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A NOTE ON DRAZIN INVERSES

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D is the Drazin inverse of T if $TD = DT$, $D = TD^2$, and $T^k = T^{k+1}D$ for some k.

In recent years, there has been a great deal of interest in generalized inverses of matrices (**[2]**, **[4]**, **[5]**) and many of the concepts can be formulated in Banach space. In particular, if X is a Banach space and $B(X)$ denotes the algebra of bounded operators on X, then we make the following definitions:

DEFINITION 1. An operator S in $B(X)$ is called a *generalized inverse* of T if $TST = T$ and $STS = S$.

DEFINITION 2. An operator T in $B(X)$ is called *generalized Fredholm* if both the range $R(T)$ and the null space $N(T)$ are closed complemented subspaces of X.

Let an operator D in $B(X)$ be the *Drazin inverse* of T. Then $T^k = T^{k+1}D$ for some nonnegative integer k.

DEFINITION 3. The smallest k for which the latter is valid is called the *index* of T.

In fact, if an operator T in $B(X)$ has a Drazin inverse then it has only one (**[2]**, Theorem 1).

REMARKS. (1) It is well known and easy to prove that T is a generalized Fredholm operator if and only if it has a generalized inverse. Some properties of the operator thus defined are obtained in **[1]** but generally there remain unsatisfactory features. For example, in Banach space there is no obvious way of defining a unique generalized inverse and there is no useful relation between the spectrum of an operator and of any of its generalized inverse.

(2) The Drazin inverse was introduced in **[2]** in a very general context and avoids the two defects mentioned above. Note also that if the index is equal to 1, then D is a generalized inverse of T.

We will now proceed to obtain some properties of operators with a Drazin inverse including an exact characterization of such operators. In order to simplify the proof of Theorem 1, we prove the following lemma:

LEMMA 1. *Let T be an operator in $B(X)$. Then T has a generalized inverse S such that $TS = ST$ if and only if X can be written $X = R(T) \oplus N(T)$.*

Proof. Let $X = R(T) \oplus N(T)$ and let P be the projection from X onto $R(T)$ along $N(T)$. Let

$$Q = T \mid R(T)$$

then $N(Q) = (0)$ and Q is bounded with closed range. Hence, Q has a bounded inverse on $R(T)$. We define

$$S = Q^{-1}P.$$

It is easy to see that S is a commuting generalized inverse of T.

Conversely, if T has a commuting generalized inverse S then TS is a projection from X onto $R(T)$. Let

$$X = R(T) \oplus X_1,$$

where $X_1 = N(TS)$. For each $x \in X_1$, $TSx = 0$ and

$$Tx = TSTx = TTSx = 0;$$

this implies $x \in N(T)$. On the other hand, for each $x \in N(T)$ then $Tx = 0$ and

$$TSx = STx = 0;$$

this says $x \in X_1$. Consequently, $N(T) = X_1$.

In fact, $TS = ST$ implies $N(T) = N(S)$ and $R(T) = R(S)$. Thus,

$$X = R(T) \oplus N(T) = R(S) \oplus N(S).$$

THEOREM 1. *Suppose T is an operator in $B(X)$ with generalized inverse S such that $TS = ST$. Then the nonzero points in $\rho(T)$, the resolvent set of T are precisely the reciprocals of the nonzero points in $\rho(S)$.*

Proof. By Lemma 1, X can be decomposed into

$$X = R(T) \oplus N(T).$$

Assume $\lambda \neq 0$ in $\rho(T)$ then

$$(T-\lambda I)^{-1}(T-\lambda I)=I$$
$$T(T-\lambda I)^{-1}(T-\lambda I)S=TS,$$

which yields

$$-T(T-\lambda I)^{-1}\left(S-\frac{1}{\lambda}TS\right)=TS.$$

Since TS is the identity on $R(T)$, for each $x\in R(T)$,

$$-\lambda T(T-\lambda I)^{-1}\left(S-\frac{1}{\lambda}I\right)x=x.$$

This implies $(S-(1/\lambda)I)$ has a bounded inverse on $R(T)$ for all $\lambda\neq 0$ in $\rho(T)$.

On the other hand, for each $x\in N(T)$

$$\left(S-\frac{1}{\lambda}I\right)x=-\frac{1}{\lambda}x$$

or

$$-\lambda\left(S-\frac{1}{\lambda}I\right)x=x.$$

Thus $(S-\lambda^{-1}I)$ also has a bounded inverse on $N(T)$ for all $\lambda\neq 0$ in $\rho(T)$.

Because $(S-\lambda^{-1}I)R(T)=(S-\lambda^{-1}I)R(S)\subseteq R(S)=R(T)$ and $(S-\lambda^{-1}I)N(T)=(S-\lambda^{-1}I)N(S)\subseteq N(S)=N(T)$, so $1/\lambda\in\rho(S)$.

The converse statement is established with T replaced by S and S by T. The proof is complete.

REMARK. The commutativity condition in Theorem 1 is essential, for consider the shift operator $S:(x_1,x_2,x_3,\cdots)\ (0,x_1,x_2,\cdots)$ in l^2. Then $SS^*S=S$ and $S^*SS^*=S^*$ so that S^* is a generalized inverse of S. But $\rho(S)=\rho(S^*)=\{\lambda:|\lambda|=1\}$.

THEOREM 2. *Let T be an operator in $B(X)$ with Drazin inverse D and index k. Then D^k is a generalized inverse of T^k and D^k commutes with T^k.*

Proof. Obviously D^k and T^k commute. Then

$$D^kT^kD^k=D^{2k}T^k=(D^2T)^k=D^k$$

and

$$\begin{aligned} T^k D^k T^k &= T^{k+1} D^{k+1} T^k \\ &= T^{k+1}(D^2 T) D^{k-1} T^{k-1} \\ &= T^{k+1} D^k T^{k-1} \\ &= \cdots \\ &= T^{k+1} D \\ &= T^k. \end{aligned}$$

COROLLARY. *If D is the Drazin inverse of T with index k, then* $X = R(T^k) \oplus N(T^k)$.

THEOREM 3. *If T in $B(X)$ has a Drazin inverse D and λ is a nonzero point in $\rho(T)$, then λ^{-1} belongs to $\rho(D)$.*

Proof. $(TD)^2 = TDTD = TD$, so TD is a projection. It is easy to verify that $R(D) = R(TD)$ and $N(D) = N(TD)$. Hence $R(D)$ and $N(D)$ are closed complemented in X.

Since

$$D(T^2D)D = T^2D^3 = TD^2 = D$$

and

$$(T^2D)D(T^2D) = T^4D^3 = T^3D^2 = T^2D,$$

this shows that T^2D is a commuting generalized inverse of D. Then, by Lemma 1,

$$X = R(D) \oplus N(D).$$

The rest of the proof is analogous to the first part of Theorem 1 since TD is identity and zero on $R(D)$ and $N(D)$ respectively.

Recall the definition of ascent $a(T)$ and descent $d(T)$ for operator T in $B(S)$: an operator has finite ascent if the chain $N(T) \subseteq N(T^2) \subseteq N(T^3) \subseteq \cdots$ becomes constant after a finite number of steps. The smallest integer k such that $N(T^k) = N(T^{k+1})$ is then defined to be $a(T)$. The descent is defined similary for the chain $R(T) \supseteq R(T^2) \supseteq R(T^3) \supseteq \cdots$. If T has finite ascent and descent, then they are equal (**[6]**, Theorem 5.41–E).

THEOREM 4. *An operator T in $B(X)$ has a Drazin inverse if and only if it has finite ascent and descent. In such a case, the index of T is equal to the common value of $a(T)$ and $d(T)$.*

Proof of sufficiency. Let $k = a(T) = d(T)$ be finite. Then (**[6]**, Theorem 5.41–G) T is completely reduced by the pair of closed complemented subspaces $R(T^k)$ and $N(T^k)$ of X and

$$X = R(T^k) \oplus N(T^k).$$

Let P be the projection from X onto $R(T^k)$ along $N(T^k)$. Then

$$PT^k = T^kP. \tag{1}$$

For each x in X, x can be written as $x = y + z$ where $y \in R(T^k)$ and $z \in N(T^k)$.

$$T^kPx = T^kp(y+z) = T^kPy = T^ky$$
$$PT^kx = PT^k(y+z) = PT^ky = T^ky.$$

Since $N(T^k) = N(T^n)$ and $R(T^k) = R(T^n)$ for all $n \geqq k$, we have $X = R(T^n) \oplus N(T^n)$ for all $n \geqq k$. This implies

$$PT^n = T^nP \quad \text{for all} \quad n \geqq k.$$

$$PT = TP. \tag{2}$$

From (1), we have

$$(TP)T^k = T^{k+1}P = (PT)T^k.$$

Thus, P and T commute on $R(T^k)$. Again, for each $x = y + z$ in X,

$$PTx = PT(y+z) = PTy = TPy = TPx.$$

Therefore $PT = TP$ on X.

(3) Define $Q = T\,R(T^k)$. Q is a closed operator follows from the fact that Q is bounded with closed domain. To show Q has a bounded inverse on $R(T^k)$ we need only to prove that Q maps $R(T^k)$ in a one one manner onto itself. Because T maps $R(T^k)$ onto itself, so does Q. If $Qx = 0$ with $x \in R(T^k)$ then

$$0 = Qx = QT^ky = T^{k+1}y \quad \text{for some} \quad y \in R(T^k).$$

This implies $yN(T^{k+1}) = N(T^k)$, thus $x = T^k y = 0$. We define

$$D = Q^{-1}P.$$

(4) Now, we must show that D, defined as above, is a Drazin inverse of T, which is unique by ([**2**], Theorem 1). For every $x = y + z$ in X with $y \in R(T^k)$ and $z \in N(T^k)$ then

$$TDx = TQ^{-1}P(y+z) = TQ^{-1}Py = y$$
$$DTx = Q^{-1}PT(y+z) = Q^{-1}TP(y+z) = Q^{-1}Ty = y,$$

so that $DT = TD$.

$$D^2Tx = Q^{-1}PTQ^{-1}P(y+z) = Q^{-1}P^2x = Dx.$$

Thus, $D = TD^2$.

Finally, $(TD)^2 = TDTD = TD = P$. Hence $I - TD$ is a projection from X onto $N(T^k)$ along $R(T^k)$. For any x in X

$$(I - TD)x \qquad N(T^k).$$

This implies $T^k(I - TD)x = 0$ and then we have

$$T^k = T^{k+1}D.$$

(5) It remains only to show that k is the smallest positive integer such that $T^k = T^{k+1}D$. Suppose there is a positive integer $m < k$ such that

$$T^m = T^{m+1}D$$

then

$$T^m(I - TD)x = 0 \qquad \forall x \in X,$$

hence $(I - TD)x \in N(T^m)$. But $(I - D)x \in N(T^k)$, this contradicts the hypothesis that k is the smallest common value of $a(T)$ and $d(T)$.

Proof of necessity. In Theorem 3 we have proved that if D is the Drazin inverse of T with index k then T^2D is a commuting generalized inverse of D and $X = R(D) \oplus N(D)$. The proof will be complete if we can show that $R(D) = R(T^k)$ and $N(D) = N(T^k)$.

If $y \in R(T^k)$ then there is some $x \in X$ such that

$$y = T^k x = T^{k+1}Dx = DT^{k+1}x \in R(D).$$

Conversely, if $y \in R(D)$ then there is some $x \in X$ such that

$$y = Dx = TD^2x = T^2D^3x = \cdots = T^kD^{k+1}x \in R(T^k).$$

This shows that $R(D) = R(T^k)$. Similarly, we can show that $N(D) = N(T^k)$. Conclusion is that

$$X = R(D) \oplus N(D) = R(T^k) \oplus N(T^k).$$

This implies $T^k(I - TD)x = 0$ and then we have

$$T^k = T^{k+1}D.$$

(6) It remains only to prove that k is the smallest positive integer such that $T^k = T^{k+1}D$. Suppose there is a positive integer $m < k$ such that

$$T^m = T^{m+1}D$$

then

$$T^m(I - TD)x = 0 \qquad x \in X,$$

hence $(I - TD)x \in N(T^m)$. But $(I - TD)x \in N(T^k)$, which contradicts the hypothesis that k is the smallest common value of $a(T)$ and $d(T)$.

The proof of the necessary part is included in Theorem 1.

The operator T can be written as

$$T = Tp + T(I - p), \tag{*}$$

since T and p commute, then for each $x \in X$

$$T(I - p)^k x = T^k(I - p)x = 0.$$

This shows that $T(I - p)$ is nilpotent of order k. As mentioned earlier $T^2D = TP$ is a commuting generalized inverse of D, so that TP has index 0 or 1 (it is zero when T is invertible). The following theorem is proved by Greville (**[4]**, Theorem 9.3) in finite dimensional space. It can be extended to the general case without changing the proof. We merely state:

THEOREM 5. *The decomposition* (*) *is the only decomposition of* T *of the form*

$$T = A + B,$$

where A *has index* 0 *or* 1, B *is nilpotent of order* k *and* $AB = BA = 0$.

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