PACIFIC JOURNAL OF MATHEMATICS
Vol. 71, No. 1, 1977

PERMUTATIONS OF THE POSITIVE INTEGERS WITH RESTRICTIONS ON THE SEQUENCE OF DIFFERENCES

PETER J. SLATER AND WILLIAM YSLAS VÉLEZ

Let $\{a_k\}$ be a sequence of positive integers and $d_k = |a_{k+1} - a_k|$. We say that $\{a_k\}$ is a permutation if every positive integer appears once and only once in the sequence, $\{a_k\}$. We prove the following: Let $\{m_i\}$ be any sequence of positive integers, then there exists a permutation $\{a_k\}$ such that $|\{k|d_k = i\}| = m_i$.

By a permutation $\{a_k | k \in N\}$, where N denotes the set of positive integers, we shall mean a sequence of positive integers such that every element of N appears once and only once in the sequence $\{a_k | k \in N\}$. Set $d_k = |a_{k+1} - a_k|$. The purpose of this paper is to answer, in the affirmative, two questions which were raised by Roger Entringer at the University of New Mexico.

Question 1. Can one construct a permutation $\{a_k | k \in N\}$ such that given any interger n, $|\{k | d_k = n\}| \leqq C$, where C is some fixed constant which is independent of n?

Question 2. Can one construct a permutation $\{a_k | k \in N\}$ such that $\{d_k | k \in N\}$ is also a permutation?

These questions are similar in nature to a problem described in [2] as having been solved by M. Hall. A solution by J. Browkin appears in [1], and the problem is to find a subset A of N such that every natural number is the difference of precisely one pair of numbers of the set A. Note that in this problem one considers all differences and not just differences formed by adjacent members in a sequence.

Let us consider the following procedure for constructing a sequence. Let $a_1 = 1$, $a_2 = 2$. We define a_3 as follows: Let a_3 be the smallest integer, which has not already appeared in the sequence, such that the difference $|a_3 - a_2|$ has also not appeared. Clearly, $a_3 = 4$. Assume that $a_1, a_2, \cdots, a_t$ have been defined in this way. Define a_{t+1} by the following conditions: (i) $|a_{t+1} - a_t| \neq d_i$, $i < t$, (ii) $a_{t+1} \neq a_i$, $i < t + 1$, and (iii) a_{t+1} is the smallest positive integer with properties (i) and (ii).

Clearly, every integer appears at most once in the sequences $\{a_k | k \in N\}$ and $\{d_k | k \in N\}$. But are these sequences permutations? The next theorem settles this question for the sequence $\{a_k | k \in N\}$.

THEOREM 1. *The sequence, $\{a_k | k \in N\}$, constructed above is a permutation.*

Proof. Assume that this sequence is not a permutation. Let i be the smallest integer which does not appear in the sequence. Choose k so that $\{1, 2, \cdots, i-1\} \subset \{a_1, \cdots, a_k\}$. Choose subscripts $k_1, k_2, \cdots, k_{i+1}$ such that $k+1 \leqq k_1 < k_2 < \cdots < k_{i+1}$ and $a_{k_j} > a_l$, for $l < k_j$, that is, a_{k_j} is the largest integer to appear in $\{a_1, \cdots, a_{k_j}\}$. Let $M = \max \{d_j \mid j = 1, \cdots, k_{i+1} - 1\}$, $M_1 = \max \{d_j \mid j = 1, \cdots, k_1 - 1\}$, $M_2 = \max \{d_j \mid j = k-1, \cdots, k_{i+1} - 1\}$. Then $M = \max \{M_1, M_2\}$. But $M_1 \leqq a_{k_1} - 1$ and $M_2 \leqq a_{k_{i+1}} - (i+1)$, since the smallest integer appearing in the sequence $\{a_{k+1}, a_{k+2}, \cdots, a_{k_{i+1}}\}$ is larger than or equal to $(i+1)$. Hence $M \leqq \max\{a_{k_1}-1, a_{k_{i+1}}-(i+1)\}$. But $a_{k_1}-1 \leqq a_{k_2}-2 \leqq \cdots \leqq a_{k_{i+1}} -(i+1)$. So $M \leqq a_{k_{i+1}} - (i+1) < a_{k_{i+1}} - i$. Hence $a_{k_{i+1}} - i > d_j$, $j = 1, \cdots, k_{i+1} - 1$, and i is the smallest integer which has not been used, so we must have that $a_{k_{i+1}+1} = i$, which is a contradiction.

We have not been able to determine whether or not the sequence $\{d_k \mid k \in N\}$ is a permutation.

We next consider another way of constructing permutations so that the differences are also a permutation.

We say that $\{a_1, \cdots, a_t\}$ has property 1 if the a_i are distinct and the $d_i = |a_{i+1} - a_i|$, $i = 1, \cdots, t-1$, are also distinct.

Let i_t be the smallest integer not appearing in the set $\{a_1, \cdots, a_t\}$, e_t is the smallest integer not appearing in the set $\{d_1, \cdots, d_{t-1}\}$, $I_t = \max \{a_1, \cdots, a_t\}$, $E_t = \max\{d_1, \cdots, d_{t-1}\}$. Clearly $E_t < I_t$.

REMARK. Note that either $e_t < E_t$ or $e_t = E_t + 1$. In either case we have that $e_t \leqq I_t$.

Rule 1. Set $a_{t+1} = 2I_t + 1$. If $e_t \leqq i_t$, then set $a_{t+2} = a_{t+1} - e_t$. If $e_t > i_t$, then set $a_{t+2} = i_t$.

LEMMA 1. *If* $\{a_1, \cdots, a_t\}$ *has property* 1 *and* a_{t+1}, a_{t+2} *are constructed according to Rule* 1, *then* $\{a_1, \cdots, a_t, a_{t+1}, a_{t+2}\}$ *also has property* 1.

Proof. Clearly $a_{t+1} \cap \{a_1, \cdots, a_t\} = \varnothing$ and $d_t = a_{t+1} - a_t = 2I_t + 1 - a_t = I_t + 1 + (I_t - a_t) \geqq I_t + 1 > E_t$, so $d_t \cap \{d_1, \cdots, d_{t-1}\} = \varnothing$.

Assume that $e_t \leqq i_t$. Then $a_{t+2} = 2I_t + 1 - e_t = I_t + 1 + (I_t - e_t) \geqq I_t + 1$. Hence $\{a_{t+2}\} \cap \{a_1, \cdots, a_t\} = \varnothing$, so $\{a_1, \cdots, a_t, a_{t+1}, a_{t+2}\}$ are $t+2$ distinct integers. Further $d_{t+1} = |a_{t+2} - a_{t+1}| = e_t$, so $\{d_1, \cdots, d_{t+1}\}$ are $t+1$ distinct differences, hence $\{a_1, \cdots, a_{t+2}\}$ has property 1.

Assume that $i_t < e_t$. Then $a_{t+2} = i_t$, so $\{a_1, \cdots, a_{t+2}\}$ are $t+2$ distinct integers. Further $d_{t+1} = 2I_t + 1 - i_t = I_t + 1 + (I_t - i_t) > (I_t + 1) + (I_t - e_t) \geqq I_t + 1 > E_t$. So $\{d_{t+1}\} \cap \{d_1, \cdots, d_t\} = \varnothing$, hence $\{a_1, \cdots, a_t, a_{t+1}, a_{+2}\}$ has property 1.

Since $\{a_1, \cdots, a_{t+2}\}$ now has property 1, we can apply Rule 1 to this sequence and obtain the sequence $\{a_1, \cdots, a_{t+4}\}$, which again has property 1.

THEOREM 2. *Let* $\{a_1, \cdots, a_t\}$ *have property* 1 *and assume that the infinite sequence* $\{a_1, \cdots, a_t, a_{t+1}, \cdots\}$ *is obtained from* $\{a_1, \cdots, a_t\}$ *by applying Rule* 1 *successively, then the sequences* $\{a_k \mid k \in N\}$ *and* $\{d_k \mid k \in N\}$ *are both permutations.*

Proof. If $e_t \leqq i_t$, then $d_{t+1} = e_t$. Hence the smallest difference which has not appeared in $\{d_1, \cdots, d_{t+1}\}$ is larger than e_t, while i_t is still the smallest integer which has not appeared in $\{a_1, \cdots, a_{t+2}\}$. If $i_t < e_t$, then just the opposite happens. We have that $a_{t+2} = i_t$ while the smallest difference which has not appeared in $\{d_1, \cdots, d_{t+1}\}$ is still e_t. From these remarks the theorem follows by induction.

Let $\{m_1, m_2, \cdots\}$ be any sequence of positive integers. Then by a slight variation we can obtain a permutation $\{a_k \mid k \in N\}$ such that $|\{i \mid d_i = j\}| = m_j$.

We say that $\{a_1, \cdots, a_t\}$ has property 2 if the a_i are distinct and $|\{i \mid d_i = j, i < t\}| \leqq m_j$, for all j.

Let i_t, I_t, E_t be defined as before. Let e_t be the smallest integer such that $|\{i \mid d_i = j, i<t\}| = m_j$, for $j < e_t$, and $|\{i \mid d_i = e_t, i<t\}| < m_{e_t}$. As before, we have that $E_t < I_t$ and $e_t \leqq I_t$.

LEMMA 2. *Assume that* $\{a_1, \cdots, a_t\}$ *has property* 2 *and that* a_{t+1}, a_{t+2} *are defined according to Rule* 1, *then* $\{a_1, \cdots, a_t, a_{t+1}, a_{t+2}\}$ *also has property* 2.

Proof. The proof is exactly the same as Lemma 1.

THEOREM 3. *Let* $\{m_1, m_2, \cdots\}$ *be any infinite sequence of positive integers and let* $\{a_1, a_2, \cdots, a_t\}$ *be a sequence which satisfies property* 2. *If the sequence* $\{a_1, \cdots, a_t, a_{t+1}, \cdots\}$ *is obtained by successively applying Rule* 1, *then this sequence is a permutation and it also has the property that* $|\{i \mid d_i = j\}| = m_j$.

Proof. The proof follows by induction.

REMARK. There are sequences which satisfy property 2, for example, $\{a_1, a_2\}$, where $a_1 \neq a_2$.

REFERENCES

1. J. Browkin, *Solution of a certain problem of A. Schinzel*, Prace Mat., **3** (1959), 205-207.

2. W. Sierpinski, *Elementary Theory of Numbers*, WARSAW, (1964), 411-412.

Received April 28, 1976 and in revised form November 22, 1976. This work was supported by the U. S. Energy Research and Development Administration (ERDA) under Contract No. AT (29-1)-789. By acceptance of this article, the publisher and/or recipient acknowledges the U.S. Government's right to retain a nonexclusive, royalty-free license in and to any copyright covering this paper.

SANDIA LABORATORIES
ALBUQUERQUE, NM 87115