PACIFIC JOURNAL OF MATHEMATICS
Vol. 71, No. 2, 1977

A NOTE ON THE GROUP STRUCTURE OF UNIT REGULAR RING ELEMENTS

ROBERT E. HARTWIG AND JIANG LUH

Local properties of unit regular ring elements are investigated. It is shown that an element of a ring R with unity is regular if and only if there exists a unit $u \in R$ and a group G such that $a \in uG$.

1. Introduction. It is well-known that [15, 7] a ring R is strongly regular if and only if every $a \in R$ is a group member. In this note we shall use the basic theorem for group members in a ring to show locally that a ring element $a \in R$ (with unity) is unit regular exactly when there is a unit $u \in R$ and a group G in R such that $a \in uG$. Hence unit regular rings are, as it were locally a "rotated" version of strongly regular rings.

We remind the reader that a ring R is called regular if for every $a \in R$, $a \in aRa$; strongly regular if for every $a \in R$, $a \in a^2R$, and unit regular if for every $a \in R$, there is a unit $u \in R$ such that $aua = a$ [3]. Similar definitions hold locally. A ring with unity is called finite if $ab = 1$ implies $ba = 1$. Any solution a^- to $axa = a$ is called an inner or 1-inverse of [1], while any solution a^+ to $axa = a$ and $xax = x$ is called a reflexive or 1-2 inverse of a.

For idempotents e and f in R, $e \sim f$ denotes the equivalence in the sense of Kaplansky [13] as contrasted with $a \overset{u}{\sim} b$ which denotes that $a = pbq$ with p and q invertible.

As usual, similarity will be denoted by $\approx$, the right and left annihilators of $a \in R$ will be denoted by $a^0 = \{x \in R: ax = 0\}$, ${}^0a = \{x \in R: xa = 0\}$ respectively, while interior direct sums and isomorphisms are denoted by $\dotplus$ and $\cong$ respectively. A ring R is called faithful if $aR = (0)$ implies $a = 0$.

We shall make use of the following fundamental theorem for group members.

THEOREM 1. *Let S be a semigroup and $a \in S$. The following are equivalent.*

1. *a is a group member.*
2. *a has a group inverse $a^\#$ in S which satisfies $axa = a$, $xax = x$ and $ax = xa$.*
3. *a has a commutative inner inverse a^- which satisfies $axa = a$, and $ax = xa$.*
4. *$aS = eS$, $Sa = Se$ and $a \in eSe$ for some idempotent $e \in S$.*
5. *$a \in a^2S \cap Sa^2$.*

6. *$a \in a^- aSaa^=$ for some inner inverses a^-, $a^=$ in S.*
7. *$aS = a^+S$ for some reflexive inverse a^+ in S.*
7a. *$Sa = Sa^+$ for some reflexive inverse a^+ in S.*
8. *$aS = a^-aS$ for some inner inverse a^- in S.*
8a. *$Sa = Saa^-$ for some inner inverse a^- in S.*

If in addition $S = R$ is a faithful ring, these are equivalent to

9. $R = aR \dotplus a^0$.
9a. $R = Ra \dotplus {}^0a$.

In any of the above cases $a^\#$ and $e = aa^\#$ are unique and the maximal subgroup containing a is given by

$$(1.1)\qquad \begin{aligned} H_a &= \{x \in S\colon x^\# \text{ exists, } xx^\# = aa^\# = e\} \\ &= \{x \in S\colon xS = aS,\ Sx = Sa,\ x \in aSa\}\ . \end{aligned}$$

Proof. For a proof of the equivalence of (1)—(5); we refer to [14, 7, 8].

(1) $\Rightarrow$ (6): Clearly, $a = a^\# a^3 a^\#$.

(6) $\Rightarrow$ (7): Let $a = a^-azaa^=$ for some $z \in S$ and set $a^+ = a^-aa^=$. Then $a = a^-aa^=azaa^= = a^+azaa^= \in a^+S$.

On the other hand, since $a^3 = a(a^-azaa^=)a = aza$, we have $a = a^-a^3a^=$, and $a^3a^= = a^2 = a^-a^3$. Hence $a^+ = a^-aa^= = a^-(a^-a^3a^=)a^= = a^-a^2a^=a^= = a^-a^3(a^=)^3 = a(a^=)^2 \in aS$, and so $a^+S = aS$.

(7) $\Rightarrow$ (8): Obvious, since $a^+S = a^+aS$.

(8) $\Rightarrow$ (1): If $aS = a^-aS$, then $a^2 = a^-ax$ for some x. Hence $a^-aa^2 = a^2$ or $a^-a^3 = a^2$.

Similarly, $a^-a = ay$ for some y, and so $a = a^2y$. By a result of Drazin [2] the index of a equals one and $a^\#$ exists.

The results 7a and 8a follow by symmetry.

We remark that an element $a \in R$ for which $aR = a^+R$ or $Ra = Ra^+$ for some a^+, generalizes so called EP elements [16, 7, 1] for which $aR = a^-R = a^\dagger R$, R $*$-regular, where $a^\dagger$ is the Moore-Penrose inverse of a. Thus in a $*$-regular ring, an EP element belongs to some group G.

For a proof of (9) $\Rightarrow$ (1) for the case where R has a unity 1 or is regular, we refer to [7]. When R is faithful we have to proceed as follows. $R = aR \dotplus a^0 \Rightarrow a = ar + n,\ an = 0 \Rightarrow a = a(as + m) + n$, for some $s \in R$, $m \in a^0$. Hence $a^2 = a^4b$, for some $b \in aR$. Also $a(ax) = 0 \Rightarrow ax \in aR \cap a^0 = (0)$, so that $(a^2)^0 = a^0$. Hence $R = a^2R \dotplus (a^2)^0$. It then follows that $b = (a^2)^\#$, since

$$a^2(a^2 - a^2ba^2) = a^2(a^2b - ba^2) = a^2(ba^2b - b) = 0\ .$$

Because a^2 commutes with a, it follows by a result of Drazin [2] that $(a^2)^\#a = a(a^2)^\#$. Now $(a - a^2(a^2)^\#a)R = (a - a^2(a^2)^\#a)aR = (0)$ and hence if R is faithful, $a = a^2(a^2)^\# a = a^2(a(a^2)^\#)$. One may now repeat the above argument to show that $a^\# = a(a^2)^\# = a^\# a a^\#$.

That $(1) \Rightarrow (9)$ is clear.

Before giving our main result several remarks should be made here.

REMARK 1. The condition "faithful" may be replaced by the weaker condition

$$\text{(1.2)} \qquad \text{for every} \quad r \in R \quad r^0 \cap {}^0R = (0) .$$

This may not be dropped entirely as seen from the example

$$R = \left\{\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} : \alpha \text{ is a real number}\right\},\ a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ aR = (0),\ a^0 = R$$

and ${}^0R = R$. Here $R = aR + a^0$, yet $a^\#$ clearly does not exist since $a^2 = 0$.

REMARK 2. For a regular ring R with unity, (1.1) may be written as [10]

$$\text{(1.3)} \qquad H_a = \{x \in R : x = pa = aq \quad \text{for some units } p \text{ and } q\} .$$

REMARK 3. If a has a unique reflexive inverse a^+ then $a^\#$ exists, and if a has a unique idempotent of the form aa^+ then $a \in a^2R$. Hence if either of them hold globally, then R is stronly regular. These results are easy consequences of the fact that the class $\{a^+\}$ of *all* reflexive inverses of a is given by [9],

$$[a^+ + (1 - a^+a)R]a[a^+ + R(1 - aa^+)] .$$

2. Main results.

We begin with several preliminary results which will be used in our main theorem.

LEMMA 1. *If R is a ring with unity 1, and if $\phi: aR \to bR$ is a module isomorphism, where a and $p = \phi(a)$ are regular elements, then $Ra = Rp$ and $pR = bR$.*

Proof. $\phi(a) = \phi(aa^-a) = \phi(aa^-)a$ and $\phi(a) = pp^-p \Rightarrow a = \phi^{-1}(pp^-)p = \phi^{-1}(pp^-)\phi(a)$. The following is given in [10].

LEMMA 2. *If a and b are regular elements in a ring R with unity 1, then*

$$aR = bR \quad \textit{and} \quad Ra = Rb \Longleftrightarrow b = ua = av$$

for some units u, v in R.

LEMMA 3. *Let R be a ring with unity* 1 *and a and b be regular elements in R. Then the following are equivalent*:

(i) $b \overset{u}{\sim} a$;

(ii) $aa^- \approx bb^-$ *and* $a^-a \approx b^-b$, *for some, and hence all* a^-, b^-;

(iii) $aa^- \sim bb^-, 1 - aa^- \sim 1 - bb^-$, *and* $1 - a^-a \sim 1 - b^-b$, *for some and hence all* a^-, b^-;

(iv) $aR \cong bR$ *and* $R/aR \cong R/bR$, $R/Ra \cong R/Rb$.

Proof. (i) $\Rightarrow$ (ii): If $b = paq$, for some units p and q, then for any particular a^-, $q^{-1}a^-p^{-1} \in \{b^-\}$, and hence $paa^-p^{-1} \in \{bb^-\}$, $q^{-1}a^-aq \in \{b^-b\}$. Now for any $a^= \in \{a^-\}$, $b^= \in \{b^-\}$, $aa^= \sim aa^-$, $bb^= \sim bb^-$ and thus $aa^= \approx aa^- \approx paa^{-1}p^{-1} \approx bb^=$.

(ii) $\Rightarrow$ (i): Let $aa^- = ubb^-u^{-1}$, $a^-a = v^{-1}b^-bv$. Then $aR = ubvR$, $Ra = Rubv$. Lemma 2 now ensures that $a = ubvp = qubv$ for some units p, q and thus $a \overset{u}{\sim} b$.

The equivalence of (ii) and (iii) is well-known since $aa^- \approx bb^- \Leftrightarrow aa^- \sim bb^-$ and $1 - aa^- \sim 1 - bb^-$, while $aa^- \sim bb^- \Leftrightarrow a^-a \sim b^-b$, [11].

(i) $\Rightarrow$ (iv): If $b = paq$ where p and q are units, then $aR \cong bR$ and $1 - bb^- = p(1 - aa^-)p^{-1} \Rightarrow (1 - bb^-)R = p(1 - aa^-)R \cong (1 - aa^-)R$. Lastly, since $bR \dotplus (1 - bb^-)R = R = aR \dotplus (1 - aa^-)R \Rightarrow R/aR \cong (1 - aa^-)R$ and $R/bR \cong (1 - bb^-)R$, the results follows.

(iv) $\Rightarrow$ (ii): If $aR \cong bR$ and $R/aR \cong R/bR$, then $(1 - aa^-)R \cong R/aR \cong R/bR \cong (1 - bb^-)R$ and so $aa^- \sim bb^-, 1 - aa^- \sim 1 - bb^-$. It follows that $aa^- \approx bb^-$. Similarly, $a^-a \approx b^-b$.

We note in passing that the statement $R/aR \cong R/bR$ is clearly equivalent to the statement "aR and bR have all direct summands isomorphic."

LEMMA 4. *If $a \in R$ is a regular element of R and $1 \in R$, then for all units $u, v \in R$, $\{(uav)^-\} = v^{-1}\{a^-\}u^{-1}$.*

Proof. This is an easy consequence of the fact that the class of *all* inner inverses of b is given by $\{b^-\} = b^- + (1 - b^-b)R + R(1 - bb^-)$.

We now come to the main theorem of this paper, which gives numerous conditions for a ring element to be unit regular.

THEOREM 2A. *Let R be a ring with unity* 1 *and let $a \in R$. Then the following are equivalent*:

1. $aua = a$ *for some unit u in R.*

2. $(au)^\#$ *exists for some unit* u *in* R.

2a. $(ua)^\#$ *exists for some unit* u *in* R.

3. au *has a commutative inner inverse for some unit* u *in* R.

3a. ua *has a commutative inner inverse for some unit* u *in* R.

4. $auR = eR$ *and* $Rau = Re$ *for some unit* u *and idempotent* e *in* R.

4a. $uaR = eR$ *and* $Rua = Re$ *for some unit* u *and idempotent* e *in* R.

5. $a \in auaR \cap Raua$ *for some unit* u *in* R.

6. $R = aR \dotplus u(a^0)$ *for some unit* u *in* R.

6a. $R = Ra \dotplus ({}^0a)u$ *for some unit* u *in* R.

Proof. (1) $\Rightarrow$ (2): Clearly, $aua = a \Rightarrow (au)^2 = au \Rightarrow (au)^\#$ exists.

(2) $\Rightarrow$ (1): Observe that $au[(au)^\# + (1-(au)^\# au)]au = au \Rightarrow auva = a$, where $v = (au)^\# + (1-(au)^\# au)$ and $v^{-1} = au + 1 - (au)^\# au$.

(2) $\Leftrightarrow$ (2a): $ua = u(au)u^{-1}$ and so $(ua)^\#$ exists exactly when $(au)^\#$ exists.

Since idempotents clearly are group members, it is obvious that a is unit regular precisely when $a \in uG$ for some group G and unit u in R. The equivalence of (2) through (6a) follows immediately from Theorem 1, applied to the group members au, and ua. For example, $au \in (au)^2R \cap R(au)^2 \Leftrightarrow a \in auaR \cap Raua$ and $(ua)^\#$ exists $\Leftrightarrow R = uaR \dotplus (ua)^0 \Leftrightarrow R = aR \dotplus u^{-1}(a^0)$. If we are given in addition that $a \in R$ is a *regular* element, then several important additional conditions may be given for a to be *unit* regular.

THEOREM 2B. *If* R *is a ring with unity* 1 *and* $a \in R$ *is a regular element, then the following are equivalent to a being unit regular.*

(7) $a \in u^{-1}a^- aRaa^= u^{-1}$ *for some unit* u *and some inner inverses* a^-, $a^=$ *in* R.

(8) $a^- xa = y$, $aya^= = x$, *where* a^-, $a^=$ *are inner inverses of* $a \Rightarrow x \approx y$.

(9) $ca = ac$, $c \in R \Rightarrow caa^= \approx a^- ac$ *for some and hence all inner inverses* a^-, $a^=$ *in* R.

(10) $aa^- \approx a^- a$ *for some and hence all inner inverses* a^- *in* R.

(11) $aR = ua^- aR$ *for some unit* u *and some inner inverse* a^- *in* R.

(12) $aR = ua^+R$ *for some unit* u *and some reflexive inverse* a^+ *in* R.

(13) $aR = eR$, *with* $e^2 = e \Rightarrow au = e$ *for some unit* u *in* R.

(14) $aR = bR$ *with* b *unit regular* $\Rightarrow ag = b$ *for some unit* g *in* R.

(15) $aR \overset{\phi}{\cong} bR$, *with* $\phi(a)$, b *unit regular* $\Rightarrow a \overset{u}{\sim} b$.

(16) $aR \overset{\phi}{\cong} bR$, *with* $\phi(a)$, *bunit regular* $\Rightarrow R/aR \cong R/bR$,

together with their left analogues.

Proof. (2) $\Leftrightarrow$ (7): By Theorem 1(6), $(au)^{\#}$

$$\text{exists} \Longleftrightarrow au \in (au)^{-}auRau(au)^{=} \Longleftrightarrow au \in u^{-1}a^{-}aRaa^{=} \Longleftrightarrow a \in u^{-1}a^{-}aRaa^{=}u^{-1}\,,$$

for some inner inverses a^{-}, $a^{=}$ of a. It should be noted that Lemma 4 was also used.

(1) $\Rightarrow$ (8): Let $aua = a$, where u is a unit. Then $y = a^{-}xa = a^{-}aya^{=}a \Rightarrow y = ya^{=}a = a^{-}ay = ya^{-}a$, and $x = aya^{=} = aa^{-}xaa^{=} \Rightarrow aa^{=}x = x = aa^{-}x = xaa^{=}$. Also, clearly, $ay = xa$ and $yua = ya^{=}aua = ya^{=}a = y$, $aux = x$. Now note that $y = a^{-}ay \approx uay$ since $uay(1 - a^{-}a + ua) = uay = (1 - a^{-}a + ua)a^{-}ay$ and so, $y = a^{-}ay \approx uay = uxa = u(xau)u^{-1}$. Next, again $xua \approx xaa^{=} = x$, for

$$(1 - aa^{=} + au)xau = xau = xaa^{=}(1 - aa^{=} + au)\,.$$

And so, $y = q^{-1}xq$, where $q = (1 - aa^{=} + au)u^{-1}(1 - a^{-}a + ua)$.

(8) $\Rightarrow$ (9): Since $a^{-}(caa^{=})a = a^{-}ac$ and $a(a^{-}ac)a^{=} = caa^{=}$, the result follows at once from (9).

(9) $\Rightarrow$ (10): Because $aa^{-} \approx aa^{=}$ for any $a^{-}, a^{=}$, we simply set $c = 1$ in (9).

(10) $\Rightarrow$ (11): $aa^{-} \approx a^{-}a \Rightarrow aa^{-} = ua^{-}au^{-1}$ for some unit $u \Rightarrow aR = ua^{-}aR$ as desired.

(11) $\Rightarrow$ (12): $aR = ua^{-}aa^{-}aR = ua^{+}aR = ua^{+}R$, where $a^{+} = a^{-}aa^{-}$.

(12) $\Rightarrow$ (2a): Let $aR = ua^{+}R$. Then $u^{-1}aR = a^{+}R = a^{+}uR = (u^{-1}a)^{+}R$, and hence by Theorem 1(7), $(u^{-1}a)^{\#}$ exists.

(1) $\Rightarrow$ (13): If $aR = eR$ and $aua = a$, u unit, $e^2 = e$, then $auR = eR \Rightarrow aue = e$. Hence $auv = e$, where $v = 1 - au + e$, $v^{-1} = 1 + au - e$. Thus a and e are right associates.

(13) $\Rightarrow$ (14): If $aR = bR$, $bvb = b$ and v is a unit, then $aR = eR$, where $e = bv$. By (13), $au = e = bv$ for some unit e. Hence $auv^{-1} = b$ as desired.

(14) $\Rightarrow$ (1): Since $aR = aa^{-}R$, and aa^{-} is unit regular, (14) implies that $ag = aa^{-}$ for some unit g. Hence $aga = a$ as requested. It is now clear by symmetry, that the *left* analogues of the above results also are equivalent to element a being unit regular.

(14) $\Rightarrow$ (15): Suppose that (14) and hence its left analogue (14a) both hold.

Now let $aR \cong \phi(a)R = bR$ and $p = \phi(a)$. Then by Lemma 1, $Ra = Rp$ and $pR = bR$, so that by (14) and (14a), $pv = b$ and $ua = p$ for some units u and v. These are in fact given by $u = (p^{=})^{-1}(1 + p^{=}p - a^{=}a)a^{=}$, $v = p^{=}(1 - pp^{=} + bb^{=})(b^{=})^{-1}$, in which $a^{=}$, $b^{=}$, and $p^{=}$ are unit inner inverses. Hence $b = uav$, as desired.

(15) $\Rightarrow$ (16): This follows immediately from Lemma 3.

(16) $\Rightarrow$ (1): Since $aR \overset{\phi}{\cong} a^-aR$, where $\phi(a) = a^-a = b$, it follows that $aa^- \sim b, 1 - aa^- \sim 1 - b$, so that $aa^- \approx b$.

Hence, by Lemma 3, $uav = b = a^-a$ for some units u, v, which implies that $uavuav = uav$ *or* $a(vu)a = a$, as desired. Alternatively, (10) could be used.

The remaining results follow again by symmetry.

REMARK 1. In (8), we proved the conjecture made in [12] that pseudosimilarity implies similarity in a unit regular ring. Pseudo-similarity, $\overline{\sim}$, is defined by

DEFINITION 1. $x \overline{\sim} y$ if $a^-xa = y, aya^= = x$ for some a and its inner inverses $a^-, a^=$.

REMARK 2. The equivalence of (1) and (6) was also proved by Ehrlich [4] who used endomorphism rings. As shown above it is actually a simple consequence of the fundamental Theorem 1.

REMARK 3. Part (10) should be compared with the *global* result of Vidav [17] and Fuchs [5], which state that a regular ring R is unit regular exactly when $e^2 = e \sim f = f^2 \Rightarrow e \approx f$ [17] or when $aR \cong bR \Rightarrow R/aR \cong R/bR$ [5].

REMARK 4. The global analogue of (16) is that a regular ring R is unit regular exactly when $aR \cong bR$ implies that aR and bR have a *common* direct summand [6].

One final remark is here needed, namely, if R is a unit regular ring and if $\phi: aR \rightarrow bR$ is *any* isomorphism, then, by Lemma 1, $Ra = R\phi(a)$ and hence by (14a) $\phi(a) = ua$ for some unit u.

We have thus shown:

COROLLARY 1. *In a unit regular ring R, all right module isomorphisms $\phi: aR \rightarrow bR$, are of the form $\phi(ar) = uar$, where u is a unit. Similarly, all left module isomorphisms $\phi: Ra \rightarrow Rb$ are of the form $\phi(ra) = rav$, for some unit $v \in R$.*

The converse of these statements always hold.

3. The unit inner inverses. We shall now examine more closely the class $\mathscr{U}_a$ of unit inner inverses of a given element a of a unit regular ring.

We begin by noting that if $aua = a$, with u invertible then $\mathscr{U}_a$ can be represented as

$$\mathscr{U}_a = u\mathscr{U}_{au} = \mathscr{U}_{ua}u \ . \tag{3.1}$$

Indeed, if $w \in \mathscr{U}_{au}$, then $auwau = au$ which implies that $auwa = a$ and hence $uw \in \mathscr{U}_a$, while conversely, if $awa = a$, w a unit, then $au(u^{-1}w)au = au$ which implies that $u^{-1}w \in \mathscr{U}_{au}$ and hence $w \in u\mathscr{U}_{au}$. The second identity follows similarly.

Since $u\mathscr{U}_{au}$ is independent of the choice of the unit inner inverse u of a, we have, for any unit inner inverses u and v of a,

$$\mathscr{U}_a = u\mathscr{U}_{au} = v\mathscr{U}_{av} \ , \tag{3.2}$$

so that in particular, $u^{-1}v \in \mathscr{U}_{au}$.

Consequently, the set $\mathscr{U}_a$ is determined by the set of unit inner inverses $\mathscr{U}_e$ of the idempotent element $e = au$. When $e^2 = e$, there are several representations for $\mathscr{U}_e$. In fact, $\mathscr{U}_e$ is the set of *all* units of the form:

$$\begin{array}{lll} \text{(i)} & 1 + (1-e)x + y(1-e) & \text{for some } x, y; \\ \text{(ii)} & e + (1-e)v + s(1-e) & \text{for fome } v, s; \\ \text{(iii)} & 1 + h - ehe & \text{for some } h; \\ \text{(iv)} & e + k - eke & \text{for some } k. \end{array} \tag{3.3}$$

In general, the set $\mathscr{U}_a$ or even $\mathscr{U}_e$ will not be a union of semigroups. For example, if $e = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in \boldsymbol{R}_{2\times 2}$, where $\boldsymbol{R}_{2\times 2}$ denotes the two by two matrix ring over the real field, then it is easy to see that $\begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix} \in \mathscr{U}_e$, but $\begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}^2 \notin \mathscr{U}_e$.

In fact, it is only for idempotent elements possible to possess union of semigroups of unit inner inverses.

PROPOSITION 3. *Let a be a unit regular element of a ring R with unity* 1.

(i) *If the set $\mathscr{U}_a$ of unit inner inverses of a is a union of semigroups then $a^2 = a$.*

(ii) *If R is a prime ring and if $\mathscr{U}_a$ forms a semigroup, then $a = 0$ or $a = 1$.*

Proof. (i) Let $aua = a$ with u a unit. Then $u^2 \in \mathscr{U}_a$ and $au^2a = a$. Now consider: $au(1 + au(1 - au))a = aua + au(1 - au)a = a + a - a = a$, which implies that $u(1 + a(1 - au)) \in \mathscr{U}_a$. Thus $(u(1 - a(1 - au)))^2 \in \mathscr{U}_a$. That is, $a = a(u(1 - a(1 - au)))^2a = (au - a(1 - au))u(1 - a(1 - au))a = (au^2 - au + a^2u^2)(a - a^2 + a^2) = au^2a - aua + a^2u^2a = a - a + a^2 = a^2$.

(ii) Now suppose that $a = e = e^2$. Then clearly $1 + eR(1 - e)$ and $1 + (1 - e)Re$ are contained in $\mathscr{U}_e$. Hence by the semigroup assumption, $e(1 + eR(1 - e))(1 + (1 - e)Re)e = e$ which implies that

$$eR(1 - e)Re = 0 . \tag{3.4}$$

Since R is prime, it follows that either $e = 0$ or $e = 1$ as desired.

REMARK 1. In (ii), the primeness cannot be dropped as seen from the example of semiprime ring $R = Z_2 \oplus Z_2$, where Z_2 denotes the Galois field of order 2. Here $\mathscr{U}_{(1,0)} = \mathscr{U}_{(0,1)} = \{(1, 1)\}$ is a semigroup, yet $(1, 0)$ and $(0, 1)$ are neither zero element nor unity element.

REMARK 2. The *same* conclusions may be drawn if the element is just regular and the set $\{a^-\}$ of inner inverses forms a semigroup. In fact, if $aba = a$ then $ab^2a = a$ and also $a(b - ba + ba^2b)a = a \Rightarrow a(b - ba + ba^2b)ba = a \Rightarrow a = a - aba + a^2b^2a = a^2$.

The rest follows as in part (ii).

REMARK 3. For an invertible element $1 + h - ehe$ in a unit regular ring, ehe need not lie in H_e. For example, if $e = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in \boldsymbol{R}_{2\times 2}$ and $h = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then $1 + h - ehe = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is invertible but $ehe = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin H_e$.

There are five sets of units that appear naturally in the study of $\mathscr{U}_e$. These are:

1. $P_e = 1 + (1 - e)Re = \{u \in \mathscr{U}_e: e(1 - u)(1 - e) = 0\}$,
2. $Q_e = 1 + eR(1 - e) = \{u \in \mathscr{U}_e: (1 - e)(1 - u)e = 0\}$,
3. $V_e = \{v \in \mathscr{U}_e: ev = e\}$,
4. $W_e = \{w \in \mathscr{U}_e: we = e\}$, and
5. $C_e = \{z \in R: ez = ze, z \text{ is a unit}\}$.

For example, $1 - aa^= + aa^{\equiv} \in W_{aa^-}$ for any inner inverses $a^-, a^=, a^{\equiv}$ of a.

It is easily seen that

(i) all these sets are semigroups (in fact monoids).

(ii) $P_e \subseteq V_e \subseteq \mathscr{U}_e$, $Q_e \subseteq W_e \subseteq \mathscr{U}_e$, $V_e \cap W_e = \{1 + (1 - e)x(1 - e) \in \mathscr{U}_e: x \in R\}$.

(iii) $P_e \cap Q_e \subseteq V_e \cap W_e = \mathscr{U}_e \cap C_e \subseteq C_e$.

In addition it is known that [14]

(iv) $eC_e = H_e$ is the maximal subgroup containing e.

Moreover, it is easily shown that

(v) $V_e \mathscr{U}_e W_e = \mathscr{U}_e = P_e \mathscr{U}_e Q_e$, for let $u \in \mathscr{U}_e$, $v \in V_e$, $w \in W_e$, then $evuwe = eue = e$, while conversely $u = 1 \cdot u \cdot 1$ ensures the first equality. The second equality follows similarly.

It should be remarked here that in general $P_e \neq V_e$, $Q_e \neq W_e$, for again let $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}$ in $\boldsymbol{R}_{2\times 2}$ with $x_1 \neq 0$ and $1 + x_1$ invertible. Then $1 + (1 - e)x = \begin{bmatrix} 1 + x_1 & x_3 \\ 0 & 1 \end{bmatrix} \in V_e$, while $\begin{bmatrix} x_1 & x_3 \\ 0 & 0 \end{bmatrix} \neq ye$ for any $y \in \boldsymbol{R}_{2\times 2}$.

Before examining the subgroup H_e, let us first prove a global conjecture made in [11]. We start with

LEMMA 4. *Let R be a ring with unity 1. Then the following two conditions are equivalent.*

(i) *R is unit regular such that every nonzero element in R has a unique inner inverse;*

(ii) *R contains only idempotent elements and invertible elements.*

Proof. (i) $\Rightarrow$ (ii): Suppose $a^2 \neq a \in R$ and $aua = a$, u a unit $u \neq 1$. Then

$$au(1 - a(1 - au))a = a = a(1 - (1 - ua)a)ua$$

where $(1 - a(1 - au))^{-1} = 1 + a(1 - au)$ and

$$(1 - (1 - ua)a)^{-1} = 1 + (1 - ua)a \ .$$

Hence by uniqueness, $u(1 - a(1 - au)) = u = (1 - (1 - ua)a)u$ or $a(1 - au) = 0 = (1 - ua)a$. Now $a^2u = a = ua^2$ implies by Theorem 1, that a has a group inverse $a^\# = uau$. Consequently, $au = aa^\# = a^\# a = ua$. Since $a(a^\# + 1 - aa^\#)a = a$ and $(a^\# + 1 - aa^\#)^{-1} = a + 1 - aa^\#$, it follows by uniqueness that $u = uau + 1 - au$ or $u(1 - au) = 1 - au$. Multiplying this by $1 - au$, we obtain

$$(1 - au)u(1 - au) = 1 - au \ . \tag{3.7}$$

Now either $1 - au = 0$ or $1 - au \neq 0$. Since $1 - au \neq 0$ is idempotent and $(1 - au)1(1 - au) = 1 - au$, uniqueness implies that $u = 1$, which is impossible. Hence $au = 1 = ua$ and a is a unit.

(ii) $\Rightarrow$ (i): It is clear that R is a regular ring. Now let $a \in R$ and $a \neq 0$. First suppose $a = 1$. Then $aua = a$ implies that $u = 1$ and so is unique. Next, suppose $a \neq 1$. If $a^2 = a$ and $aua = a$, where u is a unit $\neq 1$, then $1 - u$ is also a unit. For otherwise $(1 - u)^2 = 1 - u$ would imply that $u^2 = u$ which forces u to equal 1. Now, since a is not a unit, $a(1 - u)$ is not a unit. Hence $[a(1 - u)]^2 = a(1 - u)$. This implies that $a = a(1 - u)a = a^2 - aua = a^2 - a = 0$, a contradiction. Hence $u = 1$ and the unit inner inverse of a is unique. If a is a not idempotent then a is a unit and clearly a^{-1} is the only unit inner inverse of a, completing the proof.

We may now sharpen this to the following.

THEOREM 4. *Let R be a unit regular ring. If every nonzero element of R has a unique unit inner inverse then either R is a Boolean ring or R is a division ring.*

Proof. Suppose R is neither Boolean nor a division ring. Then there exists $a \in R$ such that $a^2 \neq a$ and there are $x \neq 0, y \neq 0$ in R such that $xy = 0$, (since it is well-known that a regular integral domain must be a field). By Lemma 4, a is a unit and x and y are idempotents. Now, consider element ax. If $(ax)^2 = ax$ then

$$a(xa-1)x = 0 \Longrightarrow (xa-1)x = 0 \Longrightarrow x = xax \Longrightarrow a = 1 ,$$

by the uniqueness of unit inner inverses of x. This yields a contradiction. On the other hand, if $(ax)^2 \neq ax$ then ax must be a unit which implies that x is a unit and thus that $y = 0$, which again is a contradiction. Thus R must be either a division ring or a Boolean ring.

Let us now consider briefly the maximal subgroup

$$H_e = \{x \in R: xR = eR, Rx = Re\}$$

which contains the idempotent element $e \in R$. We begin with a global result.

PROPOSITION 5. *If R is a regular ring with unity* 1 *and e is an idempotent element in R, then*

$$H_e = \{eue: eueve = e = eveue, u, v \text{ units in } R\} . \tag{3.8}$$

This says that the e-units in eRe are all of the form eue for some 1*-unit $u \in R$.*

Proof. It is well-known that

$$\begin{aligned} H_e &= \{ere: erese = e = esere; r, s \in R\} \\ &= \{ere: ereR = eR, Rere = Re\} . \end{aligned}$$

By Lemma 3, for $ere \in H_e$ there are units u, v in R such that $ereu = e = vere$, which implies that $(ere)(eue) = e = (eve)(ere)$. The uniqueness of e-inverses ensures that $eue = eve$.

Now again by Lemma 3, since $eueR = eR$ *and* $Reue = Re$, there are units w, z in R, such that $euew = e = zeue$. Consequently, $euewe = e = ezeue$. And so, by uniqueness, $ewe = eze = ere$. Hence we may replace in each element ere the element r by a 1-unit $w \in R$.

Conversely, it is easily seen that this set is contained in H_e.

We remark that when R is a finite regular ring [11] we may shorten this to

$$H_e = \{eue\colon eueve = e;\ u, v \text{ units in } R\}\,. \tag{3.9}$$

Suppose now again that $aua = a = ava$, with u, v units in R. Then if we set $e = au$, $f = av$, we have $a \in H_{au}u^{-1}$, and more generally, $a \in \bigcap \{H_{au}u^{-1}\colon u \in \mathscr{U}_a\}$. Since $eR = fR = aR$, it follows that $ef = f$, $fe = e$ and that $e \approx f$. In fact, if $w = 1-e+f = (1+e-f)^{-1} = 1 - a(u - v)$, then $ew = wf = f$ and thus

$$wH_f w^{-1} = H_e\ , \tag{3.10}$$

that is, the subgroups H_{au} and H_{av} are isomorphic. It follows similarly that

$$H_{ua} = uH_{au}u^{-1}\ , \tag{3.11}$$

because $x \in H_{ua} \Leftrightarrow u^{-1}xu \in H_{au}$. And so, the subgroups H_{au}, H_{ua}, H_{av}, H_{va} are all isomorphic.

4. Conclusions.

We have seen that an element $a \in R$ is unit regular exactly when $a \in uG$ for some unit u and group G in R. In the same way that the concept of a Drazin inverse a^d(see [1, 2]) generalizes that of a group inverse $a^\#$ to the case that $(a^k)^\#$ exists for some $k \geqq 1$, we may generalize the concept of a unit regular element.

DEFINITION 2. (i) An element $a \in R$ is k-unit regular if a^k is unit regular for some $k \geqq 1$.

(ii) An element $a \in R$ is unit-Drazin invertible if there is a unit $u \in R$ such that $(ua)^k$ is a group member for some $k \geqq 1$.

By Theorem 2, the former is equivalent to $R = a^kR \dotplus u(a^k)^0$, while the latter reduces to the existence of $(ua)^d$.

In closing we mention of few open problems relating to $\mathscr{U}_a$ in a unit regular ring. Let e be an idempotent element.

1. For what h is $1 + h - ehe$ invertible?
2. For what x is $1 + (1 - e)x$ invertible?
3. How are $\mathscr{U}_e$ and H_e related?
4. What sort of subgroup is $\cap \{H_{au}\colon u \in \mathscr{U}_a\}$?
5. For what type of regular semigroups does Theorem 2, 1-2 remain valid?

REFERENCES

1. A. Ben Israel and T. N. E. Greville, *Generalized Inverse Theory and Applications*, Wiley, New York, 1974.
2. M. P. Drazin, *Pseudo-inverses in associative rings and semigroups*, Amer. Math. Monthly, **65** (1958), 506–514.
3. G. Ehrlich, *Unit regular rings*, Portugal Math., **27** (1969), 209–212.
4. ———, *Units and one-sided units in regular rings*, Trans. Amer. Math. Soc., **216** (1976), 81–90.
5. L. Fuchs, *On the substitution property for modules*, Monatsh. Math., (1971), 198–204.
6. D. Handelman, *Perspectivity and cancellation in regular rings*, to appear.
7. R. E. Hartwig, *Block generalized inverses*, Arch. Rath. Mech. Anal., **61** (1976), 187–251.
8. ———, *More on the Souriau-frame Algorithm and the Drazin inverse*, SIAM J. Appl. Math., **31** (1976), 42–46.
9. ———, 1–2 *Inverses and the invariance of* $BA^{+}C$, Linear Algebra Appl., **11** (1975), 271–275.
10. ———, *Generalized inverses*, Ep Elements and Associates, to appear, in Revue Roumaine.
11. R. E. Hartwig and Jiang Luh, *On finite regular rings*, Pacific J. Math., **69** (1977), 73–95.
12. R. E. Hartwig and F. Hall, *Pseudo-similarity for matrices over fields*, submitted for publication.
13. I. Kaplansky, *Rings of Operators*, W. A. Benjamin Inc., New York, 1968.
14. G. L. Losey and H. Schneider, *Group membership in rings and semigroups*, Pacific J. Math., **11** (1961), 1089–1098.
15. Jiang Luh, *A note on strongly regular rings*, Proc, Japan Acad., **40** (1964), 74–75.
16. H. Schwerdtfeger, *Introduction to Linear Algebra and the Theory of Matrices*, 2nd Ed., P. Noordhoff, Grominger, 1962.
17. I. Vidav, *Modules over regular rings*, Mathematica Balkamica, **1** (1971), 287–292.

Received August 17, 1976 and in revised form March 3, 1977.

UNIVERSITY OF GRAZ
GRAZ, AUSTRIA 8010
AND
NORTH CAROLINA STATE UNIVERSITY
RALEIGH, NC 27607