t)], \cdots, g[\tau_{i\mu_i}(t)]).$$

From the proof of our theorem, this is obvious under additional conditions on G, which ensure that for every bounded nonoscillatory and n-times continuously r-differentiable function u on an interval $[T, \infty)$,

$$\int^{\infty} \rho(t) | G(t; u\langle\tau_0(t)\rangle; (D_r^{(1)}u)\langle\tau_1(t)\rangle, \cdots, (D_r^{(n-1)}u)\langle\tau_{n-1}(t)\rangle) | \, dt < \infty,$$

where ρ is the function introduced in conditions (C_1) and (C_2) of our theorem. For a such related result to Corollary 1 see Chen [1].

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