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REMARKS ON SINGULAR ELLIPTIC THEORY FOR COMPLETE RIEMANNIAN MANIFOLDS

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This paper is about a C^*-algebra $\mathfrak{A}$ of 0-order pseudo-differential operators on $L^2(\Omega)$, where Ω is a complete Riemannian manifold which need *not* be compact. This algebra is designed to handle singular elliptic theory for certain Nth-order differential operators. In particular, this paper studies the maximal ideal space M of the (commutative) algebra $\mathfrak{A}/\mathcal{K}$, where $\mathcal{K}$ denotes the compact ideal. The space M contains the bundle of cospheres as a subspace, and in general will contain additional points at infinity of the manifold. The significance of this for elliptic theory lies in the fact that an operator $A \in \mathfrak{A}$ is Fredholm if and only if the associated continuous function $\sigma_A \in C(M)$ is never zero.

1. Introduction. Let Ω be an n-dimensional paracompact C^∞-manifold with *complete* Riemannian metric $ds^2 = g_{ij}dx^i dx^j$ and surface measure $d\mu = \sqrt{g}\, dx$ where $g = \det(g_{ij})$. As in **[5]** we define $\Lambda = (1-\Delta)^{-1/2}$ as a positive-definite operator in $\mathcal{L}(\mathfrak{h})$, the bounded operators over the Hilbert space $\mathfrak{h} = L^2(\Omega, d\mu)$, and define the Sobolev spaces $\mathfrak{h}_N \subset \mathfrak{h}$ for $N = 0, 1, \cdots$ by requiring $\Lambda^N : \mathfrak{h} \to \mathfrak{h}_N$ to be an isometric isomorphism. It was shown in **[3]** that $C_0^\infty(\Omega)$ is then dense in each $\mathfrak{h}_N$.

In **[5]** we defined classes of bounded functions and vector fields, **A** and **D**, whose successive covariant derivatives with respect to a symmetric affine connection ∇ *vanish at infinity* in the special sense that for $f \in C(\Omega)$ we write $\lim_{x\to\infty} f = 0$ if for every $\epsilon > 0$ there exists a compact set $K \subset \Omega$ such that

$$|f(x)| < \epsilon \quad \text{for} \quad x \in \Omega \setminus K. \tag{1.1}$$

Let $\mathbf{L}^N$ denote the class of Nth-order differential operators generated by taking sums of products of elements in **D** and **A**. The connection ∇ need not be the Riemannian connection ∇g, but must satisfy *Condition* (r_0) of **[5]** that it does not differ drastically from ∇g at infinity. We also require *Condition* $(\mathbf{L}^2)$ that $1\text{-}\Delta \in \mathbf{L}^2$, a condition which was seen in **[5]** to imply the curvature tensor R tends to zero as $x \to \infty$ in the sense of (1.1). Under these two conditions it was shown that the operators $L\Lambda^N$ and $\Lambda^N L$ for $L \in \mathbf{L}^N$ are bounded over $\mathfrak{h}$ and thus generate an algebra $\mathfrak{A}^0 \subset \mathcal{L}(\mathfrak{h})$. Moreover it was found that after adding the compact ideal $\mathcal{K}$ to

$\mathfrak{A}^0$ and taking the norm closure, we obtain a C^*-algebra $\mathfrak{A}$ with compact commutators.

In this paper we focus our attention on the maximal ideal space $\mathbf{M}$ of the commutative C^*-algebra $\mathfrak{A}/\mathcal{K}$. If we define the *symbol* σ_A to be the continuous function on $\mathbf{M}$ associated with the coset $A+\mathcal{K}$, then a necessary and sufficient condition for A to be Fredholm is that σ_A never vanish on $\mathbf{M}$ (c.f. **[1]**). Thus a further analysis of $\mathbf{M}$ and the symbols σ_A is desirable for the Fredholm theory of differential operators in $\mathbf{L}^N$. For compact manifolds Ω it was shown in **[8]** that $\mathbf{M}$ is just the bundle of unit co-spheres $S^*\Omega \subset T^*\Omega$. For the special noncompact manifold $\Omega = \mathbf{R}^n$ it was shown in **[4]** that $\mathbf{M}$ contains $S^*\Omega = \mathbf{R}^n \times S^{n-1}$ as a proper subset: in fact $\mathbf{M} = \partial P^*\Omega = P^*\Omega \setminus T^*\Omega$ where $P^*\Omega$ is a certain compactification of $T^*\Omega$. In both **[4]** and **[8]** explicit formulas for σ_A were obtained. For general Ω, the main result of this paper (c.f. Theorem 2.2) asserts the inclusions $S^*\Omega \subset \mathbf{M} \subset \partial P^*\Omega$. Although we do not achieve a complete description of $\mathbf{M}$ and σ_A, this theorem yields many results (e.g. criteria for "weak = strong" and characterizations of Fredholm essential spectra) of classical elliptic theory (c.f. **[2]**). For example, if $L \in \mathbf{L}^N$ is uniformly elliptic (see §2) and formally self-adjoint, then L is essentially self-adjoint (with domain $C_0^\infty(\Omega)$). A discussion of this and further applications of the result of this paper is planned for a subsequent publication.

2. The formal algebra symbol. Let $\mathfrak{A}_M$ denote the function algebra obtained by closing $\mathbf{A}$ under uniform norm. Since $\mathfrak{A}_M$ is a subalgebra of the bounded continuous functions on Ω, the Gelfand isomorphism yields $\mathfrak{A}_M \cong C(\bar{\Omega})$ where $\bar{\Omega}$ is some compactification of Ω. On the other hand, considering $\mathfrak{A}_M \subset \mathfrak{A}$ we obtain a canonical injection $i: \mathfrak{A}_M \to \mathfrak{A}/\mathcal{K}$ whose associated dual map $p = i^*$ provides a continuous surjection $p: \mathbf{M} \to \bar{\Omega}$. Let us denote the open subset $p^{-1}(\Omega) \subset \mathbf{M}$ by $\mathbf{S}$. The following theorem, which is an immediate consequence of Theorem 2.2 although we state it first for purposes of exposition, extends the corresponding well-known result for compact Ω.

THEOREM 2.1. *Let $\pi: S^*\Omega \to \Omega$ denote the fibre bundle of unit cospheres $S^*\Omega \subset T^*\Omega$. There is a (surjective) homeomorphism $\theta: \mathbf{S} \to S^*\Omega$ such that $\pi \circ \theta = p$ on $\mathbf{S}$ and for $m \in \mathbf{S}$ and $\theta(m) = (x,\xi) \in S^*\Omega$ we have*

$$\sigma_a(m) = a(x) \qquad \sigma_{D\Lambda}(m) = b^j(x)\xi_j \tag{2.1}$$
$$\sigma_\Lambda(m) = 0 \qquad \sigma_K(m) = 0$$

where $K \in \mathcal{K}$, $a \in \mathbf{A}$, and $D \in \mathbf{D}$ is given in local coordinates by $-ib^j(\partial/\partial x^j)$ and $\xi = \xi_j dx^j$.

For $a \in \mathbf{A}$ and $D \in \mathbf{D}$ with local expression $-ib^j(\partial/\partial x^j)$, the following *formal symbols* define continuous functions on $T^*\Omega$.

$$\tilde{\sigma}_a(x, \xi) = a(x) \qquad \tilde{\sigma}_D(x, \xi) = b^j(x)\xi_j \tag{2.2}$$
$$\tilde{\sigma}_\Lambda(x, \xi) = (1 + |\xi|^2)^{-1/2}$$

and we may extend algebraically to sums and products. In particular, the formal symbols $\tilde{\sigma}_a$, $\tilde{\sigma}_{D\Lambda}$, and $\tilde{\sigma}_\Lambda$ for $a \in \mathbf{A}$ and $D \in \mathbf{D}$ generate a C^*-algebra, $\mathfrak{A}_\sigma$, of continuous bounded functions on $T^*\Omega$. The maximal ideal space of $\mathfrak{A}_\sigma$ is a compactification, $P^*\Omega$, of $T^*\Omega$, and we define the boundary $\partial P^*\Omega = P^*\Omega \setminus T^*\Omega$. The associated dual map to the injection $\mathfrak{A}_M \to \mathfrak{A}_\sigma$ provides a surjection of $P^*\Omega$ onto $\bar{\Omega}$, and the restriction of this map to the boundary is denoted by $\pi: \partial P^*\Omega \to \bar{\Omega}$. Using (2.1) of **[5]**, the formal symbols of $L\Lambda^N$ and $\Lambda^N L$ for $L \in \mathbf{L}^N$ defined by algebraic extension of (2.2) are unique when restricted to $\partial P^*\Omega$. Thus we are lead to defining the *formal algebra symbol* as the algebra homomorphism

$$\dot{\sigma}: \mathfrak{A}^0 \to C(\partial P^*\Omega) \tag{2.3}$$

obtained by this restriction of $\tilde{\sigma}$.

It is evident that $S^*\Omega$ is homeomorphic to $\pi^{-1}\Omega$ by the map $(x, \xi) \to \lim_{r\to\infty}(x, r\xi) \in \partial P^*\Omega$. Theorem 2.1 may be interpreted as providing a continuous injection $\theta: \mathbf{S} \to \partial P^*\Omega$ such that

$$\dot{\sigma}_A(\theta(m)) = \sigma_A(m) \tag{2.4}$$

for $m \in \mathbf{S}$ and operators $A = a$ or $A = D\Lambda$. The main result of this paper extends this formula as follows.

THEOREM 2.2. *Under Conditions* (r_0) *and* $(\mathbf{L}^2)$, *there exists a continuous injection* $\theta: \mathbf{M} \to \partial P^*\Omega$ *such that*

$$\begin{array}{ccc} \mathbf{M} & \xrightarrow{\ \theta\ } & \partial P^*\Omega \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & \bar{\Omega} & \end{array} \tag{2.5}$$

is commutative, surjective on fibres over Ω, *and* (2.4) *holds for all* $m \in \mathbf{M}$ *and* $A \in \mathfrak{A}^0$.

If $L \in \mathbf{L}^N$ with $\dot{\sigma}_{L\Lambda^N}$ bounded away from zero on $S^*\Omega = \theta(\mathbf{S})$, we say L is *uniformly elliptic*.

3. Proof of Theorem 2.2. Condition ($\mathbf{L}^2$) implies that we may write

$$1 - \Delta = \sum_{\nu=1}^{M} C_\nu D_\nu + \text{lower order terms} \tag{3.1}$$

with $2M$ vector fields $C_\nu, D_\nu \in \mathbf{D}$. Taking real and imaginary parts in (3.1), we may assume C_ν and D_ν are real. Let $B_\nu \in \mathbf{D}$, $\nu = 1, \cdots, N$, be a basis for the module spanned by $C_1, \cdots, C_M, D_1, \cdots, D_M$ over the algebra of real-valued functions in $\mathbf{A}$. In local coordinates, let G denote the $n \times n$ matrix $((g^{ij}))$ and B denote the $n \times N$ matrix $((b^i_\nu))$ where b^i_ν are the components of B_ν. Let B^T be the matrix transpose of B. Considering principal parts in (3.1), there is a symmetric $N \times N$ matrix-valued function $A = ((a_{\nu\mu}))$ whose coefficients $a_{\nu\mu}$ are all real-valued functions of $\mathbf{A}$, such that

$$G = BAB^T. \tag{3.2}$$

Let us introduce the $N \times N$ matrix-valued function $P = B^T G^{-1} BA$. Observe that P does not depend on local coordinates and $P^2 = P$ implies that P is a projection matrix with rank n. Let $\Gamma = \binom{N}{n}$, the binomial coefficient. We shall require the following lemma from linear algebra.

LEMMA 3.1. *For any $N \times N$ projection matrix P with rank n, there exists an $n \times n$ diagonal matrix minor $\tilde{P}$ such that $|\det \tilde{P}| \geqq \Gamma^{-1}$.*

Proof.. Since $\det(P - \lambda) = (1-\lambda)^n(-\lambda)^{N-n}$, the coefficient of λ^{N-n} is ± 1. But this coefficient equals the sum of all $n \times n$ diagonal minors. Since there are precisely Γ such minors, at least one must have absolute value not less than Γ^{-1}.

Applying the lemma, we see that at each point $x \in \Omega$ there is a matrix minor $\tilde{P}$ of P, $\tilde{P} = B^T_{(\gamma)} G^{-1} B\tilde{A}$ where $B_{(\gamma)}$ denotes one of the Γ distinct $n \times n$ matrix minors of B and $\tilde{A}$ denotes a certain $N \times n$ matrix minor of A, such that

$$|\det \tilde{P}| > (2\Gamma)^{-1}. \tag{3.3}$$

The matrix $\tilde{A}^T B^T G^{-1} B\tilde{A}$ has coefficients in $\mathbf{A}$ so that $|\det \tilde{A}^T B^T G^{-1} B \tilde{A}|$ is uniformly bounded over Ω. Thus $|\det G^{-1/2} B\tilde{A}| = |\det \tilde{A}^T B^T G^{-1} B\tilde{A}|^{1/2}$ is also uniformly bounded. So by (3.3), there exists a constant $C > 0$ such that at each $x \in \Omega$, $|\det B^T_{(\gamma)} G^{-1/2}| > C$ holds for at least one $\gamma = 1, \cdots, \Gamma$. Observe that $d_\gamma = \det B^T_{(\gamma)} G^{-1/2}$ is a C^∞-

function on Ω and we have a finite open cover of Ω by the Γ sets

$$\Omega_\gamma = \{x \in \Omega : |d_\gamma(x)| > C\}. \tag{3.4}$$

Let us suppose that we have chosen C such that also the sets

$$\Omega''_\gamma = \{x \in \Omega : |d_\gamma(x)| > 2C\}. \tag{3.5}$$

cover Ω. Also define

$$\Omega'_\gamma = \{x \in \Omega : |d_\gamma(x)| > \tfrac{4}{3}C\}. \tag{3.5$'$}$$

Observe that $\Omega''_\gamma \subset \Omega'_\gamma \subset \Omega_\gamma$. Let $\overline{\Omega'_\gamma}$ and $\overline{\Omega''_\gamma}$ denote the closures of Ω'_γ and Ω''_γ respectively in $\bar{\Omega}$.

In each set Ω_γ, $\det B_{(\gamma)} > 0$ so we may define the $n \times N$ matrix-valued function $Q_\gamma = B_{(\gamma)}^{-1} B$. Let us also define an $n \times n$ matrix-valued function on Ω_γ

$$A_\gamma = Q_\gamma A Q_\gamma^T = (B_{(\gamma)}^{-1} G^{1/2})(G^{1/2} B_{(\gamma)}^{-1T}). \tag{3.6}$$

Clearly A_γ is coordinate invariant and positive definite with spectrum bounded uniformly (over Ω_γ) below by $\epsilon > 0$. Since $|\det A_\gamma| < C^{-2}$ on Ω_γ, we conclude that the spectrum of A_γ is contained in a fixed (independent of $x \in \Omega_\gamma$) compact subset of $(0, \infty)$. Thus we may define $A_\gamma^{1/2}$ by a resolvent integral. A computation shows that the coefficients of $A_\gamma^{1/2}$ are bounded over Ω_γ and have covariant derivatives tending to zero in Ω_γ outside large compact sets of Ω. Thus if we define $\tilde{B}_{(\gamma)} = B_{(\gamma)} A^{1/2}$ we have $G = \tilde{B}_{(\gamma)} \tilde{B}_{(\gamma)}^T$ in Ω_γ. In other words we have diagonalized the metric in Ω_γ as follows.

PROPOSITION 3.2. *Under Condition* ($\mathbf{L}^2$), *there is a finite open cover of* Ω *by open sets* $\{\Omega_\gamma\}_{\gamma=1}^{\Gamma}$ *such that in each set* Ω_γ *we may express*

$$g^{ij} = \sum_{\nu=1}^{n} \tilde{b}^i_\nu \tilde{b}^j_\nu \tag{3.7}$$

where the n real vector fields $\tilde{B}_\nu$ *with components* $\tilde{b}^i_\nu$ *are bounded over* Ω_γ *and satisfy: for every* $n \geqq 1$ *and* $\epsilon > 0$ *there exists a compact set* $K_\epsilon \subset \Omega$ *such that*

$$|\nabla^n \tilde{B}_\nu| < \epsilon \quad \textit{for all} \quad x \in \Omega_\gamma \backslash K_\epsilon. \tag{3.8}$$

Now let $\chi \in C^\infty(\mathbf{R})$ with $\chi(t) = 0$ for $t \leqq 0$, $\chi(t) = 1$ for $t \geqq C$ and $0 \leqq \chi \leqq 1$ for $0 < t$. Define $\varphi_\gamma(x) = \chi(3|d_\gamma(x)| - 4C)$, $\psi_\gamma(x) =$

$\chi(3|d_\gamma(x)|-3C)$, and $\mu_\gamma(x)=\chi(3|d_\gamma(x)|-2C)$. Dividing each φ_γ by $\Sigma_{\gamma=1}^{\Gamma}\varphi_\gamma$, we may assume $\Sigma_{\gamma=1}^{\Gamma}\varphi_\gamma\equiv 1$ on Ω. Observe $\varphi_\gamma, \psi_\gamma, \mu_\gamma \in \mathbf{A}$, $\varphi_\gamma\equiv 1$ on Ω_γ'' and $\mu_\gamma\equiv 0$ on $\Omega\backslash\Omega_\gamma$. In fact $\psi_\gamma\equiv 1$ on $\overline{\Omega_\gamma'}=\operatorname{supp}\varphi_\gamma$ and $\mu_\gamma\equiv 1$ on $\operatorname{supp}\psi_\gamma$, so $\varphi_\gamma=\varphi_\gamma\psi_\gamma$ and $\psi_\gamma=\psi_\gamma\mu_\gamma$. Observe that $D_{\gamma,\nu}=-i\mu_\gamma\tilde{B}_\nu\in\mathbf{D}$ (in particular, a vector field defined on all of Ω).

LEMMA 3.3. *Vector fields of the form $\varphi_\gamma D$ and $\psi_\gamma D$ with $D\in\mathbf{D}$ may be written as $\varphi_\gamma D=\Sigma_{\nu=1}^n a_{\gamma,\nu}\varphi_\gamma D_{\gamma,\nu}$ and $\psi_\gamma D=\Sigma_{\nu=1}^n a_{\gamma,\nu}\psi_\gamma D_{\gamma,\nu}$* with $a_{\gamma,\nu}\in\mathbf{A}$.

Proof. If D is given in local coordinates by $b^j\partial/\partial x^j$, simply define $a_{\gamma,\nu}=ib^jg_{jk}\tilde{b}^k_\nu\mu_\gamma$.

We now invoke some of the results of **[5]**. Condition (r_0) implies that the formal adjoint of $\psi_\gamma D_{\gamma,\nu}$ is of the form $(\psi_\gamma D_{\gamma,\nu})'=\psi_\gamma D_{\gamma,\nu}+a$ with $\lim_{x\to\infty}a=0$ (c.f. (2.2) of **[5]**). Thus if we let $T_{\gamma,\nu}=\psi_\gamma D_{\gamma,\nu}\Lambda\in\mathfrak{A}^0$, we have by Remark 2.3, Theorem 3.1, and Proposition 4.4 of **[5]** that

$$T^*_{\gamma,\nu}=\Lambda(\psi_\gamma D_{\gamma,\nu})'\equiv\Lambda\psi_\gamma D_{\gamma,\nu}\equiv\psi_\gamma D_{\gamma,\nu}\Lambda=T_{\gamma,\nu}\ (\operatorname{mod}\mathcal{K}). \tag{3.9}$$

Also observe

$$\sum_{\nu=1}^n(\psi_\gamma D_{\gamma,\nu})'(\psi_\gamma D_{\gamma,\nu})=-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}g^{ij}\psi_\gamma^2\sqrt{g}\frac{\partial}{\partial x^j}=-\psi_\gamma^2\Delta-D \tag{3.10}$$

with $\lim_{x\to\infty}|D|=0$. Thus using Corollary 3.6 of **[5]** together with (3.9) and (3.10) above

$$\begin{aligned}\psi_\gamma^2&=\psi_\gamma^2\Lambda(1-\Delta)\Lambda\equiv(\psi_\gamma\Lambda)^2-\Lambda\psi_\gamma^2\Delta\Lambda \quad (\operatorname{mod}\mathcal{K})\\ &\equiv(\psi_\gamma\Lambda)^2-\sum_{\nu=1}^n\Lambda(\psi_\gamma D_{\gamma,\nu})'(\psi_\gamma D_{\gamma,\nu})\Lambda \quad (\operatorname{mod}\mathcal{K})\\ &\equiv\sum_{\nu=0}^n T^2_{\gamma,\nu} \quad (\operatorname{mod}\mathcal{K})\end{aligned} \tag{3.11}$$

where we have also defined $T_{\gamma,0}=\psi_\gamma\Lambda$. Similarly, let us define $S_{\gamma,\nu}=\varphi_\gamma T_{\gamma,\nu}$ for all γ and ν.

Let $\mathbf{M}_\gamma'=p^{-1}(\overline{\Omega_\gamma'})\subset\mathbf{M}$. Also let S^n be the half-sphere

$\left\{\sigma=(\sigma_0,\sigma_1,\cdots,\sigma_n)\in\mathbf{R}^{n+1}\colon \sum_{\nu=0}^n\sigma_\nu^2=1 \quad\text{and}\quad \sigma_0\geqq 0\right\}$, and $S^{n-1}=\partial S^n_+$.

PROPOSITION 3.4. *For each $\gamma=1,\cdots,\Gamma$ there is a continuous injection*

$$\mathbf{M}_\gamma'\to\overline{\Omega_\gamma'}\times S^n_+ \tag{3.12}$$

where $m \mapsto (x, \sigma)$ *such that* $\sigma_a(m) = a(x)$ *and* $\sigma_{S_{\gamma,\nu}}(m) = \varphi_\gamma(x)\sigma_\nu$ *for* $\nu = 0, 1, \cdots. n.$ *In addition,* (3.12) *maps* $\mathbf{M}'_\gamma \cap \mathbf{S}$ *onto* $\Omega'_\gamma \times S^{n-1}$.

Proof. Let $\mathfrak{A}^\#_\gamma$ denote the smallest C^*-algebra with unit containing $\mathcal{K}$ and $T_{\gamma,\nu}$ for $\nu = 0, 1, \cdots, n$. Let $\mathfrak{A}_\gamma$ denote the smallest C^*-algebra containing $\mathfrak{A}_M$ and $\mathfrak{A}^\#_\gamma$. Since $\mathfrak{A}_\gamma/\mathcal{K}$ is a commutative C^*-algebra, let $\mathbf{N}_\gamma$ denote its maximal ideal space, and let $\sigma^\gamma : \mathfrak{A}_\gamma \to C(\mathbf{N}_\gamma)$ be the symbol homomorphism. Also let $p_\gamma : \mathbf{N}_\gamma \to \bar{\Omega}$ be the associated dual map to the inclusion $\mathfrak{A}_M \to \mathfrak{A}_\gamma$. For $a \in \mathbf{A}$ and $D \in \mathbf{D}$, define $\rho_\gamma(a) = a$, $\rho_\gamma(D\Lambda) = \psi_\gamma D\Lambda$, and $\rho_\gamma(\Lambda) = \psi_\gamma\Lambda$. By Lemma 3.3, ρ_γ extends to a continuous algebra homomorphism of $\mathfrak{A}$ *onto* $\mathfrak{A}_\gamma$. Since $\rho_\gamma(\mathcal{K}) \subset \mathcal{K}$, there is an induced surjective homomorphism $\bar{\rho}_\gamma : \mathfrak{A}/\mathcal{K} \to \mathfrak{A}_\gamma/\mathcal{K}$. Thus the associated dual map $i_\gamma = \bar{\rho}^*_\gamma$ provides a continuous injection such that

$$\begin{array}{ccc} \mathbf{N}_\gamma & \xrightarrow{\;i_\gamma\;} & \mathbf{M} \\ & {\scriptstyle p_\gamma}\searrow \quad \swarrow{\scriptstyle p} & \\ & \bar{\Omega} & \end{array} \tag{3.13}$$

commutes and

$$\sigma^\gamma_{\rho_\gamma(A)}(n) = \sigma_A(i_\gamma n) \tag{3.14}$$

for all $A \in \mathfrak{A}$ and $n \in \mathbf{N}_\gamma$. The restriction of i_γ to $\mathbf{N}'_\gamma = p_\gamma^{-1}(\bar{\Omega}'_\gamma)$ may easily be seen to provide a surjection of $\mathbf{N}'_\gamma$ onto $\mathbf{M}'_\gamma$. Thus we may consider $\mathbf{N}'_\gamma \subset \mathbf{M}$.

On the other hand, $\mathfrak{A}^\#_\gamma/\mathcal{K}$ is also a commutative C^*-algebra with unit whose maximal ideal space will be denoted $\mathbf{M}^\#_\gamma$. But by a well-known theorem concerning C^*-algebras generated by a finite number of elements (c.f. [7]), $\mathbf{M}^\#_\gamma$ is homeomorphic to the joint spectrum of the $n + 1$ cosets $T_{\gamma,\nu} + \mathcal{K}$ of $\mathfrak{A}^\#_\gamma/\mathcal{K}$. Using (3.11) and the non-negativity of $T_{\gamma,0}$, this implies that $\mathbf{M}^\#_\gamma \subset B^{n+1}_+ = \{r\sigma : \sigma \in S^n_+ \text{ and } 0 \leqq r \leqq 1\}$. Since $\mathfrak{A}_\gamma$ is generated by $\mathfrak{A}_M$ and $\mathfrak{A}^\#_\gamma$, Herman's Lemma (c.f. [6] Theorem 1) implies the existence of a continuous injection

$$\mathbf{N}_\gamma \to \bar{\Omega} \times B^{n+1}_+ \tag{3.15}$$

such that $n \mapsto (x, \psi_\gamma(x)\sigma)$ where $\sigma^\gamma_a(n) = a(x)$ and $\sigma^\gamma_{T_{\gamma,\nu}}(n) = \psi_\gamma(x)\sigma_\nu$ for $\nu = 0, 1, \cdots, n$. But since $\psi_\gamma \equiv 1$ on Ω'_γ, the image of $\mathbf{N}'_\gamma$ under (3.15) is contained in $\bar{\Omega}'_\gamma \times S^n_+$. Combining this with (3.13) and (3.14) yields (3.12) with $\sigma_a(m) = \sigma^\gamma_a(i_\gamma^{-1}m) = a(x)$ and $\sigma_{S_{\gamma,\nu}}(m) = \sigma^\gamma_{\varphi_\gamma}(i_\gamma^{-1}m) \cdot \sigma^\gamma_{T_{\gamma,\nu}}(i_\gamma^{-1}m) = \varphi_\gamma(x)\sigma_\nu$.

Finally, let $m^1 \in \mathbf{M}'_\gamma$ with $p(m^1) = x \in \Omega'_\gamma$. Let $\varphi \in C_0^\infty(\Omega'_\gamma)$ with $\varphi(x) = 1$. Then $\sigma_{T_{\gamma,0}}(m^1) = \sigma_{\varphi T_{\gamma,0}}(m^1) = 0$ since $\varphi T_{\gamma,0} = \varphi \Lambda \in \mathcal{K}$ by Theorem 3.1 of **[5]**. Thus $m^1 \mapsto (x, \sigma^1)$ with $\sigma^1 \in S^{n-1}$. Let σ^2 be arbitrary in S^{n-1} and $0 = ((r_{\nu\mu}))$ an orthogonal $n \times n$ matrix such that $\sigma^2 = 0\sigma^1$. Defining $\tau(a) = a$, $\tau(T_{\gamma,0}) = T_{\gamma,0}$, $\tau(T_{\tau,\nu}) = \Sigma r_{\nu\mu} T_{\gamma,\mu}$ induces a surjective automorphism $\bar{\tau}$: $\mathfrak{A}_\gamma / \mathcal{K} \to \mathfrak{A}_\gamma / \mathcal{K}$. The associated dual map $\bar{\tau}^*$: $\mathbf{M}_\gamma \to \mathbf{M}_\gamma$ is a homeomorphism such that

$$\sigma_{\tau(A)}(m) = \sigma_A(\bar{\tau}^* m)$$

for all $A \in \mathfrak{A}_\gamma$ and $m \in \mathbf{M}_\gamma$. In particular, for $A = T_{\gamma,\nu}$ and $m^2 = \bar{\tau}^* m^1$,

$$\sigma_{T_{\gamma,\nu}}(m^2) = \sum r_{\nu\mu} \sigma_{T_{\gamma,\mu}}(m^1) = \sigma_\nu^2$$

implies $m^2 \mapsto (x, \sigma^2)$. Hence (3.12) provides a homeomorphism of $\mathbf{M}'_\gamma \cap \mathbf{S}$ onto $\Omega'_\gamma \times S^{n-1}$.

Let $\mathfrak{A}_{\sigma,\tau}$ denote the C^*-subalgebra of $CB(T^*\Omega'_\gamma)$ obtained by restricting functions in $\mathfrak{A}_\sigma$ to $T^*\Omega'_\gamma$. Let $P^*\Omega'_\gamma$ denote the compactification of $T^*\Omega'_\gamma$ induced by $\mathfrak{A}_{\sigma,\gamma}$, and consider the functions $\tilde{\sigma}_A$ extended to $P^*\Omega'_\gamma$ without change in notation.

PROPOSITION 3.5. *For each* $\gamma = 1, \cdots, \Gamma$ *there is a continuous injection*

$$P^*\Omega'_\gamma \to \overline{\Omega'_\gamma} \times S_+^n \tag{3.16}$$

such that $(p) \mapsto (x, \sigma)$ with $\tilde{\sigma}_a(p) = a(x)$ and $\tilde{\sigma}_{S_{\gamma,\nu}}(p) = \varphi_\gamma(x)\sigma_\nu$ *for* $\nu = 0, 1, \cdots n$. *In fact* (3.16) *is surjective.*

Proof. Restricting the formal symbols $\tilde{\sigma}_{T_{\gamma,\nu}}$ for $\nu = 0, 1, \cdots, n$ to $T^*\Omega'_\gamma$ generates a C^*-algebra with maximal ideal space S_+^n and which together with $\mathfrak{A}_M|_{\Omega'_\gamma}$ generates $\mathfrak{A}_{\sigma,\gamma}$. Herman's Lemma yields the injection (3.16) which may easily be seen to be surjective.

Now we may prove our main result.

Proof of Theorem 2.2. For each $\gamma = 1, \cdots, \Gamma$, (3.12) together with (3.16) yield a map θ_γ: $\mathbf{M}'_\gamma \to P^*\Omega'_\gamma$ such that $\sigma_a(m) = \tilde{\sigma}_a(\theta_\gamma(m))$ and $\sigma_{S_{\gamma,\nu}}(m) = \tilde{\sigma}_{S_{\gamma,\nu}}(\theta_\gamma(m))$ for $\nu = 0, 1, \cdots, n$. Since each $P^*\Omega'_\gamma \subset P^*\Omega$, and $m \in \mathbf{M}'_\gamma \cap \mathbf{M}'_\delta$ implies $\tilde{\sigma}_{S_{\gamma,\nu}}(\theta_\gamma(m)) = \tilde{\sigma}_{S_{\delta,\nu}}(\theta_\delta(m))$ for every $\nu = 0, 1, \cdots, n$, the θ_γ induce a continuous injection θ: $\mathbf{M} \to P^*\Omega$ such that

$$\sigma_A(m) = \tilde{\sigma}_A(\theta(m)) \tag{3.17}$$

for all $A = a$ or $S_{\gamma,\nu}$, $a \in \mathbf{A}$ and $\gamma = 1, \cdots, \Gamma$ and $\nu = 0, 1, \cdots, n$. Now let $D \in \mathbf{D}$, and write $D\Lambda = \Sigma_\gamma \varphi_\gamma D\Lambda = \Sigma_{\gamma,\nu} a_{\gamma,\nu} S_{\gamma,\nu}$ by Lemma 3.3. Then

$$\sigma_{D\Lambda}(m) = \Sigma_{\gamma,\nu}\sigma_{a_{\gamma,\nu}}(m)\sigma_{S_{\gamma,\nu}}(m) = \Sigma_{\gamma,\nu}\tilde{\sigma}_{a_{\gamma,\nu}}(\theta(m))\tilde{\sigma}_{S_{\gamma,\nu}}(\theta(m)) = \tilde{\sigma}_{D\Lambda}(\theta(m)).$$

Similarly $\sigma_\Lambda(m) = \tilde{\sigma}_\Lambda(\theta(m))$. In fact the second statement of Proposition 3.4 implies that the image of θ is contained in $\partial P^*\Omega$, and (3.17) becomes

$$\sigma_A(m) = \dot{\sigma}_A(\theta(m)) \tag{3.18}$$

for all $A = a$, $D\Lambda$, or Λ with $a \in \mathbf{A}$ and $D \in \mathbf{D}$. The extension of (3.18) to all $A \in \mathfrak{A}^0$ follows from the algebraic properties of σ and $\tilde{\sigma}$.

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