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THE MAXIMAL RIGHT QUOTIENT SEMIGROUP OF A STRONG SEMILATTICE OF SEMIGROUPS

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Let S be a strong semilattice Y of monoids. If S is right nonsingular then Y is nonsingular. The converse is true when S is a sturdy semilattice Y of right cancellative monoids. Should S have trivial multiplication then each monoid of more than one element has as its index an atom of Y. Finally, if S is a right nonsingular strong semilattice Y of principal right ideal Ore monoids with onto linking homomorphisms then $Q(S)$, the maximal right quotient semigroup of S, is a semilattice $Q(Y)$ of groups.

1. **Introduction.** Let Y be a semilattice and let $\{S_\alpha\}_{\alpha\in Y}$ be a collection of pairwise disjoint semigroups. For each pair α, $\beta\in Y$ with $\alpha\geqq\beta$, let $\psi_{\alpha,\beta}\colon S_\alpha\rightarrow S_\beta$ be a semigroup homomorphism such that $\psi_{\alpha,\alpha}$ is the identity mapping and if $\alpha>\beta>\gamma$ then $\psi_{\alpha,\gamma}=\psi_{\beta,\gamma}\psi_{\alpha,\beta}$. Let $S=\bigcup_{\alpha\in Y}S_\alpha$ with multiplication

$$a*b=\psi_{\alpha,\alpha\beta}(a)\psi_{\beta,\alpha\beta}(b)$$

for $a\in S_\alpha$ and $b\in S_\beta$. The semigroup S is called a *strong semilattice Y of semigroups S_α*. If, in addition, each $\psi_{\alpha,\beta}$ is one-to-one then S is called a *sturdy semilattice of semigroups*. The basic terminology in use throughout this paper can be found in [1], [7], and [9]. Note that a semilattice of groups [1, p. 128] is a strong semilattice of semigroups. In [6], McMorris showed that if M is a semilattice X of groups G_δ, then $Q(M)$, the maximal right quotient semigroup of M, is also a semilattice of groups. Hinkle [2] constructed $Q(M)$ and showed that its indexing semilattice is $Q(X)$.

Let S be a semigroup with 0. A right ideal D of S is *dense* if for each s_1, s_2, $s\in S$ with $s_1\neq s_2$, there exists an element $d\in D$ such that $s_1d\neq s_2d$ and $sd\in D$. A right ideal L of S is *$\cap$-large* if for each nonzero right ideal R of S, $R\cap L\neq\{0\}$. It is easy to see that dense implies $\cap$-large. If each $\cap$-large right ideal of S is also dense then S is said to be *right nonsingular*. If a semigroup is commutative or each one-sided ideal is two-sided then we will use the term nonsingular. Let T be a right S-system with 0[5] then the *singular congruence* ψ_T on T is a right congruence defined for a, $b\in T$ by $a\psi_Tb$ if and only if $as=bs$ for all s in an $\cap$-large right ideal of S. McMorris [8] showed that $\psi_S=i_S$, the identity congruence on S, if and only if S is right nonsingular.

Recently it has been shown [4], [5] that if S is a commutative nonsingular semigroup then $Q(S)$ is a semilattice of groups. However, since S is commutative it is uniquely expressible as a semilattice Y of archimedian semigroups [1, p. 135]. Thus we investigate right nonsingular strong semilattices of semigroups.

Henceforth we require that both S and Y be semigroups with 0. If for $\alpha \in Y$, S_α is a monoid then the identity will be denoted by e_α. Also a semigroup homomorphism which takes the identity of one semigroup to the identity of the other is called a *monoid homomorphism.*

LEMMA 1.1. *If S is a strong semilattice Y of right cancellative monoids S_α, then for each $\alpha, \beta \in Y$ with $\alpha \geqq \beta$, $\psi_{\alpha,\beta}$ is a monoid homomorphism and Y is isomorphic to the semilattice E of idempotents of S.*

LEMMA 1.2. *Let S be a strong semilattice Y of monoids S_α with $\psi_{\alpha,\beta}$ a monoid homomorphism for $\alpha \geqq \beta \in Y$. If L is an $\cap$-large right ideal of S, then $A = \{\sigma \in Y \mid L \cap S_\sigma \neq \varnothing\}$ is an $\cap$-large ideal of Y.*

Proof. To see that A is $\cap$-large let R be a nonzero ideal of Y and define $B = \bigcup_{\tau \in R} S_\tau$. Let $t \in B \cap S_\beta$ and $s \in S_\sigma$ for some $\beta \in R$ and $\sigma \in Y$. Then $t*s = \psi_{\beta,\sigma\beta}(t)\psi_{\sigma,\sigma\beta}(s) \in S_{\sigma\beta}$. But $S_{\sigma\beta} \subseteqq B$ since $\beta \in R$ an ideal of Y. Dually we can show that $s*t \in S_{\sigma\beta}$ and so B is a two-sided ideal of S. Since L is an $\cap$-large right ideal of S then $L \cap B \neq \{0\}$ so there exists $0 \neq r \in L \cap B$. But then $r \in S_\delta$ for $0 \neq \delta \in R$ and so $0 \neq \delta \in A \cap R$ and A is $\cap$-large. It is easy to show that A is an ideal of Y.

LEMMA 1.3. *Let S be a strong semilattice Y of monoids S_α with $\psi_{\alpha,\beta}$ a monoid homomorphism for $\alpha \geqq \beta \in Y$. If T is an $\cap$-large ideal of Y, then $L = \bigcup_{\beta \in T} S_\beta$ is an $\cap$-large ideal of S.*

Proof. We saw in the proof of Lemma 1.2 that L is an ideal of S. To see that L is $\cap$-large we let B be a nonzero right ideal of S, and define $R = \{\sigma \in Y \mid B \cap S_\sigma \neq \varnothing\}$. Since R is a nonzero ideal of Y and T is $\cap$-large then $R \cap T \neq \{0\}$. Thus there exists $0 \neq \delta \in R \cap T$ for which $S_\delta \subseteqq L$, and so there exists $0 \neq t \in B \cap L$.

2. Right nonsingular strong semilattices of semigroups. In studying a semigroup M which is a semilattice X of groups G_δ, Johnson and McMorris [3] showed that if M is nonsingular then the set E of idempotents of M is a nonsingular semilattice. Note that under these conditions the idempotents of M are central, every one-sided ideal is two-sided, and X is isomorphic to E. Here we consider a weaker structure and obtain the results of Johnson and McMorris.

THEOREM 2.1. *Let S be a strong semilattice Y of monoids S_α with $\psi_{\alpha,\beta}$ a monoid homomorphism for $\alpha \geqq \beta \in Y$. If S is right nonsingular, then Y is nonsingular.*

Proof. Let T be an $\cap$-large ideal of Y and define $L = \bigcup_{\beta \in T} S_\beta$. Since S is right nonsingular then L is a dense right ideal of S for, by Lemma 1.3, L is an $\cap$-large right ideal of S. Let α, $\beta \in Y$ such that $\alpha \neq \beta$. Then $e_\alpha \neq e_\beta$ and there exists an $x \in L$ such that $e_\alpha * x \neq e_\beta * x$ where $x \in S_\delta$. Thus $\delta \in T$ and $\alpha\delta \neq \beta\delta$ for if otherwise

$$e_\alpha * x = \psi_{\alpha,\alpha\delta}(e_\alpha)\psi_{\delta,\alpha\delta}(x) = \psi_{\delta,\alpha\delta}(x)$$
$$\psi_{\delta,\beta\delta}(x) = \psi_{\beta,\beta\delta}(e_\beta)\psi_{\delta,\beta\delta}(x) = e_\beta * x$$

which is a contradiction. Thus T is dense in Y.

THEOREM 2.2. *Let S be a sturdy semilattice Y of right cancellative monoids S_α. If Y is nonsingular, then S is right nonsingular.*

Proof. Let L be an $\cap$-large right ideal of S and let $x \neq y$, $z \in S$. Since L is $\cap$-large then $z^{-1}L = \{s \in S \mid z * s \in L\}$ is an $\cap$-large right ideal of S and so is $L^* = L \cap z^{-1}L$. By Lemma 1.2, $A = \{\sigma \in Y \mid L^* \cap S_\sigma \neq \varnothing\}$ is an $\cap$-large ideal of Y, and since Y is nonsingular then A is dense in Y. We now consider the following two cases:

Case 1. Suppose that $x \in S_\alpha$ and $y \in S_\beta$ with $\alpha \neq \beta$. Since A is dense there exists $\delta \in A$ such that $\alpha\delta \neq \beta\delta$. Hence there exists a $t \in L^* \cap S_\delta$ such that $z * t \in L$ and $t \in L$. Since $\alpha\delta \neq \beta\delta$ then $S_{\alpha\delta} \cap S_{\beta\delta} = \varnothing$ and so $x * t \neq y * t$.

Case 2. Suppose that $x, y \in S_\alpha$ and define $[0, \alpha] = \{\sigma \in Y \mid 0 \leqq \sigma \leqq \alpha\}$. Since $[0, \alpha]$ is a nonzero ideal of Y, then there exists $0 \neq \delta \in A \cap [0, \alpha]$. Thus there is a $t \in L^*$ with $t \in L$ and $z * t \in L$. Now $x * t \neq y * t$ for if otherwise then $\psi_{\alpha,\delta}(x)t = \psi_{\alpha,\delta}(y)t$. But S_δ is right cancellative so $\psi_{\alpha,\delta}(x) = \psi_{\alpha,\delta}(y)$. Since $\psi_{\alpha,\delta}$ is one-to-one then $x = y$ which is a contradiction.

Thus in both cases L is a dense right ideal of S.

COROLLARY 2.3. *Let S be a sturdy semilattice Y of right* *cancellative monoids S_α. Then S is right nonsingular if and only if Y is nonsingular.*

If each $\psi_{\alpha,\beta}(\alpha > \beta)$ is the trivial homomorphism; that is, it takes all elements to the identity, we say that S has *trivial multiplication.*

THEOREM 2.4. *Let S be a strong semilattice Y of monoids S_α and let S have trivial multiplication. If S is right nonsingular, then $|S_\alpha| > 1$ implies α is an atom* (a minimal nonzero element) *of Y.*

Proof. Let $|S_\alpha| > 1$ and let $x, y \in S_\alpha$ with $x \neq y$. Also let L be an $\cap$-large right ideal of S. Since S is right nonsingular, L is dense and so there exists $z \in S$ such that $x*z \neq y*z$ and $e_\alpha * z \in L$. We claim that if $z \in S_\beta$ then $\alpha \leqq \beta$. To see this we consider the following two cases:

Case 1. If α is not related to β then $\alpha > \alpha\beta$ and $\beta > \alpha\beta$. Thus $x*z = \psi_{\alpha,\alpha\beta}(x)\psi_{\beta,\alpha\beta}(z) = e_{\alpha\beta}e_{\alpha\beta} = e_{\alpha\beta}$ and $y*z = \psi_{\alpha,\alpha\beta}(y)\psi_{\beta,\alpha\beta}(z) = e_{\alpha\beta}e_{\alpha\beta} = e_{\alpha\beta}$. This is a contradiction since $x*z \neq y*z$.

Case 2. If $\beta \leqq \alpha$ then $x^*z = \psi_{\alpha,\beta}(x)\psi_{\beta,\beta}(z) = e_\beta z = z$ and $y*z = \psi_{\alpha,\beta}(y)\psi_{\beta,\beta}(z) = e_\beta z = z$. Again this is a contradiction.

Let B be an $\cap$-large ideal, L^* and z as before. Then $\alpha \leqq \beta$ implies $\alpha\beta = \alpha \in \beta$.

Finally, we suppose that α is not an atom of Y. Then there exists $\delta \in Y$ such that $0 < \delta < \alpha$. Define $I = \{\sigma \in Y \mid \sigma\delta = 0 \text{ or } \sigma \leqq \delta\}$. It is easy to see that I is an $\cap$-large ideal of Y but $\alpha \notin I$ which is a contradiction.

THEOREM 2.5. *Let S be a strong semilattice Y of right cancellative monoids S_α. If Y is nonsingular and $|S_\alpha| > 1$ implies α is an atom of Y, then S is right nonsingular.*

Proof. Let $x \neq y$, $z \in S$ and let L be an $\cap$-large right ideal of S. If $x \in S_\alpha$ and $y \in S_\beta$ with $\alpha \neq \beta$ by the same argument as in Theorem 2.2, Case 1 there exists $t \in L$ such that $x*t \neq y*t$ and $z*t \in L$. Hence assume that $x, y \in S_\alpha$, then since $|S_\alpha| > 1$, α is an atom of Y and $[0, \alpha]$ is a nonzero ideal of Y. Thus there exists $t \in L \cap S_\alpha$ such that $z*t \in L$ and $x*t \neq y*t$, for if otherwise $x = y$ since S_α is right cancellative and this would be a contradiction.

Note that if $|S_\alpha| > 1$ implies α is an atom of Y, then S has trivial multiplication.

COROLLARY 2.6. *Let S be a strong semilattice Y of right cancellative monoids S_α and assume S has trivial multiplication. Then S is right nonsingular if and only if E is nonsingular and $|S_\alpha| > 1$ implies that e_α is an atom of E.*

3. The maximal right quotient semigroup. Since McMorris [6] showed that the maximal right quotient semigroup of a semilattice of groups is a semilattice of groups, a natural question arises; which strong semilattices of semigroups have for their maximal right quotient semigroup a semilattice of groups? In this section, we let S be a strong semilattice Y of right cancellative principal right ideal monoids S_α with the linking homomorphisms onto.

LEMMA 3.1. *If aS_α is a dense principal right ideal of S_α then $\psi_{\alpha,\beta}(a)S_\beta$ is a dense principal right ideal of S_β for $\alpha \geqq \beta$.*

Proof. The proof is straightforward and is omitted.

Let $\alpha, \beta \in Y$ with $\alpha \geqq \beta$ and let $Q(S_\alpha)$, $Q(S_\beta)$ be the maximal right quotient semigroup of S_α and S_β respectively. The members of these equivalence classes will be denoted $[f]_\alpha$ and $[g]_\beta$ with the subscripts being dropped if there is no confusion.

We can extend $\psi_{\alpha,\beta}: S_\alpha \to S_\beta$ to a mapping $\phi_{\alpha,\beta}: Q(S_\alpha) \to Q(S_\beta)$ defined by $[f]_\alpha \to [\hat{f}]_\beta$ where if $f: aS_\alpha \to S_\alpha$ then $\hat{f}: \psi_{\alpha,\beta}(a)S_\beta \to S_\beta$ is defined by $\psi_{\alpha,\beta}(a)s \to \psi_{\alpha,\beta}(f(a))s$ for $s \in S_\beta$. Note that $\hat{f}$ is an S_β-homomorphism since if $t \in S_\beta$ then $\hat{f}(\psi_{\alpha,\beta}(a)s)t = (\psi_{\alpha,\beta}(f(a))s)t = \psi_{\alpha,\beta}(f(a))(st) = \hat{f}(\psi_{\alpha,\beta}(a)(st)) = \hat{f}((\psi_{\alpha,\beta}(a)s)t)$.

We next show that $\phi_{\alpha,\beta}$ is independent of the representative we choose from $[f]$. Hence let $[f] = [g]$, then f and g agree on a dense right ideal of S_α, call it D, found in the intersection of their domains D_f and D_g respectively. Since S_α is a principal right ideal semigroup then $D_f = aS_\alpha$, $D_g = cS_\alpha$ and $D = xS_\alpha$ for some $a, c, x \in S_\alpha$. Now $\phi_{\alpha,\beta}([f]) = [\hat{f}]$ where $\hat{f}: \psi_{\alpha,\beta}(a)S_\beta \to S_\beta$ defined by $\psi_{\alpha,\beta}(a)s \to \psi_{\alpha,\beta}(f(a))s$, and $\phi_{\alpha,\beta}([g]) = [\hat{g}]$ where $\hat{g}: \psi_{\alpha,\beta}(c)S_\beta \to S_\beta$ defined by $\psi_{\alpha,\beta}(c)s \to \psi_{\alpha,\beta}(g(c))s$. We claim $\hat{f}$ and $\hat{g}$ agree on the dense right ideal $\psi_{\alpha,\beta}(x)S_\beta \subseteqq \psi_{\alpha,\beta}(a)S_\beta \cap \psi_{\alpha,\beta}(c)S_\beta$. Since $xS_\alpha \subseteqq aS_\alpha \cap cS_\alpha$ it is easy to see that $\psi_{\alpha,\beta}(x)S_\alpha \subseteqq \psi_{\alpha,\beta}(a)S_\beta \cap \psi_{\alpha,\beta}(c)S_\beta$. Furthermore, since xS_α is dense in S_α then by Lemma 3.1, $\psi_{\alpha,\beta}(x)S_\beta$ is dense in S_β. Hence let $\psi_{\alpha,\beta}(x)s \in \psi_{\alpha,\beta}(x)S_\beta$ then $\hat{f}(\psi_{\alpha,\beta}(x)s) = \hat{f}(\psi_{\alpha,\beta}(x)\psi_{\alpha,\beta}(t))$ where $t \in S_\alpha$ since $\psi_{\alpha,\beta}$ is onto. Since $\psi_{\alpha,\beta}$ is a semigroup homomorphism, it follows that $\hat{f}(\psi_{\alpha,\beta}(x)s) = \hat{f}(\psi_{\alpha,\beta}(xt)) = \psi_{\alpha,\beta}(f(xt)) = \psi_{\alpha,\beta}(g(xt)) = \hat{g}(\psi_{\alpha,\beta}(xt)) = \hat{g}(\psi_{\alpha,\beta}(x)\psi_{\alpha,\beta}(t)) = \hat{g}(\psi_{\alpha,\beta}(x)s)$. Thus the claim is established.

THEOREM 3.2. *Let $S = \bigcup_{\alpha \in Y} S$ be a strong semilattice Y of right cancellative principal right ideal monoids S_α with $\psi_{\alpha,\beta}$ onto for $\alpha \geqq \beta \in Y$. If $T = \bigcup_{\alpha \in Y} Q(S_\alpha)$ with multiplication defined by*

$$[f]_\alpha [g]_\beta = \phi_{\alpha,\alpha\beta}([f]_\alpha)\phi_{\beta,\alpha\beta}([g]_\beta)$$

where $[f]_\alpha \in Q(S_\alpha)$, $[g]_\beta \in Q(S_\beta)$ and $\phi_{\alpha,\alpha\beta}$, $\phi_{\beta,\alpha\beta}$ are defined as above, then T is a strong semilattice Y of monoids $Q(S_\alpha)$.

Proof. Note that since $S_\alpha \cap S_\beta = \varnothing$ for $\alpha \neq \beta$ then $Q(S_\alpha) \cap Q(S_\beta) = \varnothing$, and that $\phi_{\alpha,\alpha}$ is the identity mapping. We now show that $\phi_{\alpha,\beta}: Q(S_\alpha) \rightarrow Q(S_\beta)$ is a semigroup homomorphism. Let $[f]$, $[g] \in Q(S_\alpha)$ then we must show that $\phi_{\alpha,\beta}([f][g]) = \phi_{\alpha,\beta}([f])\phi_{\alpha,\beta}([g])$. To this end we let $\phi_{\alpha,\beta}([f]) = [\hat{f}]$ and $\phi_{\alpha,\beta}([g]) = [\hat{g}]$ where if $f: aS_\alpha \rightarrow S_\alpha$ and $g: cS_\alpha \rightarrow S_\alpha$ then $\hat{f}: \psi_{\alpha,\beta}(a)S_\beta \rightarrow S_\beta$ defined by $\psi_{\alpha,\beta}(a)s \rightarrow \psi_{\alpha,\beta}(f(a))s$ and $\hat{g}: \psi_{\alpha,\beta}(c)S_\beta \rightarrow S_\beta$ defined by $\psi_{\alpha,\beta}(c)s \rightarrow \psi_{\alpha,\beta}(g(c))s$. Since $[f][g] = [fg]$ where $fg: g^{-1}(aS_\alpha) \rightarrow S_\alpha$ and $g^{-1}(aS_\alpha) = \{x \in cS_\alpha \mid g(x) \in aS_\alpha\}$, then for some $h \in S_\alpha$, $hS_\alpha = g^{-1}(aS_\alpha)$ and so $\widehat{fg}: \psi_{\alpha,\beta}(h)S_\beta \rightarrow S_\beta$ defined by $\psi_{\alpha,\beta}(h)s \rightarrow \psi_{\alpha,\beta}(fg(h))s$. Thus $\phi_{\alpha\beta}([f][g]) = \phi_{\alpha,\beta}([fg]) = [\widehat{fg}]$. On the other hand, $\phi_{\alpha,\beta}([f])\phi_{\alpha,\beta}([g]) = [\hat{f}][\hat{g}] = [\hat{f}\hat{g}]$ where $\hat{f}\hat{g}: \hat{g}^{-1}(\psi_{\alpha,\beta}(a)S_\beta) \rightarrow S_\beta$ and $\hat{g}^{-1}(\psi_{\alpha,\beta}(a)S_\beta) = \{y \in \psi_{\alpha,\beta}(c)S_\beta \mid \hat{g}(y) \in \psi_{\alpha,\beta}(a)S_\beta\}$. Hence we must show that $[\widehat{fg}] = [\hat{f}\hat{g}]$; that is, $\widehat{fg}$ and $\hat{f}\hat{g}$ agree on a dense right ideal found in the intersection of their domains. Now $\psi_{\alpha,\beta}(h)S_\beta \subseteqq g^{-1}(\psi_{\alpha,\beta}(a)S_\beta)$ for if $\psi_{\alpha,\beta}(h)s \in \psi_{\alpha,\beta}(h)S_\beta$ then $\psi_{\alpha,\beta}(h)s = \psi_{\alpha,\beta}(h)\psi_{\alpha,\beta}(t)$ where $t \in S_\alpha$ since $\psi_{\alpha,\beta}$ is onto. Thus $\psi_{\alpha,\beta}(h)s = \psi_{\alpha,\beta}(ht) = \psi_{\alpha,\beta}(cr)$ since $ht \in cS_\alpha$ and so $ht = cr$ for some $r \in S_\alpha$. Hence $\psi_{\alpha,\beta}$ being a semigroup homomorphism implies $\psi_{\alpha,\beta}(h)s = \psi_{\alpha,\beta}(c)\psi_{\alpha,\beta}(r) \in \psi_{\alpha,\beta}(c)S_\beta$. Now $\hat{g}(\psi_{\alpha,\beta}(h)s) = \psi_{\alpha,\beta}(g(h))s = \psi_{\alpha,\beta}(g(h))\psi_{\alpha,\beta}(t) = \psi_{\alpha,\beta}(g(h)t) = \psi_{\alpha,\beta}(g(ht)) = \psi_{\alpha,\beta}(ax)$ since $g(ht) \in aS_\alpha$ and so $g(ht) = ax$ for some $x \in S_\alpha$. Again since $\psi_{\alpha,\beta}$ is a semigroup homomorphism we have that $\hat{g}(\psi_{\alpha,\beta}(h)s) = \psi_{\alpha,\beta}(a)\psi_{\alpha,\beta}(x) \in \psi_{\alpha,\beta}(a)S_\beta$. We now claim that $\widehat{fg}$ and $\hat{f}\hat{g}$ agree on $\psi_{\alpha,\beta}(h)S_\beta$. Let $\psi_{\alpha,\beta}(h)s \in \psi_{\alpha,\beta}(h)S_\beta$ then $\widehat{fg}(\psi_{\alpha,\beta}(h)s) = \psi_{\alpha,\beta}(fg(h))s = \psi_{\alpha,\beta}(f(g(h)))s = \hat{f}(\psi_{\alpha,\beta}(g(h)))s = \hat{f}(\psi_{\alpha,\beta}(g(h))s) = \hat{f}(\hat{g}(\psi_{\alpha,\beta}(h))s) = \hat{f}\hat{g}(\psi_{\alpha,\beta}(h)s)$.

Finally, we show that if $\alpha > \beta > \delta$ then $\phi_{\beta,\delta}\phi_{\alpha,\beta} = \phi_{\alpha,\delta}$. Let $[f] \in Q(S_\alpha)$ with $f: aS_\alpha \rightarrow S_\alpha$ and let $\phi_{\alpha,\delta}([f]) = [\bar{f}] \in Q(S_\delta)$ where $\bar{f}: \psi_{\alpha,\delta}(a)S_\delta \rightarrow S_\delta$ defined by $\psi_{\alpha,\delta}(a)s \rightarrow \psi_{\alpha,\beta}(f(a))s$. Let $\phi_{\alpha,\beta}([f]) = [\hat{f}] \in Q(S_\beta)$ where $\hat{f}: \psi_{\alpha,\beta}(a)S_\beta \rightarrow S_\beta$ defined by $\psi_{\alpha,\beta}(a)t \rightarrow \psi_{\alpha,\beta}(f(a))t$. Hence $\phi_{\beta,\delta}(\phi_{\alpha,\beta}([f])) = \phi_{\beta,\delta}([\hat{f}]) = [\tilde{f}]$ where $\tilde{f}: \psi_{\beta,\delta}(\psi_{\alpha,\beta}(a))S_\delta \rightarrow S_\delta$ is defined by $\psi_{\beta,\delta}(\psi_{\alpha,\beta}(a))s \rightarrow \psi_{\beta,\delta}(\hat{f}(\psi_{\alpha,\beta}(a)))s$. To see that $\tilde{f} = \bar{f}$, we note that $\psi_{\beta,\delta}\psi_{\alpha,\beta} = \psi_{\alpha,\delta}$ so $\psi_{\alpha,\delta}(a)S_\delta = \psi_{\beta,\delta}(\psi_{\alpha,\beta}(a))S_\delta$. Hence if $\psi_{\beta,\delta}(\psi_{\alpha,\beta}(a))s \in \psi_{\beta,\delta}(\psi_{\alpha,\beta}(a))S_\delta$ then $\tilde{f}(\psi_{\beta,\delta}(\psi_{\alpha,\beta}(a))s) = \psi_{\beta,\delta}(\hat{f}(\psi_{\alpha,\beta}(a)))s = \psi_{\beta,\delta}(\psi_{\alpha,\beta}(f(a)))s =$

$\psi_{\beta,\delta}\psi_{\alpha,\beta}(f(a))s = \psi_{\alpha,\delta}(f(a))s$.

THEOREM 3.3. *Under the hypothesis of Theorem* 3.2, *S can be embedded into T.*

Proof. Define $\Phi: S \to T$ by $s \to [\lambda_s]$ where if $s \in S_\alpha$ then $[\lambda_s]_\alpha \in Q(S_\alpha)$ and $\lambda_s: S_\alpha \to S_\alpha$ is defined by $t \to st$. The mapping Φ is one-to-one for suppose $\Phi(s) = \Phi(r)$ where $s \in S_\alpha$ and $r \in S_\beta$.

Case 1. If $\alpha \neq \beta$ then $\Phi(s) \neq \Phi(r)$ since $Q(S_\alpha) \cap Q(S_\beta) = \varnothing$.

Case 2. If $\alpha = \beta$ then $[\lambda_s]_\alpha = [\lambda_r]_\alpha$ and so λ_s and λ_r agree on a dense right ideal of $S_{\alpha'}$ say D. Hence for $d \in D$, $sd = \lambda_s(d) = \lambda_r(d) = rd$ and since S_α is right cancellative then $s = r$.

Next we show that Φ is a semigroup homomorphism. Let $x \in S_\alpha$, $y \in S_\beta$ then $\Phi(x*y) = [\lambda_{x*y}]_{\alpha\beta}$ where $\lambda_{x*y}: S_{\alpha\beta} \to S_{\alpha\beta}$ defined by $s \to (x*y)s = \psi_{\alpha,\alpha\beta}(x)\psi_{\beta,\alpha\beta}(y)s$. Now $\Phi(x)\Phi(y) = [y_x]_\alpha[\lambda_y]_\beta = \phi_{\alpha,\alpha\beta}([\lambda_x]_\alpha)\phi_{\beta,\alpha\beta}([\lambda_y]_\beta) = [\hat{f}][\hat{g}] = [\hat{f}\hat{g}]$ where $[\hat{f}]$, $[\hat{g}] \in Q(S_{\alpha\beta})$ and $\hat{f}: S_{\alpha\beta} \to S_{\alpha\beta}$ defined by $s \to \psi_{\alpha,\alpha\beta}(x)s$ and $\hat{g}: S_{\alpha\beta} \to S_{\alpha\beta}$ defined by $s \to \psi_{\beta,\beta\alpha}(y)s$. If $s \in S_{\alpha\beta}$ then $\hat{f}\hat{g}(s) = \hat{f}(\hat{g}(s)) = \hat{f}(\psi_{\beta,\alpha\beta}(y)s) = \hat{f}(\psi_{\beta,\alpha\beta}(y))s = \psi_{\alpha,\alpha\beta}(x)\psi_{\beta,\alpha\beta}(y)s = \lambda_{x*y}(s)$.

We identify S with its image in T and note that if S is right nonsingular we have the diagram

$$\begin{array}{ccc} T & \longrightarrow & T/\psi_T \\ \cup\!\!\shortparallel & & \cup\!\!\shortparallel \\ S & = & S/\psi_s \, . \end{array}$$

THEOREM 3.4. *Let* $R = T/\psi_T$. *Under the hypothesis of Theorem* 3.2 *and if S is right nonsingular then* $\psi_R = i_R$.

Proof. Suppose that $t_1^*\psi_R t_2^*$. Let $t_1 \in t_1^*$ and $t_2 \in t_2^*$ then $(t_1d)\psi_T(t_2d)$ for all $d \in D$ a dense right ideal of S. Hence for each $d \in D$ there exists X_d dense in S such that $t_1dx = t_2dx$ for all $x \in X_d$. Let $W = \bigcup_{d \in D} dX_d$, then $t_1w = t_2w$ for all $w \in W$. If W is dense in S then $t_1\psi_T t_2$ and so $t_1^* = t_2^*$. To see that W is dense in S, we let $s_1 \neq s_2$, $s_3 \in S$. Since D is dense then there exists $d \in D$ such that $s_1d \neq s_2d$ and $s_3d \in D$. Since X_{s_3d} is dense then there exists $x \in X_{s_3d}$ such that $(s_1d)x \neq (s_2d)x$ and $(s_3d)x \in (s_3d)X_{s_3d}$. But then $s_1(dx) \neq s_2(dx)$ and $s_3(dx) \in W$. Since $dx \in D$ and X_{dx} is dense there exists $y \in X_{dx}$ such that $s_1((dx)y) \neq s_2((dx)y)$ and $s_3((dx)y) \in X_{dx}$. But W is a right ideal so $s_3((dx)y) \in W$ with $(dx)y \in W$. This shows that W is dense in S.

A *right Ore semigroup* is a right cancellative semigroup all of whose nonzero right ideals are $\cap$-large. The maximal right quotient semigroup of a right Ore semigroup R is a group $Q(R) = \{ab^{-1} \mid a, b \in R\}$[2].

THEOREM 3.5. *Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a strong semilattice Y of principal right ideal Ore monoids S_α with $\psi_{\alpha,\beta}$ onto for $\alpha \geqq \beta \in Y$. If S is right nonsingular then $Q(S)$ is a semilattice of groups.*

Proof. By Theorem 3.2, $T = \bigcup_{\alpha \in Y} Q(S_\alpha)$ is a strong semilattice and since each $Q(S_\alpha)$ is a group then T is a semilattice Y of groups $Q(S_\alpha)$ and so regular with idempotents in the center of T [1, pp. 128–129]. Hence T/ψ_T is regular and its idempotents are in the center of T/ψ_T, which makes T/ψ_T a semilattice of groups. McMorris [6] showed that $Q(T/\psi_T)$ is also a semilattice of groups. By Theorem 3.4, $Q(S) \approx Q(T/\psi_T)$ and so is a semilattice of groups.

THEOREM 3.6. *Under the hypothesis of Theorem 3.5, T/ψ_T can be taken to be the union of the same semilattice Y of groups.*

Proof. Since $T = \bigcup_{\alpha \in Y} Q(S_\alpha)$ where each $Q(S_\alpha)$ is a group, we let $e_\alpha = [e_\alpha] \in Q(S_\alpha)$. If $e_\alpha \psi_T e_\beta$ when $\alpha \neq \beta$ then $e_\alpha * x = e_\beta * x$ for all $x \in L$ an $\cap$-large right ideal of S. Since S is right nonsingular then Y is right nonsingular by Theorem 2.1. Furthermore, $A = \{\sigma \in Y | L \cap S_\sigma\} \neq \varnothing$ is dense in Y. Hence since $\alpha \neq \beta$ there exists $\delta \in A$ such that $\alpha\delta \neq \beta\delta$. Let $t \in L \cap S_\delta$ then $e_\alpha * t = e_\beta * t$ which implies that $e_{\alpha\delta}\psi_{\delta,\alpha\delta}(t) = e_{\beta\delta}\psi_{\delta,\beta\delta}(t)$ or that $\phi_{\delta,\alpha\delta}(t) = \phi_{\delta,\beta\delta}(t)$. This is a contradiction since for $\alpha\delta \neq \beta\delta$, $Q(S_{\alpha\delta}) \cap Q(S_{\beta\delta}) \neq \varnothing$. Hence $e_\alpha\psi_T \neq e_\beta\psi_T$ when $\alpha \neq \beta$. Thus in T/ψ_T there are at least as many idempotents as there are in T. Now suppose that $g\psi_T$ is an idempotent of T/ψ_T. Since $g \in Q(S_\alpha)$ a group then $g\psi_T \in Q(S_\alpha)/\psi_T$, also a group. The only idempotent of $Q(S_\alpha)/\psi_T$ is $e_\alpha\psi_T$ so $g\psi_T = e_\alpha\psi_T$. Hence in T/ψ_T there are no new idempotents.

Hinkle [2] showed that $Q(T/\psi_T)$ is a semilattice $Q(Y)$ of groups. Thus $Q(S)$ is a semilattice $Q(Y)$ of groups where Y is the semilattice of both S and T/ψ_T. The next theorem is a restatement of the above results.

THEOREM 3.7. *Let S be a strong semilattice Y of principal right ideal Ore monoids with onto linking homomorphisms. If S is right nonsingular then $Q(S)$ is a semilattice $Q(Y)$ of groups.*

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