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ON INTEGRAL REPRESENTATIONS OF PIECEWISE HOLOMORPHIC FUNCTIONS

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Let D be the interior of the unit circle in C, D^c its exterior and T the unit circumference. We consider certain piecewise holomorphic functions that are holomorphic in D and also in D^c. This paper deals with those piecewise holomorphic functions that are representable by means of complex Poisson-Stieltjes integrals on T; we call this set of functions P_1. The set of all piecewise holomorphic functions (holomorphic in D and in D^c) we call P. Earlier work—see Rolf Nevanlinna, Eindeutige Analytische Funktionen, Springer, Berlin, 1953 and references there—dealt with positive (Herglotz-Riesz) or real (Nevanlinna) measures; we shall use here the entire space M of bounded complex Borel measures on T. This gives the theory more flexibility. We consider characterizations of functions in P representable by means of complex Poisson-Stieltjes integrals, uniqueness questions, the nature of the mapping between the subset P_1 of P of representable functions and M, as well as the ring structures in M (under convolution) and P_1 (Hadamard products), and questions of derivatives and integrals. We end with an application to Fourier-Stieltjes moments relative to measues in M.

We call a function $F \in P$ *representable* if there is a measure $m \in M$ so that $F = \int P_C dm + k$ where $P_C = P_C(z) = (e^{it} + z)/(e^{it} - z)$ is the complex Poisson kernel, k is a piecewise constant function in P, and where the limits of integration are omitted when they are 0 and 2π respectively. A function $F \in P$ is said to be of *real type* if $F(\bar{z}^{-1}) = -\overline{F(z)}$ for all $z \in D \cup D^c$. The functions

$$(1)\quad G = G_F(z) = \frac{1}{2}(F(z) - \overline{F(\bar{z}^{-1})}),\ H = H_F(z) = -\frac{1}{2}(iF(z) + i\overline{F(\bar{z}^{-1})})$$

are of real type; we have $F = G + iH$ and $F \in P_1$ if and only if G and H are in P_1.—The decomposition of the complex measure m into its real and imaginary parts is given by $m = (1/2(m + \bar{m})) + i((1/2i)(m - \bar{m})) = (\operatorname{Re} m) + i(\operatorname{Im} m)$ where $\bar{m}$ is defined as usual by $\int \bar{g}\, dm = \overline{\int g\, dm}$ for continuous functions g on T. If the representable function $F \in P_1$ is given by $F = \int P_C dm + k$, then $G_F = \int P_C d(\operatorname{Re} m) + 1/2(k - \bar{k})$ and $H_F = \int P_C d(\operatorname{Im} m) + (1/2i)(k + \bar{k})$. — If $m \in M$, we write $\hat{m}_j = \int e^{-ijt} dm$.

The following theorem characterizes the elements of P_1 among those of P.

THEOREM 1. *The function $F \in P$ is representable if and only if there is a constant B_F such that*

$$\int |F(r\, e^{it}) - F(r^{-1}e^{it})|\, dt \leqq B_F \text{ for all } r \in [0, 1)\,. \tag{2}$$

Note that if F is of real type this becomes Nevanlinna's condition $\int |\mathrm{Re}\, F(r\, e^{it})|\, dt \leqq B_F$ for all $r \in [0, 1)$; we deduce our theorem from Nevanlinna's.

Proof. The representability of F implying that of G and H, Nevanlinna's theorem asserts the existence of constants B_G and B_H such that

$$\int |\mathrm{Re}\, G(r\, e^{it})|\, dt \leqq B_G,\ \int |\mathrm{Re}\, H(r\, e^{it})|\, dt \leqq B_H \tag{3}$$

for all $r \in [0, 1)$. Thus, since $2\,\mathrm{Re}\, G(z) = \mathrm{Re}\, F(z) - \mathrm{Re}\, F(\bar{z}^{-1})$ and $2\,\mathrm{Re}\, H(z) = \mathrm{Im}\, F(z) - \mathrm{Im}\, F(\bar{z}^{-1})$, (3) implies (2). Conversely, let F satisfy (2). Then G and H given by (1) satisfy (3): $1/2 \int |\mathrm{Re}\, F(re^{it}) - \mathrm{Re}\, F(r^{-1}e^{it})|\, dt = \int |\mathrm{Re}\, G(r\, e^{it})|\, dt \leqq 1/2 \int |F(re^{it}) - F(r^{-1}e^{it})|\, dt \leqq B_F$ and similarly $1/2 \int |\mathrm{Im}\, F(r\, e^{it}) - \mathrm{Im}\, F(r^{-1}e^{it})|\, dt = \int |\mathrm{Re}\, H(re^{it})|\, dt \leqq B_F$ so that by Nevanlinna's theorem there exist measures m_1 and m_2 in M which are real such that $G = \int P_C dm_1 + k_1$ and $H = \int P_C dm_2 + k_2$ (for suitable constants k_1, k_2) so that $F = \int P_C(dm_1 + idm_2) + (k_1 + ik_2) = \int P_C dm + k$ with $m = m_1 + im_2$, $k = k_1 + ik_2$ and $F \in P_1$.

Representations $F = \int P_C dm + k$ are clearly not unique: adding a multiple aL of Lebesgue measure ($a \in C$) to m merely changes the constant: $F = \int P_C d(m + aL) + k'$ with $k' = k - 2\pi a$ in D and $k' = k + 2\pi a$ in D^c. It is, however, possible to standardize, and thus to make unique, the representations. This is done in the following theorem which also presents an inversion formula expressing m and k in terms of F.

THEOREM 2. *If $F = \int P_C dm_1 + k_1 = \int P_C dm_2 + k_2$ with k_1 the same constant in D and in D^c and similarly for k_2, then $m_1 = m_2$ and $k_1 = k_2$. If $F \in P_1$ and if we define*

$$m_F(t) = (1/4\pi) \lim_{r \uparrow 1} \int_0^t (F(re^{is}) - F(r^{-1}e^{is}))ds \tag{4}$$

$$(4) \qquad k_F = \frac{1}{2}(F(0) + F(\infty)) \textit{ both in } D \textit{ and } D^c ,$$

then

$$(5) \qquad F = \int P_C dm_F + k_F$$

uniquely.

Thus all functions $F \in P_1$ have a (unique) representation with the constant the same in D and D^c. We do not wish to confine ourselves to this representation in view of Theorems 4-7 below.

Proof. If $F = G + iH$ as in (1), then Nevanlinna's theory says that there are measures m_G and m_H and constants k_G and k_H given by $m_G(t) = (1/2\pi)\lim_{r\uparrow 1}\int_0^t \operatorname{Re} G(r\, e^{is})\,ds = (1/4\pi)\lim_{r\uparrow 1}\int_0^t (\operatorname{Re} F(r\, e^{is}) - \operatorname{Re} F(r^{-1}e^{is}))ds$, $G = \int P_C dm_G + k_G$ with $k_G = 1/2(G(0) + G(\infty)) = 1/2(G(0) - \overline{G(0)}) = i \operatorname{Im} G(0)$ with similar expressions for m_H, k_H, H. Thus m_F and k_F are as given in (4) and (5) is therefore true. The uniqueness results from this: If $F = \int P_C dm + k$ where k is the same constant in D and D^c, then $F(0) = m(T) + k$, $F(\infty) = -m(T) + k$ so that $2k = F(0) + F(\infty)$. If now $F = \int P_C dm_1 + k = \int P_C dm_2 + k$, then $\int P_C dm = 0$ where $m = m_1 - m_2$ so that $\hat{m}_j = 0$ for all integers j and $m = 0, m_1 = m_2$.—Note that (4) and (5) can also be deduced directly from our hypothesis (2) and the expressions

$$F(z) = (1/4\pi)\int (e^{it} + r^{-1}z)/(e^{it} - r^{-1}z)(F(re^{it}) - F(r^{-1}e^{it}))dt + k_F(|z|<r<1)$$

$$F(z) = (1/4\pi)\int (e^{it} + rz)/(e^{it} - rz)(F(re^{it}) - F(r^{-1}e^{it}))dt + k_F(|z|>r^{-1}>1)\,.$$

Condition (2) which characterizes representability can be used to introduce a natural norm in P_1. If $F \in P_1$ define $\|F\|_0 = \sup_{0\leq r<1}\int |F(re^{it}) - F(r^{-1}e^{it})|\,dt$ and $\|F\| = \|F\|_0 + |k_F|$. The following lemma relates $\|F\|_0$ to $\|m_F\|$ for $m_F \in M$.

LEMMA. $\|F\|_0 \leqq 24\pi\, \|m_F\| \leqq 6\, \|F\|_0$.

Proof. (1) We have for $m \in M$ the definition $\|m\| = \operatorname{Var}|_0^{2\pi}[m] = \sup_E \sum_K |m(t_k) - m(t_{k-1})|$ over all partitions E: $0 = t_0 < t_1 < \cdots < t_n = 2\pi$ with $I_k = [t_{k-1}, t_k]$. Let $D_F(r, s) = D(r, s) = F(re^{is}) - F(r^{-1}e^{is})$. Then $\|m_F\| = (1/4\pi)\sup_E \lim_r \sum_k \left|\int_{I_k} D(r, s)ds\right|$ where we have used (4). Thus $\|m_F\| \leqq (1/4\pi)\sup_E \lim_r \sum_k^k \int_{I_k} |D(r, s)|\,ds = (1/4\pi)\lim_r \int |D(r, s)|\,ds = (1/4\pi)\, \|F\|_0$.

(2) When F_1 and F_2 are in P_1 we have $\|F_1 + F_2\|_0 \leqq \|F_1\|_0 + \|F_2\|_0$ since

$$(6)\qquad |F_1 + F_2\|_0 = \sup_{0 \leqq r < 1} \int |D_{F_1} + D_{F_2}| \leqq \sup \int |D_{F_1}| + \sup \int |D_{F_2}| = \|F_1\|_0 + \|F_2\|_0 .$$

Thus if $F = G + iH$ as in (1), we have $\|F\|_0 \leqq \|G\|_0 + \|H\|_0$ and since $2m_G = m_F + \bar{m}_F$ and $2im_H = m_F - \bar{m}_F$ we have $\|m_G\| \leqq \|m_F\|$ and $\|m_H\| \leqq \|m_F\|$. We next establish the inequality $\|G\|_0 \leqq 12\pi \|m_G\|$. We have $G = G_1 - G_2$ corresponding to a decomposition $m_G = m_1 - m_2$ for positive measures m_1 and m_2. We also have $\|G\|_0 \leqq \|G_1\|_0 + \|G_2\|_0$. If some function $F_0 \in P_1$ has nonnegative real part and so corresponds to a positive measure m_0, we have $\|F_0\|_0 = 2 \lim_r \int \operatorname{Re} F_0(re^{it})\,dt = 4\pi \operatorname{Re} F_0(0) = 4\pi m_0(T) = 4\pi \|m_0\|$. Let now $m_1(t) = \operatorname{Var}|_0^t[m_G]$ and $m_2(t) = m_1(t) - m_G(t)$. Then $\|m_1\| = \|m_G\|$ and $\|m_2\| \leqq 2\|m_G\|$ so that $\|G\|_0 \leqq \|G_1\|_0 + \|G_2\|_0 \leqq 12\pi \|m_G\| \leqq 12\pi \|m_F\|$ and similarly $\|H\|_0 \leqq 12\pi \|m_F\|$, i.e., the first inequality asserted in the lemma is proved.

THEOREM 3. *The function* $F \mapsto \|F\| = \|F\|_0 + |k_F|$ *is a norm on* P_1. *The map* $\phi: M \times C \to P_1$ *given by* $(m, k) \mapsto F = \int P_C dm + k$ *is a 1-1 linear bicontinuous map of the Banach space* $M \times C$ (with usual norm topology) *onto* P_1 (relative to the norm topology based on $\|F\|$) *so that, in particular,* P_1 *is a Banach space with its norm. A sequence* (m_j, k_j) *converges to* (m_0, k_0) *where the convergence of the measures is weak* and that of the* k_j *the ordinary convergence of complex numbers if and only if* $F_j \to F_0$ *for the corresponding functions uniformly on compact sets and there exists a constant* B *with* $\|F_j\| \leqq B$ *for all positive* j.

Note that ϕ would not be well-defined if we did not use the (unique) representation of F with k the same in D and D^c; see (4), (5).

Proof. (1) The first part of the theorem is just a summary of assertions proved earlier. (2) Suppose $m_j \to m_0$ weak*. Then $\int P_C dm_j \to \int P_C dm_0$ pointwise in $D \cup D^c$; on every compact subset of $D \cup D^c$ the family $\left\{\int P_C dm_j\right\}$ is uniformly bounded so that by virtue of normal family theory the convergence $\int P_C dm_j \to \int P_C dm_0$ is uniform on compact sets. The weak* convergence of m_j to m_0 says that the $\|m_j\|$ and hence, by the lemma, the $\|F_j\|_0$ are bounded. The convergence $\int P_C dm_j \to \int P_C dm_0$ uniformly on compacts and the convergence $k_j \to k_0$

imply that $F_j \to F_0$ uniformly on compacts and that the $||F_j||$ are bounded.—The converse is similar: If $F_j \to F_0$ uniformly on compacts and if the $||F_j||$ are bounded, then first by the lemma the $||m_j||$ are bounded and $\int t dm_j \to \int t dm_0$ on the dense subset $\{t\}$ of trigonometric polynomials on T and this, together with the boundedness of the $||m_j||$ implies the weak* convergence of m_j to m_0.

Note that if we consider the restriction ϕ_r of ϕ to M, then the image $\phi(M)$ in P_1 is the closed subspace consisting of all F with $F(0) + F(\infty) = 0$. Then map ϕ_r has of course the same properties as ϕ.

Let $F(D)$ and $F(D^c)$ be the parts of $F \in P$ in D and D^c respectively. When F is merely in P, the relation between $F(D)$ and $F(D^c)$ is of course totally arbitrary. If, however, $F \in P_1$, there is a relation. First, if we take two arbitrary functions f and g with the proviso that f be holomorphic in $\{z; |z| < 1 + a\}$ and g holomorphic in $\{z; |z| > 1 - b\}$ (for positive a and b) and then combine their restrictions to D and D^c respectively, then F with $F(D) = f$ and $F(D^c) = g$ will be in P_1; otherwise, however, the relation between $F(D)$ and $F(D^c)$ is governed by the following theorems.

THEOREM 4. *A function $F \in P$ with $F(D)$ constant is in P_1 if and only if $F = \int P_C dm + k$ where m is absolutely continuous with derivative $f_T \in L_1$ and Fourier series $\sum_0 e^{-ijt}a_j$ and which is the boundary function of $f(z)$ antiholomorphic in D given by $\sum_0 \bar{z}^j a_j$. Similarly $F(D^c)$ is constant if and only if $F = \int P_C dm + k$ with absolutely continuous m whose derivative $g_T \in L_1$ has Fourier series $\sum_0 e^{ijt}b_j$ and is the boundary function of $g(z) = \sum_0 z^j b_j$ holomorphic in D.*

Proof. This is just the F. and M. Riesz theorem—the necessities are obvious. Suppose now that $\int P_C dm + k = d$, a constant in D, then $m(T) + k = d$ and, since $P_C = 1 + 2\sum_1 e^{-ijt}z^j$ in D, we conclude that $\int P_C dm + k = \int (1 + 2\sum_1 e^{ijt}z^j) dm + k = d + 2\sum_1 \hat{m}_j z^j = d$ so that $\hat{m}_j = 0 = \int e^{-ijt} dm$ $(j = 1, \cdots)$ i.e., $\int e^{ijt} d\bar{m} = 0$ for all positive integers j, so that $\bar{m}$ is absolutely continuous with derivative $\bar{m}' = \bar{f}_T(t)$ with Fourier series $\sum_0 e^{ijt}\bar{a}_j$ and the first half of the theorem is proved. The second half proceeds the same way.

Let M_0 be the subset of M consisting of absolutely continuous measures with derivatives $g_T \in L_1$ with Fourier series $\sum_1 a_j e^{-ijt}$.

THEOREM 5. *A function f holomorphic in D is the part $F(D)$ for some function $F \in P_1$ if and only if* $\mathrm{dist}_{0 \leqq r < 1}(m_r, M_0) \leqq B < \infty$ *for* *some constant B where m_r is the absolutely continuous measure with derivative $f(re^{it})$ and dist is based on the usual norm in M.*

This criterion contains the criteria contained in the Herglotz-Riesz and Nevanlinna theorems.

Proof. For the necessity suppose we have $F \in P_1$ with $F(D) = f$ in D. To show that $\operatorname{dist}_{0 \leqq r < 1}(m_r, M_0) \leqq B$, we find for each $r \in [0, 1)$ a measure $n_r \in M_0$ with $\| m_r - n_r \| = \sup_{\|c\|_\infty = 1} \left| \int c\, d(m_r - n_r) \right| \leqq B$, taken over all continuous functions c on T. Now $m_r' = f(re^{it}) = F(D)(re^{it}) = F(re^{it})$ and our basic criterion (2) furnishes n_r with $n_r' = F(D^c)(r^{-1}e^{it}) = F(r^{-1}e^{it})$ so that $\operatorname{dist}(m_r, M_0) \leqq \| m_r - n_r \| \leqq \sup_r \int |F(re^{it}) - F(r^{-1}e^{it})|\, dt \leqq B < \infty$.

For the sufficiency, suppose that $\operatorname{dist}(m_r, M_0) \leqq B$. This implies by the weak* compactness of bounded sets in M that there exists a sequence $r_j \in [0, 1)$ with r_j increasing to 1 and measures $n_r \in M_0$ such that $m_r - n_r \to m \in M$ (weak* convergence). Write $n_r' = \sum_1 a_j(r) e^{-ijt}$. Then $L(r) = \int P_C(f(re^{it}) - \sum_1 a_j(r) e^{-ijt})\, dt \to \int P_C\, dm$ as $r \uparrow 1$. Now $L(r) = 4\pi f(rz) - 2\pi f(0)$ while $\int P_C\, dm$ furnishes a function (in D^c) that is holomorphic in D^c. Thus $\int P_C\, dm = F \in P_1$ yields a function with $F(D) = f + \text{const.}$

It is clear from this argument and from Theorem 4 that there are many functions $F \in P_1$ with $F(D) = f + \text{const.}$: the difference of any two of them is characterized in the second half of that theorem.

In addition to its Banach space structure, M has also a ring structure with respect to convolution of measures. The corresponding ring structure in P_1 is given by the Hadamard product: If f and g have expansions $\sum_0 a_j z^j$ and $\sum_0 b_j z^j$ respectively, define $f^* g$ by $\sum_0 a_j b_j z^j$, if f and g have expansions $\sum_0 c_j z^{-j}$ and $\sum_0 d_j z^{-j}$ respectively, define $f^* g$ by $-\sum_0 c_j d_j z^{-j}$. If F and G are in P, then the Hadamard product $F^* G$ is defined in D and D^c according to the rules just mentioned for D and D^c respectively.

THEOREM 6. *If F_1 and F_2 are in P_1 with $F_j = \int P_C\, dm_j + k_j$ where the k_j are piecewise constants in P_1 then $F_1{}^* F_2 = F = \int P_C\, dm + k \in P_1$ and*

$$m = 2(m_1{}^* m_2),\ k = \begin{matrix} -m_1(T)m_2(T) + k_1k_2 + k_1m_2(T) + k_2m_1(T) \\ \\ +m_1(T)m_2(T) - k_1k_2 + k_1m_2(T) + k_2m_1(T) \end{matrix} \tag{7}$$

in D and D^c respectively.

Proof. The proof is a simple calculation based on the formula $(m_1{}^*m_2)_j = \hat{m}_{1,j}\hat{m}_{2,j}$: If $z \in D$ then $\int P_C d(m_1{}^*m_2) = m_1(T)m_2(T) + 2\sum_1 \hat{m}_{1,a}\hat{m}_{2,a}z^a$ with analogous expansions for F_1 and F_2. We obtain $(F_1{}^*F_2)(z) = (m_1(T) + k_1)(m_2(T) + k_2) + 4\sum_1 \hat{m}_{1,a}\hat{m}_{2,a}z^a$ in D with a similar equation in D^c so that (7) is established.

COROLLARY. *The map* $(F, G) \mapsto F*G$ *is continuous in both variables in the norm of* P_1.

The usual Banach algebra inequality $\|F*G\| \leqq \|F\|\,\|G\|$ is not valid in P_1: take $F = G = 10 + z$ in D and equal to $10 + z^{-1}$ in D^c. The map ϕ of Theorem 3 is thus not an isometry.

If $m \in M$, define F_m by

$$(8) \qquad F_m = \frac{1}{2}\int P_C dm + k_m,\ 2k_m = -m(T) \text{ in } D \text{ and } m(T) \text{ in } D^c\,.$$

Let P_2 be the subset of P_1 consisting of all F will $F(0) = F(\infty) = 0$; it is a closed subalgebra of P_1. The following immediate consequence of the preceding theorem is worth stating separately.

THEOREM 7. *The map* $\psi: M \rightarrow P_2$ *given by* (8) *is a linear continuous open epimorphism of the Banach algebra* M *to the Banach algebra* P_2 *with kernel the constant multiples of Lebesgue measure.*

Similar statements are valid about other subalgebras of $M \times \mathrm{C}$ and P_1, e.g., for the subalgebra of M of all m with $m(T) = 0$ and the subalgeba of P_2 of all F with $k_F = 0$; the kernel of the restriction of ψ to this subalgebra of M is determined on the basis of Theorem 4.

Our using complex measures makes the following considerations possible. We define derivatives of functions in P as usual (i.e., in D and D^c separately). If $G = F'$ for functions in P we call F an integral of G. We shall use the phrase that F is differentiable (or integrable) in P_1 if F and F' are in P_1. Differentiability of F in P_1 imposes a strong restriction of F; integrability is much less restrictive although infinite integrability is of course very restrictive. In what follows all functions in P_1 will be in the standard representation (5).

THEOREM 8. *A function* $F = \int P_C dn + k$ *has a derivative* $F' = \int P_C dm + k'$ (all in P_1) *if and only if* n *is absolutely continuous with derivative* g *of bounded variation and* $g(0) = g(2\pi) = 0$. *If* F

is differentiable in P_1 *then* $m(t) = -i\int_0^t e^{-is}\,dg$, $\hat{m}_{-1} = 0$, $k' = m(T)$. *Or equally well*: *A function* $G = \int P_C dm + k'$ *has an integral* $F = \int P_C dn + k$ (all in P_1) *if and only if* $\hat{m}_{-1} = 0$ *and* $k' = m(T)$. *If* G *is integrable in* P_1 *then* n *is absolutely continuous with derivative* g *of bounded variation and* $g(0) = g(2\pi) = 0$, *and* $g(t) = i\int_0^t e^{is}dm$.

Proof. We prove the first version of the theorem. Necessity: (1) If $F' = \int(-1 - 2\sum_1 z^{-j}e^{ijt})dm(t) + k' = -\hat{m}_0 - 2\sum_1 \hat{m}_{-j}z^j + k'$ (expansion in D^c) is to be the derivative of a function in P_1, we must have $-\hat{m}_0 + k' = -m(T) + k' = 0$ and $\hat{m}_{-1} = 0$. (2) Consider $i\int P_C\int_0^t e^{is}dm(s)dt$ and expand P_C. Treat D and D^c separately. In the expansion, change the order of integration and differentiate; using $\hat{m}_{-1} = 0$ and $k' = m(T)$, we see that we have obtained F'. Thus $F = \int P_C dn + k = i\int P_C\int_0^t e^{is}dm(s)dt + \text{const.}$; the uniqueness assertion of Theorem 2 then implies that n is absolutely continuous whose derivative $g(t) = i\int_0^t e^{is}dm(s)$ which is of bounded variation with $g(0) = 0$ and $g(2\pi) = i\hat{m}_{-1} = 0$; this also shows that $m(t) = -i\int_0^t e^{-is}dg(s)$ as desired.—Sufficiency: Suppose $F = \int P_C dn + k$ with absolutely continuous n whose derivative g is of bounded variation and $g(0) = g(2\pi) = 0$: to show that F is differentiable in P_1 with $F' = \int P_C dm + m(T)$ where $m(t) = -i\int_0^t e^{-is}dg(s)$ and $\hat{m}_{-1} = 0$; the last equation is immediate: $\hat{m}_{-1} = g(2\pi) - g(0) = 0$. Consider now $\int P_C dm + m(T)$ where m is defined as above. Again we proceed by expanding P_C and treat D and D^c separately. Thus $\int P_C dm + m(T) = -i\int P_C e^{-it}dg(t) - i\int e^{-it}dg(t)$. After expanding, we integrate by parts and observe that we have obtained F' as expected.

This completes the proof of the theorem.

The second part of the following corollary is again a result of the F. and M. Riesz theorem on analytic measures.

COROLLARY. *A function* $F = \int P_C dm$ *is infinitely differentiable in* P_1 *if and only if* $m \in C_\infty$ *and* $m^{(j)}(0) = m^{(j)}(2\pi) = 0$ *for all positive integers* j. *A function in* P_1 *is infinitely integrable in* P_1 *if and only if it is zero in* D^c.

The preceding results can all be phrased in terms of Fourier-Stieltjes moments. We single out the following application. It is clear that if $\{jn_j\}$ is a moment sequence corresponding to the measure m, then $\{n_j\}$ is also a moment sequence whose measure n is absolutely continuous with derivative of bounded variation $i\,m$. This can also be read from Theorem 8; the hypothesis in the necessity part of that theorem which says that F be in P_1 can be replaced by demanding merely that $F \in P$. We consider in the following theorem a certain kind of perturbation of the multiplier sequence $\{j\}$ of $\{jn_j\}$; we obtain the same conclusion as for that latter sequence.

THEOREM 9. *If $\{a_j n_j\}$ is a moment sequence and if the analytic functions $\sum_1 (j - a_j)z^j$ and $\sum_1 (j + a_{-j})z^j$ have radii of convergence greater than* 1, *then $\{n_j\}$ is a moment sequence corresponding to an absolutely continuous measure whose derivative is of bounded variation.*

Proof. We note that the function F defined in D by $\sum_1 a_j n_j z^j$ and in D^c by $-\sum_1 a_{-j} n_{-j} z^{-j}$ is in P_1 since $\{a_j n_j\}$ is a moment sequence by hypothesis. We shall show that $\{jn_j\}$ is a moment sequence; we show first that the function G defined in D by $\sum_1 jn_j z^j$ and in D^c by $\sum_1 jn_{-j} z^{-j}$ is in P_1; we use the criterion (2) of Theorem 1. We have

$$
\begin{aligned}
G(re^{it}) - G(r^{-1}e^{it}) &= \sum_1 r^j (jn_j e^{ijt} - jn_{-j} e^{-ijt}) \\
&= \sum_1 r^j [(j - a_j) n_j e^{ijt} - (j + a_{-j}) n_{-j} e^{-ijt}] \\
&\quad + \sum_1 r^j (a_j n_j e^{ijt} + a_{-j} n_{-j} e^{-ijt})\,.
\end{aligned}
\tag{9}
$$

We show next that the sequence $\{n_j\}$ is bounded: since $\{a_j n_j\}$ is a moment sequence, it is bounded, say, $|a_j n_j| \leqq B$; since the power series mentioned in the statement of the theorem have radii of convergence greater than 1, we will have for sufficently large j the inequalities $|j - a_j| \leqq 1$ and $|j + a_{-j}| \leqq 1$ so that $|a_j| \geqq |j| - 1$ whence $|n_j| \leqq B$ as desired. We now take absolute values in (9) and integrate with respect to t. Thus

$$
\begin{aligned}
\int |G(re^{it}) - G(r^{-1}e^{it})|\, dt &\leqq \int |f(re^{it}) - \overline{g(re^{it})}|\, dt \\
&\quad + \int |F(re^{it}) - F(r^{-1}e^{it})|\, dt = T_1(r) + T_2(r)
\end{aligned}
$$

where f and g are the analytic functions with radii of convergence greater than 1 mentioned in the theorem. Thus $T_1(r) \leqq B_1$ for all $r \in (0, 1)$ and $T_2(r) \leqq B_2$ since $F \in P_1$. Thus $G \in P_1$, $\{jn_j\}$ is a moment sequence and the theorem is proved.

Analogous problems for several variables and also for regions other than D and D^c such as complementary half planes will be dealt with in another paper.

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