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OPERATORS INVERTIBLE MODULO THE WEAKLY COMPACT OPERATORS

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A continuous linear operator is a Fredholm operator if and only if it is invertible modulo the compact operators. In this note, we will generalize several theorems on Fredholm operators to theorems concerning operators invertible modulo the weakly compact operators.

1. Preliminaries. We fix the following notation.

$\boldsymbol{C}$ = the complex field

B = the category of complex Banach spaces and continuous linear operators

$B(X, Y)$ = the Banach space of continuous linear operators from X to Y (with the sup norm $|\ |$)

$WK(X, Y)$ = the closed subspace of all weakly compact operators in $B(X, Y)$

$X^* = B(X, \boldsymbol{C})$, the conjugate space

$F^* = B(F, \boldsymbol{C})$, the adjoint of $F: X \rightarrow Y$

I_X = the identity operator on X

$\bar{X} = X^{**}/n_X(X)$, where $n_X: X \rightarrow X^{**}$ is the natural injection.

If $F \in B(X, Y)$, then the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "n_X"] \arrow[d, "F"'] & X^{**} \arrow[r] \arrow[d, "F^{**}"'] & \bar{X} \arrow[r] \arrow[d, "\bar{F}"'] & 0 \\ 0 \arrow[r] & Y \arrow[r, "n_Y"] & Y^{**} \arrow[r] & \bar{Y} \arrow[r] & 0 \end{tikzcd}$$

uniquely defines an operator $\bar{F} \in B(\bar{X}, \bar{Y})$. (Here n_X, n_Y are the natural injections.)

We will need the following results (1.1)-(1.7) from [9].

1.1. X is reflexive if and only if $\bar{X} = 0$. [9, (3.1)]

1.2. $F \in WK(X, Y)$ if and only if $\bar{F} = 0$. [9, (4.1)]

1.3. $\bar{I}_X = I_{\bar{X}}$. [9, (2.3)]

1.4. If $E \in B(X, Y)$ and $F \in B(Y, Z)$, then $\overline{FE} = \bar{F}\bar{E}$. [9, (2.3)]

1.5. $|\bar{F}| \leqq |F|$. [9, (2.3)]

1.6. For any $a, b \in \boldsymbol{C}$ and $E, F \in B(X, Y)$, $\overline{aE + bF} = a\bar{E} + b\bar{F}$. [9, (2.4)]

1.7. There exists a natural topological isomorphism $N_X: (\bar{X})^* \rightarrow (\bar{X^*})$. i.e., given any $F \in B(X, Y)$, the diagram

$$\begin{array}{ccc} (\bar{X})^* & \xrightarrow{N_X} & \bar{X}^* \\ {\scriptstyle (\bar{F})^*}\uparrow & & \uparrow{\scriptstyle (\bar{F}^*)} \\ (\bar{Y})^* & \xrightarrow{N_Y} & \bar{Y}^* \end{array}$$

is commutative. [9, (2.8)] (The naturality is not explicitly stated in [9].)

THEOREM 1.8. *If* $E \in B(X, X_1)$ *and* $F \in B(Y, Y_1)$, *then* $\overline{E \oplus F} = \bar{E} \oplus \bar{F}$, *where* $\oplus$ *denotes direct sum.*

Proof. Use $(E \oplus F)^{**} = E^{**} \oplus F^{**}$ and the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X \oplus Y & \longrightarrow & X^{**} \oplus Y^{**} & \longrightarrow & \bar{X} \oplus \bar{Y} & \longrightarrow & 0 \\ & & {\scriptstyle E \oplus F}\downarrow & & {\scriptstyle E^{**} \oplus F^{**}}\downarrow & & {\scriptstyle \bar{E} \oplus \bar{F}}\downarrow & & \\ 0 & \longrightarrow & X_1 \oplus Y_1 & \longrightarrow & X_1^{**} \oplus Y_1^{**} & \longrightarrow & \bar{X}_1 \oplus \bar{Y}_1 & \longrightarrow & 0 . \end{array}$$

2. The operators invertible modulo the weakly compact operators. An operator $F \in B(X, Y)$ is *left* (*right*) *invertible modulo the weakly compact operators* if there exists an operator $E \in B(Y, X)$ such that $\overline{EF} = I_{\bar{X}}(\overline{FE} = I_{\bar{Y}})$. An operator is *invertible modulo the weakly compact operators* if it is left and right invertible modulo the weakly compact operators. Notice that this condition is quite different from merely requiring $\bar{F}$ to be invertible. We let $\Psi_l(X, Y)$, $\Psi_r(X, Y)$, denote the set of all operators left, respectively right, invertible modulo the weakly compact operators and let $\Psi(X, Y)$ denote the set of all operators invertible modulo the weakly compact operators.

THEOREM 2.1. *If* $E \in \Psi(X, Y)$ *and* $F \in \Psi(Y, Z)$. *Then* $FE \in \Psi(X, Z)$.

Proof. By assumption there exist $E_1 \in B(Y, X)$, $F_1 \in B(Z, Y)$ such that $\overline{E_1E} = I_{\bar{X}}$, $\overline{EE_1} = I_{\bar{Y}}$, $\overline{F_1F} = I_{\bar{X}}$, $\overline{FF_1} = I_Z$. Clearly,

$$(\overline{E_1F_1})(\overline{FE}) = I_{\bar{X}}$$

and $\overline{(FE)(E_1F_1)} = I_{\bar{Z}}$.

THEOREM 2.2. *Let* $E \in B(X, Y)$, $F \in B(Y, Z)$. *Assume* $FE \in \Psi(X, Z)$. *Then,*

(1) $E \in (X, Y)$ *if and only if* $F \in \Psi(Y, Z)$;

(2) *If* $F \in \Psi_l(Y, Z)$, *then* $E \in \Psi(X, Y)$ *and* $F \in \Psi(Y, Z)$;

(3) *If* $E \in \Psi_r(X, Y)$, *then* $E \in \Psi(X, Y)$ *and* $F \in \Psi(Y, Z)$.

Proof. By assumption, there exist $G \in B(Z, X)$ such that $\overline{GFE} = I_{\bar X}$ and $\overline{FEG} = I_{\bar Z}$.

(1) If $E \in \Psi(X, Y)$, then there exists $E_1 \in B(Y, X)$ such that $\overline{E_1E} = I_{\bar X}$ and $\overline{EE_1} = I_{\bar Y}$. Hence $\overline{EGF} = I_{\bar Y}$ and $\overline{F(EG)} = I_{\bar Z}$. This means $F \in \Psi(Y, Z)$. The implication in the other direction is proved similarly.

(2) If $F \in \Psi_l(Y, Z)$, then there exists $F_1 \in B(Z, Y)$ such that $\overline{F_1F} = I_{\bar Y}$. This clearly implies $\overline{(GF)E} = I_{\bar X}$, $\overline{E(GF)} = I_{\bar Y}$ and $\overline{=F(EG)}$ $I_{\bar Z}$, $\overline{(EG)F} = I_{\bar Y}$. Hence, $E \in \Psi(X, Y)$ and $F \in \Psi(Y, Z)$.

(3) is proved similarly.

THEOREM 2.3. *Let $F \in B(X, Y)$. If there exist $E_1, E_2 \in B(Y, X)$ such that E_1F and FE_2 are invertible modulo the weakly compact operators, then $F \in \Psi(X, Y)$.*

Proof. Since E_1F and FE_2 are invertible modulo the weakly compact operators, there exist $G_1 \in B(X, X)$, $G_2 \in B(Y, Y)$ such that $\overline{G_1(E_1F)} = I_{\bar X}$ and $\overline{(FE_2)G_2} = I_{\bar Y}$. Hence $F \in \Psi(X, Y)$.

THEOREM 2.4. *If $F \in \Psi(X, Y)$, then $F^* \in \Psi(Y^*, X^*)$.*

Proof. Let $F \in \Psi(X, Y)$. Then there exists $E \in B(Y, X)$ such that $\overline{EF} = I_{\bar X}$ and $\overline{FE} = I_{\bar Y}$. By 1.7, $\overline{E^*F^*} = (N_Y(\bar E)^* N_{\bar X}^{-1})(N_X(\bar F)^* N_{\bar Y}^{-1}) = I_{\bar Y^*}$ and $\overline{F^*E^*} = (N_X(\bar F)^* N_{\bar Y}^{-1})(N_Y(\bar E)^* N_{\bar X}^{-1}) = I_{\bar X^*}$. Hence $F^* \in \Psi(Y^*, X^*)$.

THEOREM 2.5. *If $F \in \Psi(X, Y)$ and $K \in WK(X, Y)$, then $F + K \in \Psi(X, Y)$.*

Proof. $\overline{F + K} = \bar F + \bar K = \bar F$.

As is shown in [9, Theorem 5.10], if the Banach spaces X and Y enjoy the property that every closed reflexive subspace of X is complemented and every closed subspace of Y with reflexive quotient is complemented, then every generalized Fredholm operator is invertible modulo the weakly compact operators.

There are, however, other kinds of operators invertible modulo the weakly compact operators. Let X be any Banach space. Let $U: X \to X$ be an invertible operator, and $K \in WK(X, X)$. Then, clearly, $U + K \in \Psi(X, X)$.

To construct a nontrivial operator invertible modulo the weakly compact operators which is not a generalized Fredholm operator, we start an operator $F = U + K \in \Psi(X, X)$ such as the one constructed above. We choose a reflexive Banach space Y and an operator $G \in B(Y, Y)$ which does not have a closed range. If we form the direct sum $F \oplus G: X \oplus Y \to X \oplus Y$, we see, by Theorem 1.8, $\overline{F \oplus G} = \bar{F} \oplus \bar{G} = \bar{F} \oplus 0$. Hence $F \oplus G$ is invertible modulo the weakly compact operators but it is definitely not a generalized Fredholm operator because it does not have a closed range. (Also see [3, V. 2.6].)

3. The operators left (right) invertible modulo the weakly compact operators.

THEOREM 3.1. (1) *If* $F \in \Psi_l(X, Y)$, $K \in WK(X, Y)$, *then* $F + K \in \Psi_l(X, Y)$.

(2) *If* $E \in \Psi_l(X, Y)$, $F \in \Psi_l(Y, Z)$, *then* $FE \in \Psi_l(X, Z)$.

(3) *If* $E \in B(X, Y)$, $F \in B(Y, Z)$ *and* $FE \in \Psi_l(X, Z)$, *then* $E \in \Psi_l(X, Y)$.

(4) *If* $F \in \Psi_r(X, Y)$, $K \in WK(X, Y)$, *then* $F + K \in \Psi_r(X, Y)$.

(5) *If* $E \in \Psi_r(X, Y)$, $F \in \Psi_r(Y, Z)$, *then* $FE \in \Psi_r(X, Z)$.

(6) *If* $E \in B(X, Y)$, $F \in B(Y, Z)$ *and* $FE \in \Psi_r(X, Z)$, *then* $F \in \Psi_r(Y, Z)$.

Proof. (1) $\overline{F + K} = \bar{F} + \bar{K} = \bar{F}$.

(2) If $E \in \Psi_l(X, Y)$ and $F \in \Psi_l(Y, Z)$, then there exist $E_1 \in B(Y, X)$, $F_1 \in B(Z, Y)$ such that $\overline{E_1E} = I_{\bar{X}}$ and $\overline{F_1F} = I_{\bar{Y}}$. Clearly, $\overline{E_1F_1FE} = I_{\bar{X}}$. Hence $FE \in \Psi_l(X, Z)$.

(3) If $FE \in \Psi_l(X, Z)$, then there exists $G \in B(Z, X)$ such that $\overline{G(FE)} = I_{\bar{X}}$. Hence $E \in \Psi_l(X, Y)$.

(4), (5), (6) are similarly proved.

THEOREM 3.2. (1) *If* $F \in \Psi_l(X, Y)$, *then* $F^* \in \Psi_r(Y^*, X^*)$.

(2) *If* $F \in \Psi_r(X, Y)$, *then* $F^* \in \Psi_l(Y^*, X^*)$.

Proof. (1) If $F \in \Psi_l(X, Y)$, then there exists $F_1 \in B(Y, X)$ such that $\overline{F_1F} = I_{\bar{X}}$. Hence $(\bar{F})^*(\bar{F}_1)^* = I_{(\bar{X})^*}$. By 1.7,

$$(N_X(\bar{F})^*N_{\bar{Y}}^{-1})(N_Y(\bar{F}_1)^*N_{\bar{X}}^{-1}) = I_{\bar{X}^*}\,,$$

whence $\overline{F^*F_1^*} = I_{X^{**}}$. This shows $F^* \in \Psi_r(Y^*, X^*)$.

(2) is proved similarly.

4. Perturbation. Let A be a Banach algebra with identity (1), A° be the group of invertible elements in A, $A_l^\circ(A_r^\circ)$ be the set of all left(right) invertible elements of A. Let

$R(A)$ = the radical of A

$= \{r \in A \mid 1 + ar \in A^\circ \text{ for every } a \in A\}$ [6, p. 163]

$= \{r \in A \mid 1 + ar \in A^\circ \text{ for every } a \in A^\circ\}$ [5, p. 4]

$Q(A) =$ the set of all quasi-nilpotent (topologically nilpotent [7, p. 12]) elements of A

$= \{q \in A \mid 1 + kq \in A^\circ$ for every $k \in C\}$ [4, p. 699]

$= \{q \in A \mid |q^n|^{1/n} \longrightarrow 0$ as $n \longrightarrow \infty\}$. [1, p. 23]

For a semigroup S in A, let

$$P(S) = \{a \in A \mid a + S \subset S\} .$$

The following theorem is proved in [5].

THEOREM 4.1. $P(A^\circ) = P(A_l^\circ) = P(A_r^\circ) = R(A)$.

THEOREM 4.2.
$Q(A) = \{q \in A \mid I + aq \in A^\circ$ *for all* $a \in A^\circ$ *such that* $aq = qa\}$.

Proof. The set on the right is obviously contained in $Q(A)$. Now take an element $q \in Q(A)$, and let $a \in A^\circ$ be such that $aq = qa$. Clearly, $|(aq)^n|^{1/n} \longrightarrow 0$ as $n \longrightarrow \infty$. Hence $aq \in Q(A)$. So $1 + aq \in A^\circ$. This shows that q is in the set on the right hand side.

THEOREM 4.3. (1) *Let* $q \in A$. *Then,* $q \in Q(A)$ *if and only if for all* $a \in A^\circ$ *such that* $aq = qa$, $a + q \in A^\circ$;

(2) *If* $q_1, q_2 \in Q(A)$ *and* $q_1q_2 = q_2q_1$, *then* $q_1 + q_2 \in Q(A)$.

Proof. (1) Clearly, $aq = qa$ is equivalent to $a^{-1}q = qa^{-1}$. Hence, $q \in Q(A) \Leftrightarrow 1 + aq \in A^\circ$ for all $a \in A^\circ$ such that $aq = qa \Leftrightarrow 1 + a^{-1}q \in A^\circ$ for all $a \in A^\circ$ such that $aq = qa \Leftrightarrow a + q = a(1 + a^{-1}q) \in A^\circ$ for all a such that $aq = qa$.

(2) Let $q_1, q_2 \in Q(A)$ be such that $q_1q_2 = q_2q_1$. Let k be an arbitrary complex number. Clearly, $(1 + kq_1) \in A^\circ$ and $kq_2 \in Q(A)$. Since $(1 + kq_1)(kq_2) = (kq_2)(1 + kq_2)$, by (1), $1 + k(q_1 + q_2) \in A^\circ$. Since k is arbitrary, $q_1 + q_2 \in Q(A)$.

We remark that (2) also follows from [7, Th. 1.4.1(v), p. 10].

Now we shall apply these theorems to the specific problem of perturbation of operators (left, right) invertible modulo the compact operators.

Let X be a Banach space. Let $B(X) = B(X, X)$, $WK(X) = WK(X, X)$, $\Psi(X) = \Psi(X, X)$, $\Psi_l(X) = \Psi_l(X, X)$, $\Psi_r(X) = \Psi_r(X, X)$, $\bar{B}(X) = B(X)/WK(X)$ and $\tau: B(X) \longrightarrow \bar{B}(X)$ be the natural projection. Notice that $\bar{B}(X)$ can be considered as a subalgebra of $B(\bar{X})$ [9, (5.11)] and we have $\Psi(X) = \tau^{-1}[(\bar{B}(X))^\circ]$, ${}_l\psi(X) = \tau^{-1}[(\bar{B}(X))_l^\circ]$, $\Psi_r(X) = \tau^{-1}[(\bar{B}(X))_r^\circ]$.

THEOREM 4.4. $P(\Psi(X)) = P(\Psi_l(X)) = P(\Psi_r(X)) = \tau^{-1}[R(\bar{B}(X))]$.

Proof. Use Theorem 4.1 and the above remark.

COROLLARY 4.5. *If* $R(\bar{B}(X)) = 0$, *then* $P(\Psi(X)) = P(\Psi_l(X)) = P(\Psi_r(X)) = WK(X)$.

Since $\bar{B}(X)$ can be considered as a subalgebra of $B(\bar{X})$ and since $R(B(\bar{X})) = 0$ [4, p. 702], $R(\bar{B}(X)) = 0$ if $\bar{B}(X) = B(\bar{X})$.

Now, let $\Omega(X) = \tau^{-1}[Q(\bar{B}(X))]$. The classical counterpart of $\Omega(X)$ is the set of all Riesz operators [2, p. 323] [8].

THEOREM 4.6. *Let* $E \in B(X)$. *Then,* $E \in \Omega(X)$ *if and only if* $E + F \in \Psi(X)$ *for all* $F \in \Psi(X)$ *such that* $\overline{EF} = \overline{FE}$.

Proof. Apply Theorem 4.3.1.

THEOREM 4.7. *If* $E_1, E_2 \in \Omega(X)$ *and* $\overline{E_1E_2} = \overline{E_2E_1}$, *then* $E_1 + E_2 \in \Omega(X)$.

Proof. Apply Theorem 4.3.2.

Note that if we apply Theorem 4.3 to the case $A = B(X)/K(X)$, where $K(X)$ is the closed 2-sided ideal of all compact operators, then we obtain Theorems 9, 12, 13, of [8]. They are the classical "relatives" of Theorems 4.6 and 4.7.

It would be an interesting problem to characterize the operators in $\Psi(X, Y)(\Psi_l(X, Y), \Psi_r(X, Y))$ intrinsically.

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