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ON THE MEASURABILITY OF CONDITIONAL EXPECTATIONS

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It is shown that for a measurable stochastic process V and a nondecreasing family of σ-algebras $\mathscr{A}_t$ there exists a measurable stochastic process V^* such that $V^*(t,\cdot)$ is a version of $E(V(t,\cdot)|\mathscr{A}_t)$ for all t.

Let $(\Omega, \mathscr{A}, P)$ be a probability space (not necessarily complete), T an interval (bounded or unbounded) of the real line and V a real-valued stochastic process defined on $T \times \Omega$ which is a measurable process, see Doob [**3**, p. 60]. Let $\mathscr{A}_t$, $t \in T$, $\mathscr{A}_t \subset \mathscr{A}$ form a nondecreasing family of σ-algebras. We shall prove in this note that under some boundedness condition on V the conditional expectations with respect to P, $E(V(t,\cdot)|\mathscr{A}_t)$ can be chosen as to define a measurable process on $T \times \Omega$. A similar statement appears in a paper by Brooks [**1**] but there it is additionally assumed that the family of σ-algebras is left-continuous, and the proof given there does not seem to carry over to a general nondecreasing family.

THEOREM. *Suppose for each* $t \in T$: $V(t,\cdot) \geqq 0$ *P-a.s. or* $\int |V(t,\cdot)|\,dP < \infty$. *Then there exists a measurable process* V^* *such that for each* $t \in T$, $V^*(t,\cdot)$ *is a version of* $E(V(t,\cdot)|\mathscr{A}_t)$.

Proof. Since for any $t \in T$

$$E(V(t,\cdot)|\mathscr{A}_t) = E(V(t,\cdot)^+|\mathscr{A}_t) - E(V(t,\cdot)^-|\mathscr{A}_t)$$

we may assume without loss of generality that for each $t \in T$ $V(t,\cdot) \geqq 0$ *P-a.s.* Using the linearity and monotone convergence property of conditional expectations the theorem now is easily reduced to the case that V is the characteristic function I_D of some subset $D = B \times A$ of $T \times \Omega$ with $A \in \mathscr{A}$ and B belonging to the Borel sets of T.

Since $E(I_D(t,\cdot)|\mathscr{A}_t) = I_B(t)E(I_A|\mathscr{A}_t)$ holds it is enough to show that $E(I_A|\mathscr{A}_t)$ can be chosen to form a measurable process. Let $\mathscr{M}$ denote the set of all random variables on $(\Omega, \mathscr{A}, P)$ taking values in $[0,1]$ with random variables that are equal *P-a.e.* identified. Then $\mathscr{M}$ is a metrizable topological space under the topology of convergence in probability. By Theorem 3 in Cohn **[2]** it is now sufficient to show that the mapping $E_A: T \to \mathcal{M}$ with $E_A(t) = E(I_A \mid \mathcal{A}_t)$ has separable range and is measurable with respect to the Borel sets of $\mathcal{M}$. $E(I_A \mid \mathcal{A}_t)$, $t \in T$, forms a uniformly integrable martingale and so it follows from Theorem 11.2 in Doob **[3]**, p. 358, that E_A is continuous at all but countably many points of T. This yields at once that E_A is measurable and furthermore—since T is separable—that the range of E_A is separable. This concludes the proof.

If the condition '$V(t,\cdot) \geqq 0$ P-*a.s.* or $\int |V(t,\cdot)|dP < \infty$' is only required to hold for μ-*a.a.* $t \in T$, μ being any measure on the Borel sets of T, then obviously there exists a measurable process V^* which is a version of $E(V(t,\cdot) \mid \mathcal{A}_t)$ for μ-*a.a.* $t \in T$.

References

1. R. A. Brooks, *Conditional expectations associated with stochastic processes*, Pacific J. Math., **41** (1972), 33–42.

2. D. L. Cohn, *Measurable choice of limit points and the existence of separable and measurable processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete., **22** (1972), 161–165.

3. J. L. Doob, *Stochastic Processes*, New York, Wiley, 1953.

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