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ATTAINING THE SPREAD AT CARDINALS OF COFINALITY ω

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Let λ be a singular cardinal of cofinality ω. We investigate the question: does every Hausdorff space with spread λ have a discrete subspace of cardinality λ? The answer is "yes" if $\lambda > 2^{\aleph_0}$ or if $\lambda < 2^{\aleph_0}$ and MA holds; however, for $\lambda < 2^{\aleph_0}$ an answer of "no" is consistent with the axioms of set theory. The proof involves showing the equivalence of the question with one about category in the real line. Similar results hold for the width of a space.

0. Introduction. All spaces are assumed Hausdorff, 2^ω is understood to have the usual product topology, and an ordinal is the set of its predecessors. The ordinal 0 is sometimes slashed ($\emptyset$). c is the cardinal $2^{\aleph_0}$.

The spread of a space X ($s(X)$) is the supremum of the cardinalities of its discrete subspaces. When $s(X)$ is a limit cardinal, we may ask whether the spread is attained—i.e., whether X has a discrete subspace of size $s(X)$. If λ is a singular strong limit cardinal spread is attained (Hajnal–Juhász; see **[1]**, Theorem 3.2); however, if $\omega < cf(\lambda) \leqq \lambda \leqq c$ spread is not (**[3]**). For results when λ is a regular limit cardinal, see **[1]**, p. 40. When $cf(\lambda) = \omega$, spread is always attained provided X is strongly T_2 (a condition between T_2 and T_3) (Hajnal–Juhász; see **[1]**, Theorem 3.3); however, the general problem for all T_2 spaces has remained open. We use $SA(\lambda)$ to abbreviate the assertion: the spread is attained in all T_2 spaces of spread λ.

Recall that a set $Y \subseteq 2^\omega$ is nowhere dense (n.w.d.) iff the closure of Y has empty interior, and Y is first category iff Y is a countable union of n.w.d. sets. We let $L(\lambda)$ be the assertion: for all $X \subseteq 2^\omega$ of cardinality λ, there is a $Y \subseteq X$ of cardinality λ such that Y is first category in 2^ω. Thus, if $\lambda > c$, $L(\lambda)$ holds vacuously. Under MA, $\neg L(c)$ (Luzin **[2]**), but $L(\lambda)$ holds for all $\lambda < c$. It is consistent that $L(\lambda)$ holds for all λ (add ω_2 random reals to a model of CH). It is also consistent with c being arbitrarily large that $L(\lambda)$ fails for all uncountable $\lambda \leqq c$. To see this, add a sequence of Cohen reals, $\langle r_\xi : \xi < \kappa \rangle$; then for $\lambda \leqq \kappa$, $\{r_\xi : \xi < \lambda\}$ is in fact a Luzin set—i.e., all first category subsets are countable. It is clear from these remarks that all the results stated in the abstract follow from:

MAIN THEOREM. *If $\omega = cf(\lambda) < \lambda$, then $SA(\lambda)$ iff $L(\lambda)$.*

The rest of the paper is devoted to a proof of this result.

1. Some remarks on the Cantor set. We collect here some facts about 2^ω to be used in §§2–3.

Let $\mathbf{P}$ be the set of functions p such that $\mathrm{dom}(p)$ is a nonempty finite subset of ω and $\mathrm{ran}(p) \subseteq 2 = \{0, 1\}$. Let $N_p = \{f \in 2^\omega : p \subseteq f\}$. The N_p for $p \in \mathbf{P}$ form a base for the usual topology on 2^ω, as do just the N_p with $\mathrm{dom}(p) \in \omega$.

Let $m_p = \min(\mathrm{dom}(p))$. For any fixed $n \in \omega$, the N_p with $m_p \geqq n$ form a base for a (non-T_2) topology on ω, a fact which is of little interest (but see §2).

If $cf(\lambda) = \omega$, both the assertions $L(\lambda)$ and $\neg L(\lambda)$ convey more information than is apparent at first sight. Fix λ with $\omega = cf(\lambda) < \lambda$.

LEMMA 1.1. If $\neg L(\lambda)$, *then there is an* $X \subseteq 2^\omega$ *of cardinality* λ *and a* $\kappa < \lambda$ *such that all first category subsets of* X *have cardinality* $\leqq \kappa$.

Proof. Fix X refuting $L(\lambda)$. Then, since a countable union of first category sets is first category, the cardinalities of such subsets of X must be bounded strictly below λ.

The rest of this section is not needed for §2, but will reappear in §3.

LEMMA 1.2 *If* $L(\lambda)$, $X \subseteq 2^\omega$, $|X| = \lambda$, *and* $\kappa < \lambda$, *then* X *has a* n.w.d. *subset of cardinality* $> \kappa$.

Proof. If all n.w.d. subsets had cardinality $\leqq \kappa$, so would all first category subsets.

LEMMA 1.3. *If* $L(\lambda)$, *then*

$$\forall \kappa < \lambda \, \exists \kappa' [\kappa < \kappa' < \lambda \quad \textit{and} \quad \forall X \subseteq 2^\omega [|X| \geqq \kappa' \Rightarrow \exists Y \subseteq X [Y \text{ n.w.d.} \quad \textit{and} \quad |Y| > \kappa]]].$$

Proof. Fix κ. If there is no such κ', fix $\kappa_n \nearrow \lambda$. Let $X_n \subset 2^\omega$ with $|X_n| \geqq \kappa_n$ and

$$\forall Y \subseteq X_n [Y \text{ n.w.d.} \Rightarrow |Y| \leqq \kappa].$$

Let $X = \bigcup_n X_n$. Then $\forall Y \subseteq X [Y \text{ n.w.d.} \Rightarrow |Y| \leqq \kappa]$, contradicting 1.2.

We conclude with a characterization of n.w.d.

LEMMA 1.4. *Let $X \subseteq 2^\omega$. Then the following are equivalent*:
(i) *X is* n.w.d.
(ii) $\forall n \in \omega \exists p \in \mathbf{P}[N_p \cap X = \emptyset$ *and* $m_p \geqq n]$.

Proof. (i)$\rightarrow$(ii): Let q_i $(i < 2^n)$ enumerate $\{q \in \mathbf{P}$: $\operatorname{dom} q = n\}$. Let $p_0 \in \mathbf{P}$ be arbitrary. Given p_i, let $p_{i+1} \supseteq p_i$ be such that $N_{q_i \cup p_{i+1}} \cap X = \emptyset$. Let $p = p_{2^n}$. Then $\forall q[\operatorname{dom} q = n \Rightarrow N_q \cap N_p \cap X = N_{q \cup p} \cap x = \emptyset]$, so $N_p \cap X = \emptyset$.

(ii)$\rightarrow$(i). For any $q \in \mathbf{P}$, there is a $p \in \mathbf{P}$ such that $\operatorname{dom} p \cap \operatorname{dom} q = \emptyset$ and $N_p \cap X = \emptyset$. Then $\emptyset \neq N_{q \cup p} \subseteq N_q$, and $N_{q \cup p} \cap x = \emptyset$. Thus, X is n.w.d.

2. Proof of $SA(\lambda) \Rightarrow L(\lambda)$. We do this direction first because it is much easier and will indicate why there is any relation at all between the two properties. Fix λ with $\omega = cf(\lambda) < \lambda$ and assume $\neg L(\lambda)$. We shall show $\neg SA(\lambda)$.

Fix an $X \subseteq 2^\omega$ and an ascending ω-sequence $\kappa_n \nearrow \lambda$ such that $|X| = \lambda$ and all first category subsets of X have cardinality $\leqq \kappa_0$. This is possible by 1.1. We may furthermore assume that

$$(*) \qquad \forall f, g \in X\,[f \neq g \Rightarrow \{n : f(n) \neq g(n)\} \quad \text{is infinite}],$$

since for any $f \in 2^\omega$, $\{g \in 2^\omega : f(n) = g(n)$ for all but finitely many $n\}$ is countable, so there is always an $X' \subseteq X$ of cardinality λ satisfying $(*)$.

We shall define a new topology, ρ on X such that (X, ρ) is a counter example to $SA(\lambda)$. Arbitrarily partition X into $X_n (n \in \omega)$ with each $|X_n| = \kappa_n$. Following the notation in the beginning of §1, define, for $f \in X_n$, $p \in \mathbf{P}$, $m_p > n$ and $p \subseteq f$,

$$M_p^f = \{f\} \cup \bigcup_{j<n} (X_j \cap N_p).$$

Let ρ be the topology whose basis is all such M_p^f.

By $(*)$, ρ is Hausdorff. Since each X_n is discrete, $s(X, \rho) = \lambda$. That the spread is not attained follows immediately from

LEMMA 2.1. *If $Y \subseteq X$, $|Y| > \kappa_0$, and $\{n$: $Y \cap X_n \neq \emptyset\}$ is infinite, then Y is not discrete in ρ.*

Proof. Fix j with $|Y \cap X_j| > \kappa_0$. Then Y is not nowhere dense in 2^ω (in the usual topology), so fix $q \in \mathbf{P}$ with $Y \cap X_j$ dense in N_q. Fix n so that $n > j$, $\operatorname{dom} q \subseteq n$, and $y \cap X_n \neq \emptyset$, and fix $f \in Y \cap$

X_n. For any $p \in \mathbf{P}$ with $m_p > n$, $N_p \cap N_q \neq \emptyset$, so $Y \cap X_j \cap N_p \neq \emptyset$. Thus, for any basis neighborhood M_p^f of f, $Y \cap X_j \cap M_p^f \neq \emptyset$, so f is a limit point of $Y \cap X_j$.

Note that ρ is not a refinement of the usual topology on X. We have declared some new sets to be discrete, but we have also thrown away some old basic neighborhoods. In fact, any refinement of a T_3 topology would be strongly T_2, and thus could not be a counterexample to $SA(\lambda)$.

3. Proof of $L(\lambda) \Rightarrow SA(\lambda)$. This is an attempt to stand §2 on its head. Fix λ with $\omega = cf(\lambda) < \lambda$. Now it is clear that $L(\lambda)$ implies that the specific construction in §2 of a counter example to $SA(\lambda)$ won't work. Thus, our strategy will be to take an arbitrary counterexample to $SA(\lambda)$ and show that it has a subspace which looks sufficiently like the space in §2 to imply $\neg L(\lambda)$. The following lemma, due essentially to Hajnal and Juhász is a step in this direction. We use $\varphi(x, X)$ to denote the least cardinality of a neighborhood of x in X. Note that $x \in Y \subseteq X$ implies $\varphi(x, Y) \leqq \varphi(x, X)$.

LEMMA 3.1. *Suppose $\neg SA(\lambda)$. Then there is an X with $s(X) = \lambda$ and no discrete subspace of cardinality λ such that*:

I. *$X = \bigcup_n X_n$, where each X_n is discrete.*

II. *For all regular $\kappa < \lambda$ and all $Y \subseteq X$ of cardinality $\geqq \kappa$, Y has a discrete subspace of cardinality κ.*

III. *For all $x \in X$, $\varphi(x, X) < \lambda$.*

IV. *For all $\kappa < \lambda$, $|\{x \in X : \varphi(x, X) < \kappa\}| < \lambda$.*

We indicate briefly the proof of 3.1 as it is not precisely the result obtained in **[1]**. Let X be any counter example to $SA(\lambda)$. Since $s(X) = \lambda$ and $cf(\lambda) = \omega$, there is an $X^{(1)} \subseteq X$ of size λ which satisfies I. Note that I then also holds for any subspace of $X^{(1)}$, and that I implies II; thus any subspace of $X^{(1)}$ of size λ is also a counter example to $SA(\lambda)$. To get III, let $X^{(2)} = \{x \in X^{(1)} : \varphi(x, X^{(1)}) < \lambda\}$. Then $X^{(2)}$ satisfies III. To see that $|X^{(2)}| = \lambda$, suppose not; let $Z = X^{(1)} \backslash X^{(2)}$; then $\forall x \in Z(\varphi(x, Z) = \lambda)$; since Z is T_2, there is an infinite sequence, $U_n (n \in \omega)$ of nonempty disjoint open sets in Z; then each $|U_n| = \lambda$, so by II, there are discrete $D_n \subseteq U_n$ with $|D_n| \nearrow \lambda$; then $\bigcup_n D_n$ would be discrete and of size λ. Finally, $X^{(2)}$ must also satisfy IV, since otherwise, as in **[1]**, p. 40 the Hajnal free set lemma would imply that $X^{(2)}$ has a discrete subspace of size λ.

Observe that our space in §2 in fact satisfied I–IV. One can, by doing more work, produce a subspace which looks still more like the one in §2. For example, one can get $|X_n| \nearrow \lambda$, each $\bigcup_{k<n} X_k$ open, and $\varphi(x, X) = |X_n|$ for all $x \in X_{n+1}$. But we do not need this here.

We now explain how to link our X with 2^ω. Let $\mathcal{V} = \{V^n_\nu : n < \omega$ and $\nu < 2\}$ be an indexed family of open subsets of X. Suppose $\mathcal{V}$ satisfies condition

$$\text{(A)} \qquad \forall n [V^n_0 \cap V^n_1 = \emptyset].$$

Then, for $x \in X$, let $f_x \in 2^\omega$ be such that $\forall n, \nu [x \in V^n_\nu \rightarrow f_x(n) = \nu]$ (so if $x \notin V^n_0 \cup V^n_1, f_x(n)$ may be 0 or 1). It is to these f_x that we shall apply $L(\lambda)$.

For $p \in \mathbf{P}$, let $V_p = \bigcap \{V^n_{p(n)}: n \in \operatorname{dom} p\}$. The V_p will play the role of the N_p in the Cantor set.

LEMMA 3.2. *Suppose X satisfies* I–IV *of* 3.1 *and there are $p_i \in \mathbf{P}$ and regular $\kappa_i \nearrow \lambda$ $(i < \omega)$ with*

$$\text{(E)} \qquad \forall i [| V_{p_i} \setminus \bigcup_{j \neq i} V_{p_j} | \geqq \kappa_i].$$

Then X has a discrete subspace of cardinality λ.

Proof. By II, fix $D_i \subseteq (V_{p_i} \setminus \bigcup_{j \neq i} V_{p_j})$ with $|D_i| = \kappa_i$ and D_i discrete. Then $\bigcup_i D_i$ is discrete and has size λ.

Unfortunately, the actual production of the p_i is rather painful. We must first find $\mathcal{V}$ so that the V_p themselves are sufficiently large, and then apply $L(\lambda)$ to obtain the p_i. But before we proceed at all, we need a further hypothesis on X, given by

LEMMA 3.3. *Suppose $\neg SA(\lambda)$. Then there is an X as in* 3.1 *which satisfies, in addition,*

(V) *For some $\theta < \lambda$,*

$$\forall \text{ regular } \kappa \geqq \theta \, \forall x \in X [\varphi(x, X) \geqq \kappa \Rightarrow \forall U (x \in U \text{ and } U \text{ open} \Rightarrow |\{y : \varphi(y, U \cup \{y\}) < \kappa\}| < \lambda)].$$

Proof. Fix X as in 3.1. Note that all subspaces of X of size λ also satisfy 3.1. We assume that no such subspace satisfies V and produce a discrete subspace of X of size λ, yielding a contradiction.

Fix $\theta_n \nearrow \lambda$. Inductively pick $x_n \in X$, neighborhoods U_n of x_n in X, $Y_n \subseteq X$, and regular $\kappa_n < \lambda$ so that

(a) $Y_0 = X$.

(b) $Y_{n+1} = \{y \in Y_n \setminus U_n : \varphi(y, (U_n \cap Y_n) \cup \{y\}) < \kappa_n\}$.

(c) $x_n \in Y_n, \kappa_n \geqq \theta_n, \varphi(x_n, Y_n) \geqq \kappa_n$, and $|Y_{n+1}| = \lambda$.

(d) $|U_n| < \lambda$ and $\forall k < n [| U_n \cap U_k \cap Y_k | < \kappa_k]$.

At stage n in this induction, x_n, κ_n, and U_n are chosen first; they may be taken to make (c) hold by our assumption on X, and U_n may be chosen small enough to make (d) hold since X satisfies III and $\forall k < n\,[x_n \in Y_{k+1}]$. Now, let $Z_k = U_k \cap Y_k \backslash (\bigcup_{n>k} U_n)$. Then $|Z_k| \geqq \kappa_k$ (by (d) and the fact that $\varphi(x_k, Y_k) \geqq \kappa_k$); and $\forall k \neq n\,[Z_k \cap U_k = \emptyset]$ (by (b)). Thus, if we let (by II) D_k be a discrete subset of Z_k of size κ_k, $\bigcup_k D_k$ will be discrete and of size λ.

LEMMA 3.4. *Assume* $L(\lambda)$, *and suppose that* X *satisfies* I–V. *Then there is a sequence of regular cardinals* $\delta_n \nearrow \lambda$ *and a* $\mathcal{V}$ *satisfying condition A plus*

(B) $\forall n, \nu(|V_\nu^n| < \delta_{n+1})$.

(C) $\forall p \in \mathbf{P}(|V_p| \geqq \delta_{m_p})$.

(D) $\forall n < \omega \quad \forall Z \subseteq 2^\omega (|Z| \geqq \delta_{n+1} \Rightarrow \exists W \subseteq Z\,(W$ *is* n.w.d. *and* $|W| \geqq \delta_n))$.

Proof. It is in D that $L(\lambda)$ first makes its appearance. By Lemma 1.3, fix regular $\delta'_n \nearrow \lambda$ satisfying D. Then any subsequence of the δ'_n also satisfies D, so our δ_n will be δ'_{k_n} for some $k_n \in \omega$. Fix θ as in V. We define by induction on n: $k_n \in \omega$, points $x_0^n, x_1^n \in X$, sets $r_n \subset X$, and V_0^n, V_1^n so that:

(1) $\delta'_{k_0} \geqq \theta$, $R_0 = \emptyset$.

(2) $x_0^n \neq x_1^n$, $x_\nu^n \notin R_n$, and $\varphi(x_\nu^n, X) \geqq \delta'_{k_n}$, $(\nu = 0, 1)$.

(3) $V_0^n \cap V_1^n = \emptyset$, $x_\nu^n \in V_\nu^n$, V_ν^n is open, and $|V_\nu^n| = \varphi(x_\nu^n, X)$ $(\nu = 0, 1)$.

(4) $R_{n+1} = R_n \cup \{y \in X : \exists \nu < 2[\varphi(y, V_\nu^n \cup \{y\}) < \delta'_{k_n}]\}$.

(5) $k_{n+1} > k_n$ and $\delta_{k_{n+1}} > \sup(|R_{n+1}|, |V_0^n|, |V_1^n|)$.

By V, each $|R_n| < \lambda$, and by III, each $|V_\nu^n| < \lambda$, so that k_{n+1} may always be chosen to satisfy (5). That the x_ν^n may be chosen to satisfy (2) follows from II.

Now, setting $\delta_n = \delta'_{k_n}$, it is immediate that A, B, and D hold. We prove C by induction on $|\mathrm{dom}\, p|$. For $|\mathrm{dom}\, p| = 1$, C is obvious from the construction. For the induction step, let $q = p \upharpoonright (\mathrm{dom}\, p \backslash \{m_p\})$, and assume C for q. Let $n = m_p$ and $\nu = p(n)$. Then $|V_q| \geqq \delta_{m_q} \geqq \delta_{n+1} > |R_{n+1}|$, so $V_q \backslash R_{n+1} \neq \emptyset$. Fix $y \in V_q \backslash R_{n+1}$. Then $\varphi(y, V_\nu^n \cup \{y\}) \geqq \delta_n$, so $|V_q \cap V_\nu^n| \geqq \delta_n$; but $V_q \cap V_\nu^n = V_p$, so C holds for p.

Finally, the fact that $L(\lambda) \Rightarrow SA(\lambda)$ follows by 3.2 plus

LEMMA 3.5. *Assume* $L(\lambda)$. *Let* X *satisfy* I–V, *and let* $\mathcal{V}$ *and* $\delta_n \nearrow \lambda$ *be as in Lemma* 3.4, *satisfying* A–D. *Then there are* $p_i \in \mathbf{P}$ *and regular* $\kappa_i \nearrow \lambda$ $(i < \omega)$ *satisfying* E.

Proof. For $Y \subseteq X$, let $Y^* = \{f_x : x \in Y\} \subseteq 2^\omega$. Define, by induction on $i \in \omega$, $p_i \in \mathbf{P}$ and $Y_i \subseteq X$ so that, setting $\kappa_i = \delta_{m_{p_{i-1}}}$, the following hold:

(i) $p_0 = \{\langle 1,0\rangle\}$, $Y_0 \subseteq V_p = V_0^1$, $|Y_0| \geqq \delta_0 = \kappa_0$, and Y_0^* is n.w.d. in 2^ω.

(ii) $m_{p_{i+1}} > \sup(\operatorname{dom} p_i)$, and $\bigcup_{j\leqq i} Y_j^* \cap N_{p_{i+1}} = \emptyset$.

(iii) $Y_{i+1} \subseteq V_{p_{i+1}} \setminus \bigcup_{j\leqq i} V_{p_j}$, $|Y_{i+1}| \geqq \kappa_{i+1}$, and Y_{i+1}^* is n.w.d. in 2^ω.

To see that Y_0 may be chosen to satisfy (i), use D plus the fact that $|V_0^1| \geqq \delta_1$; the f_x may not all be distinct, but if $|V_0^{1*}| \geqq \delta_1$, we may apply D, and if $|V_0^{1*}| < \delta_1$, we may, since δ_1 is regular, choose Y_0 with $|Y_0^*| = 1$. Similarly, Y_{i+1} may be chosen to satisfy (iii), since $|V_{p_{i+1}} \setminus \bigcup_{j\leqq i} V_{p_i}| \geqq \delta_{m_{p_{i+1}}}$ by B and C. The fact that we may find p_{i+1} to satisfy (ii) follows from Lemma 1.4.

To see that E holds, (iii) implies that each $Y_i \subseteq V_{p_i} \setminus \bigcup_{j<i} V_{p_j}$, whereas (ii) implies that for $j > i$, $Y_i^* \cap N_{p_i} = \emptyset$, whence $Y_i \cap V_{p_j} = \emptyset$, so also $Y_i \subseteq V_{p_i} \setminus \bigcup_{j>i} V_{p_j}$. Since $|Y_i| \geqq \kappa_i$, $|V_{p_i} \setminus \bigcup_{j\neq i} V_{p_j}| \geqq \kappa_i$.

Note that for $\lambda > c$, when $L(\lambda)$ holds trivially, the above argument may be considerably simplified. Each Y_i^* may be taken to be a singleton, so all reference to category in 2^ω may be omitted.

4. Remarks on height and width. Similar analyses may be done for the height (h) and the width (z) of a space (see **[1]**). Again, assume $\omega = cf(\lambda) < \lambda$, and let $HA(\lambda)$ and $ZA(\lambda)$ be the obvious things. Then the argument of §3 also establishes $L(\lambda) \Rightarrow HA(\lambda)$ and $L(\lambda) \Rightarrow ZA(\lambda)$. However, $HA(\lambda)$ is always true, essentially because all one needs from E is that $|V_{p_i} \setminus \bigcup_{j<i} V_{p_j}| \geqq \kappa_i$, which is very easy to obtain; in fact, $HA(\lambda)$ may be established directly without most of the machinery of §3.

The difficulty in getting E was to ensure that $|V_{p_i} \setminus \bigcup_{j>i} V_{p_j}| \geqq \kappa_i$; i.e., that the later, larger V_{p_j} do not engulf V_{p_i}. This is what is needed for $ZA(\lambda)$, and in fact $ZA(\lambda) \Leftrightarrow L(\lambda)$, since under $\neg L(\lambda)$, the space X in §2 has $s(X) = z(X) = \lambda$ but no left-separated sequence of type λ.

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