PACIFIC JOURNAL OF MATHEMATICS
Vol. 72, No. 2, 1977

NEW PARTIAL ASYMPTOTIC STABILITY RESULTS FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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The problem of determining sufficient conditions which that assure that all solutions of the second order equation $x'' + q(t)x = 0$ approach zero as t tends to infinity has been studied extensively since 1933. Several results have been given for generalizations of the basic linear equation. In this paper a new technique is used to obtain further results for the equation

$$(p(t)x')' + q(t)f(x) = e(t) \ . \tag{1}$$

The generality of the theorems developed is established by showing that a substantial number of previously known results are immediate consequences of the work herein. Of particular interest is the fact that three recent theorems by Burton and Grimmer, which appeared in this journal, follow from the work contained in this paper.

A substantial number of previously known results are immediate consequences of the work herein. In particular, it includes three recent theorems by Burton and Grimmer, which appeared in this journal, [1].

The following assumptions are made throughout the paper.

(I) The function f is continuous on $(-\infty, \infty)$ and satisfies the additional requirement that

$$xf(x) > 0 \quad \text{whenever} \quad x \neq 0 \ .$$

(II) The functions p and q are positive, continuous functions on $[0, \infty)$, whose product is locally of bounded variation on $[0, \infty)$. Furthermore, letting $(pq)(t) = (pq)(0) + (pq)_+(t) - (pq)_-(t)$ be the Jordan decomposition of pq, it is assumed that

$$\int_0^\infty (pq)^{-1}(\tau)d(pq)_-(\tau) < \infty \ .$$

(III) The function e is a locally integrable function on $[0, \infty)$ which satisfies the inequality

$$\int_0^\infty (pq)^{-1/2}(\tau)\,|\,e(\tau)\,|\,d\tau < \infty \ .$$

2. Major results.

THEOREM 1. *The conditions* (I), (II), *and* (III) *assure that all solutions of equation* (1) *are continuable to the interval* $[0, \infty)$. *Suppose that there is a nonnegative function* α, *locally of bounded variation on* $[0, \infty)$ *and such that* $\int_0^\infty \alpha(\tau)d\tau = \infty$ *for which the following conditions hold*:

(IV) *The function defined as* $\int_0^t (pq)^{1/2}(\tau)d(\alpha q^{-1})(\tau)$ *is of bounded variation on* $[0, \infty)$,

(V) *The negative variation of the function defined as*

$$\left[\log (pq)(t) - \int_0^t \alpha(\tau)d\tau\right] \text{ is bounded on } [0, \infty)\,,$$

and

(VI) *The function* $\alpha(p/q)^{1/2}$ *is bounded on* $[0, \infty)$.
If x *is a bounded solution of* (1), *then* x *satisfies*

$$\lim_{t\to\infty} x(t) = 0\,. \tag{2}$$

This theorem is one of three such known to the author for an equation with a forcing term. One of these which was given by the author in [9] does not appear to be comparable with Theorem 1. The other is the work of Burton and Grimmer, previously mentioned.

Proof of Theoren 1. Associated with each solution x of (1) there is a Lyapunov type function E defined by

$$E(t) = p(t)x'^2(t)/q(t) + F(x(t))\,, \tag{3}$$

where F is defined by

$$F(x) = 2\int_0^x f(\tau)d\tau\,.$$

The proof that $\lambda = \lim_{t\to\infty} E(t)$ exists and is finite for each bounded solution of (1), and the continuability of such solutions parallels that of results given by Muldowney [7].

Because of the complicated nature of the proof that all bounded solutions of (1) approach zero, a brief outline of the logic is provided. The proof is by contradiction. It is assumed that (1) possesses a bounded solution x which does not tend to zero. Condition (V) is used to establish the integrability of $\alpha p x'^2/q$ on $[0, \infty)$. Condition

(IV) is used to show the equivalence of the integrability of $\alpha p x'^2/q$ and $\alpha x f(x)$. The integrability of these functions, together with the assumption that x does not approach zero, implies the integrability of α over $[0, \infty)$, which is the desired contradiction. This procedure is based upon a technique utilized by Gollwitzer [3] to obtain some results for a linear equation. He demonstrated that for the linear equation αE was integrable on $[0, \infty)$, and consequently $\lambda = 0$, from which it followed that solutions went to zero.

Multiplying (1) by $2x'q^{-1}$ and integrating by parts from 0 to t yields, after some manipulation, the identity

$$\begin{aligned} E(t) = E(0) &+ 2\int_0^t \frac{p^{1/2}(\tau)x'(\tau)}{q^{1/2}(\tau)}(pq)^{-1/2}(\tau)e(\tau)d\tau \\ &- \int_0^t \frac{p(\tau)x'^2(\tau)}{q(\tau)}(pq)^{-1}(\tau)d(pq)(\tau)\,. \end{aligned}$$

Rewriting this identity in the form

$$\begin{aligned} \int_0^t \alpha(\tau)\frac{p(\tau)x'^2(\tau)}{q(\tau)}d\tau = E(0) - E(t) &- \int_0^t \frac{p(\tau)x'^2(\tau)}{q(\tau)}dQ(\tau) \\ &+ 2\int_0^t \frac{p^{1/2}(\tau)x'(\tau)}{q^{1/2}(\tau)}(pq)^{-1/2}(\tau)e(\tau)d\tau\,, \end{aligned}$$

where $Q(t) = \int_0^t d\left[\log (pq)(\tau) - \int_0^\tau \alpha(s)ds\right]$, provides the estimate

$$\begin{aligned} \int_0^t \alpha(\tau)\frac{p(\tau)x'^2(\tau)}{q(\tau)}d\tau < E(0) &+ \int_0^t \frac{p(\tau)x'^2(\tau)}{q(\tau)}dQ_-(\tau) \\ &+ 2\int_0^t \frac{p^{1/2}(\tau)|x'(\tau)|}{q^{1/2}(\tau)}(pq)^{-1/2}(\tau)|e(\tau)|d\tau\,. \end{aligned}$$

The convergence as t approaches infinity of the first integral in the right member of this estimate follows from the fact that E and the negative variation of Q are bounded. The convergence as t approaches infinity of the second integral follows from the boundedness of E and condition (III). Consequently, $\alpha p x'^2/q$ is integrable on $[0, \infty)$.

To demonstrate the equivalence of the integrability of $\alpha p x'^2/q$ and $\alpha x f(x)$, multiply (1) by $\alpha x q^{-1}$ and integrate by parts to obtain the identity

$$\begin{aligned} &\left[\frac{\alpha(\tau)p^{1/2}(\tau)}{q^{1/2}(\tau)}\right][x(\tau)]\left[\frac{p^{1/2}(\tau)x'(\tau)}{q^{1/2}(\tau)}\right]\Bigg|_0^t - \int_0^t \alpha(\tau)\frac{p(\tau)x'^2(\tau)}{q(\tau)}d\tau \\ &\quad - \int_0^t x(\tau)\frac{p^{1/2}(\tau)x'(\tau)}{q^{1/2}(\tau)}(pq)^{1/2}(\tau)d(\alpha q^{-1})(\tau) + \int_0^t \alpha(\tau)x(\tau)f(x(\tau))d\tau \\ &= \int_0^t \frac{\alpha(\tau)p^{1/2}(\tau)}{q^{1/2}(\tau)}x(\tau)(pq)^{-1/2}(\tau)e(\tau)d\tau\,. \end{aligned}$$

The result follows by showing that each term of this identity is bounded except possibly for

$$\int_0^t \alpha(\tau)\frac{p(\tau)x'^2(\tau)}{q(\tau)}d\tau \quad \text{and} \quad \int_0^t \alpha(\tau)x(\tau)f(x(\tau))d\tau .$$

Since each of the three factors of the first term is bounded, the product is bounded. The boundedness of the third term of the left member is a consequence of the boundedness of E and x, and condition (IV). Finally, the boundedness of the right member follows from the boundedness of $\alpha p^{1/2}/q^{1/2}$ and x, and condition (III).

Since $\bar{x} \equiv \lim_{t\to\infty} \sup |x(t)| > 0$, $\lambda = \lim_{t\to\infty} E(t)$ is positive. Let the constants η and T be determined so that

$$\begin{aligned} &F(x) < \lambda/3 \quad \text{whenever} \quad |x| \leqq \eta , \quad \text{and} \\ &|\lambda - E(t)| < \lambda/3 \qquad \text{for all} \quad t \geqq T . \end{aligned}$$

Dfiene S to be the set of $t \geqq T$ for which $|x(t)| \leqq \eta$, and define S' to be the complement of S with respect to $[T, \infty)$.

The integrability of $\alpha\ px'^2/q$ over $[0, \infty)$ implies the integrability of α over S. To see this fact, observe that $p(t)x'^2(t)/q(t) > \lambda/3$ for all $t \in S$, and consequently,

$$(\lambda/3)\int_S \alpha(\tau)d\tau < \int_S \alpha(\tau)\frac{p(\tau)x'^2(\tau)}{q(\tau)}d\tau < \infty .$$

Finally, it is shown that the integrability of $\alpha x f(x)$ on $[0, \infty)$ implies the integrability of α on S'.

Since x is bounded and $|x(t)| \geqq \eta$ when $t \in S'$, $C \equiv \min_{t\in S'} x(t)f(x(t))$ is positive. Consequently, the estimate

$$C\int_{S'} \alpha(\tau)d\tau \leqq \int_{S'} \alpha(\tau)x(\tau)f(x(\tau))d\tau < \infty$$

establishes the integrability of α on S'.

Combining the integrability of α on S and S' with the assumption that α is locally integrable yields the desired contradiction that $\int_0^\infty \alpha(\tau)d\tau < \infty$.

The proof of the theorem is complete.

COROLLARY 1. *If in addition to the hypotheses of Theorem* 1 *there are constants δ and $\bar{M}$ such that*

$$F(x) \leqq \bar{M}xf(x) \quad \textit{whenever} \quad |x| < \delta ,$$

then for any bounded solution of the homogeneous equation

$$(p(t)x')' + q(t)f(x) = 0$$

there are constants B and C such that

$$C(pq)^{-1}(t) \leqq E(t) \leqq B\left[\int_0^t \alpha(\tau)d\tau\right]^{-1} .$$

This corollary provides estimates on how quickly the Lyapunov function E approaches zero.

Proof of Corollary 1. Let x be a bounded solution of the homogeneous equation and E the associated Lyapunov function. The function $E(t)\exp\left\{-\int_0^t (pq)^{-1}(\tau)d(pq)_-(\tau)\right\}$ is nonincreasing on $[0, \infty)$ and the function $E(t)\exp\left\{\int_0^t (pq)^{-1}(\tau)d(pq)_+(\tau)\right\}$ is nondecreasing on $[0, \infty)$. These facts are a modification of a similar result by Gollwitzer [4] and can be verified in an analogous way. Since the latter function is nondecreasing it follows that

$$E(0)\exp\left\{-\int_0^t (pq)^{-1}(\tau)d(pq)_-(\tau)\right\} \leqq E(t)(pq)(t) .$$

From this inequality it is clear that letting C be the positive (condition (II)) number

$$E(0)\exp\left\{-\int_0^\infty (pq)^{-1}(\tau)d(pq)_-(\tau)\right\}$$

yields the desired result that $C(pq)^{-1}(t) < E(t)$.

It remains to show the existence of the constant B. Since x is bounded, there is a constant $\bar{\bar{M}}$ such that

$$F(x(t)) \leqq \bar{\bar{M}}x(t)f(x(t)) \quad \text{whenever} \quad |x(t)| > \delta .$$

For a proof of the existence of $\bar{\bar{M}}$ see Scott [9]. If M is defined by $M = \max\{\bar{M}, \bar{\bar{M}}\}$, then

$$F(x(t)) \leqq Mx(t)f(x(t)) \qquad \text{for all} \quad t \in [0, \infty) .$$

Consequently, the integrability of $\alpha x f(x)$ implies the integrability of $\alpha F(x)$ on $[0, \infty)$, and hence, $\int_0^\infty \alpha(\tau)E(\tau)d\tau < \infty$. Define μ by the equation

$$\mu(t) = \exp\left\{-\int_0^t (pq)^{-1}(\tau)d(pq)_-(\tau)\right\}$$

and note that $\mu(\infty) > 0$ because of (II). The constant B can easily be determined from the estimate

$$\mu(\infty)E(t)\int_0^t \alpha(\tau)d\tau \leqq \int_0^t \mu(\tau)E(\tau)\alpha(\tau)d\tau < \int_0^\infty \alpha(\tau)E(\tau)d\tau < \infty .$$

The proof is complete.

The next three corollaries are for the less general equation

$$x'' + q(t)f(x) = e(t) . \tag{4}$$

In these corollaries, conditions (II) and (III) are assumed to be valid with $p(t) \equiv 1$.

COROLLARY 2. *Let α be a nonnegative function of bounded variation on $[0, \infty)$ such that $\int_0^\infty \alpha(\tau)d\tau = \infty$, and define the function H by the equation*

$$H(t) = \log q(t) - \int_0^t \alpha(\tau)d\tau .$$

If $dH \geqq 0$, then any bounded solution of (4) *satisfies* (2).

Proof of Corollary 2. To show that this corollary follows from Theorem 1, it is sufficient to show that the function

$$\int_0^t q^{1/2}(\tau)d(\alpha q^{-1})(\tau)$$

is of bounded variation on $[0, \infty)$. This fact is an immediate consequence of the identity

$$\int_0^t q^{1/2}(\tau)d(\alpha q^{-1})(\tau) = \int_0^t q^{-1/2}(\tau)d\alpha(\tau) - \int_0^t \alpha(\tau)q^{-3/2}(\tau)dq(\tau) .$$

Condition (II) assures that $q^{-1}(t)$ is bounded, and since α is of bounded variation on $[0, \infty)$, the function defined by the first integral in the right member of this identity is of bounded variation on $[0, \infty)$. Since α is bounded and condition (II) holds, the function defined by the second integral is also of bounded variation.

The proof of the corollary is complete.

This corollary can be used to extend many previously known results for special cases of equation (4) to the more general equation. For example, using asymptotic expansions for the solutions of the equation $x'' + q(t)x^n = 0$, where n was the ratio of odd positive integers, Kiguradze [5] showed that the solutions approached zero when q satisfied

$$q'(t)/q(t) \geqq \sigma t^{-\beta}, 0 \leqq \beta \leqq 1 , \quad \text{or} \quad q'(t)/q(t) \geqq \sigma/t \log t, \sigma > 0 .$$

If q satifies either of these conditions, then by defining α via $\alpha(t) = \sigma t^{-\beta}$ or $\alpha(t) = \sigma/t \log t$, respectively, Corollary 2 can be used to show that all bounded solutions of equation (4) approach zero.

Corollary 2 also extends a special case of a result established by Meir, Willett, and Wong [6] for a homogeneous, linear equation.

They proved that the existence of a continuously differentiable function ρ satisfying

$$\int_0^\infty \rho^{-1}(\tau)d\tau = \infty ,$$

$$\liminf_{t\to\infty} \frac{\rho'(t)}{\rho(t)q^{1/2}(t)} \geqq 0 ,$$

and

$$\liminf_{t\to\infty} \frac{q'(t)\rho(t)}{q(t)} = \sigma > 0 ,$$

assures that all solutions of $x'' + q(t)x = 0$ tend to zero. If the condition involving ρ' is replaced with the more restrictive assumption that $\rho' \geqq 0$, then with α defined as σ/ρ Corollary 2 can be applied to show that all bounded solutions of (4) approach zero.

Following a procedure employed by Meir, Willett, and Wong, Corollary 2 can be used to show that if q is a continuously differentiable, concave or convex function which tends to infinity, then all bounded solutions of (4) tend to zero. If $dq' \leqq 0$, then the hypotheses of the corollary are satisfied if α is defined by $\alpha(t) = q'(t)/q(t)$. If $dq' \geqq 0$, then from the estimate

$$q(t) - q(0) = \int_0^t q'(\tau)d\tau \leqq q'(t)t$$

it follows that there are constants t_0 and $\sigma > 0$ such that

$$q'(t)/q(t) \geqq \sigma/t , \quad \text{whenever} \quad t \geqq t_0 .$$

As previously mentioned, this is a sufficient condition to assure that all bounded solutions of (4) approach zero.

Finally, Corollary 2 will be used to extend a well known theorem of Sansone [10]. Sansone showed that if q were a continuously differentiable function such that

$$q'(t) \geqq \sigma > 0 \quad \text{and} \quad \int_0^\infty \frac{d\tau}{q(\tau)} = \infty ,$$

then all solutions of the equation $x'' + q(t)x = 0$ would tend to zero. Under these conditions on q, Corollary 2 can be applied with α defined by $\alpha(t) = \sigma/q(t)$ to show that all bounded solutions of (4) approach zero.

COROLLARY 3. *If q is an absolutely continuous function which tends to infinity and if q'_+/q^β is of bounded variation on* $[0, \infty)$ *for some $\beta < 3/2$, then all bounded solutions of* (4) *satisfy* (2).

This corollary is not as inclusive as might be hope. For example, Kiguradze [5] showed that if q were a positive, absolutely continuous function such that

$$\frac{q'(t)}{q^{3/2}(t)} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty ,$$

$$\text{and} \quad \frac{1}{\log q(t)}\int_0^t \left| d\frac{q'(\tau)}{q^{3/2}(\tau)} \right| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty ,$$

then all solutions of $x'' + q(t)x = 0$ would tend to zero. The technique presented in this paper does not appear to be useful in showing that solutions of even this linear equation tend to zero under the assumption that $q'/q^{3/2}$ be of bounded variation on $[0, \infty)$. A possible explanation for the apparent inability to utilize this approach to obtain results as sharp as those obtained by Kiguradze is the following: Kiguradze utilized the form and degree of $f(x) = x^n$ in a fundamental way to develop his results However, the choice of a suitable function α has nothing to do with f, as long as condition (I) is satisified.

Although one might desire a stronger result than Corollary 3, it appears to be a new one for the general nonlinear equation with a forcing term.

Proof of Corollary 3. To prove this corollary it is sufficient to show the existence of a function α for which conditions (IV) and (V) are satisfied with $p(t) \equiv 1$. Define α via the equation $\alpha(t) = q'_+(t)/q(t)$. Since the negative variation of the function

$$\log q(t) - \int_0^t \alpha(\tau)d\tau$$

over $[0, \infty)$ is equal to $\int_0^\infty \frac{dq_-(\tau)}{q(\tau)}$, which is finite, condition (V) is satisfied. To see that condition (IV) is also satisfied, consider the identity

$$\int_0^t q^{1/2}(\tau)d(q'_+q^{-2})(\tau) = q'_+(\tau)/q^{3/2}(\tau)\Big|_0^t - (1/2)\int_0^t q'_+(\tau)q^{-5/2}(\tau)dq(\tau) .$$

It is easy to see that each term in the right member of this identity is of bounded variation on $[0, \infty)$.

The proof is complete.

COROLLARY 4. *If q is a function which tends to infinity and which has an absolutely continuous derivative and if*

$$\int_0^\infty \left| \frac{q''_+(\tau)}{q^{3/2}(\tau)} - \beta\frac{q'^2_+(\tau)}{q^{5/2}(\tau)} \right| d\tau < \infty$$

For some real $\beta \neq 3/2$, *then all bounded solutions of equation* (4) *satisfy* (2).

Corollary 4 is the consequence of Theorem 1 which facilitates the inclusion of the work by Burton and Grimmer. However before analyzing their results we prefer to make a few preliminary remarks and to prove the corollary. Observe that the integral condition in this corollary is equivalent to

$$\int_0^\infty \left| \frac{q''_+(\tau)}{q_+^{3/2}(\tau)} - \beta \frac{q'^2_+(\tau)}{q_+^{5/2}(\tau)} \right| d\tau < \infty .$$

This follows from the fact that there is a constant k such that

$$kq_+(t) \leqq q(t) \leqq q_+(t) .$$

The existence of this constant k was established by Opial [8].

The implications of these integral conditions are enlightened by the following lemma which was given by Coppel [2, p. 121]:

LEMMA. *Let a be a positive, twice continuously differentiable function for $t > 0$ and suppose*

$$\int_0^\infty [a^{-3/2}(\tau)a''(\tau) - \beta a^{-5/2}(\tau)a'^2(\tau)]d\tau$$

converges for some real $\beta \neq 3/2$. Then the following three conditions are equivalent:

$$\int_0^\infty a^{-5/2}(\tau)a'^2(\tau)d\tau < \infty ,$$

$$a^{-3/2}(t)a'(t) \longrightarrow 0 \quad as \quad t \longrightarrow \infty, \quad and$$

$$\int_0^\infty a^{1/2}(\tau)d\tau = \infty .$$

Moreover, each of these conditions hold if $\beta < 1$ or $\beta > 3/2$.

An immediate consequence of this lemma is that if $a(t) \to \infty$ and if $a^{-3/2}a'' - \beta a^{-5/2}a'^2$ is absolutely integrable, then $a^{-3/2}a'$ is of bounded variation on $[0, \infty)$. With Coppel's lemma, the proof of the corollary is trivial.

Proof of Corollary 4. As with the other corollaries, it is sufficient to show the existence of a function α for which the hypotheses of Theorem 1 are satisfied. If α is defined to be q'_+/q, then it follows as in the proof of Corollary 3 that condition (V) is satisfied.

It remains to show that the function g defined by

$$g(t) = \int_0^t q^{1/2}(\tau)d(\alpha q^{-1})(\tau)$$

is of bounded variation on $[0, \infty)$. The total variation of g is given by the identity

$$\int_0^\infty |g'(\tau)|\,d\tau = \int_0^\infty |q_+''(\tau)/q^{3/2}(\tau) - 2q_+'(\tau)q'/q^{5/2}(\tau)|\,d\tau .$$

However, this last integral is bounded by

$$\int_0^\infty |q_+''(\tau)/q^{3/2}(\tau) - 2q_+'(\tau)/q^{5/2}(\tau)|\,d\tau + \int_0^\infty 2q_+'(\tau)q_-'(\tau)/q^{5/2}(\tau)d\tau .$$

The convergence of the first of these integrals follows from the lemma and the assumption of the corollary. Establishing the convergence of the second integral is equivalent to establishing the convergence of the integral

$$\int_0^\infty 2q_+'(\tau)q_-'(\tau)/(q_+^{3/2}(\tau)q(\tau))d\tau .$$

The convergence of this integral follows from the fact that $q_+'/q_+^{3/2}$ is of bounded variation on $[0, \infty)$ and the fact that $\int_0^\infty q^{-1}(\tau)dq_-(\tau) < \infty$.

The proof is complete.

Burton and Grimmer's work can now be analyzed. They gave conditions which assure that all bounded solutions of equation (4) tend to zero. They expressed q as the product cb where the functions c and b are defined by

$$c(t) = q(0)\exp\left[\int_0^t q_+'(\tau)/q(\tau)d\tau\right],$$

and

$$b(t) = \exp\left[-\int_0^t q'(\tau)/q(\tau)d\tau\right].$$

Observe that c is a nondecreasing function and that b is a nonincreasing function, which is bounded away from zero. Burton and Grimmer proved the

THEOREM. *Suppose that* $\lim_{|x|\to\infty}\int_0^x f(\tau)d\tau = \infty$. *If* $c(t)\to\infty$ *as* $t\to\infty$, *if* $c'/c^{3/2}$ *is bounded, and if*

$$\int_0^\infty |c''(\tau)/c^{3/2}(\tau) - (5/4)c'^2(\tau)/c^{5/2}(\tau)|\,d\tau < \infty ,$$

then every solution of (4) *tends to zero.*

This theorem was stated with the assumption that e satisfies condition (III). Since $c(t) \to \infty$, it follows from the integral condition and the lemma that $c'/c^{3/2}$ is of bounded variation on $[0, \infty)$. This observation permits the conclusion that Burton and Grimmer's subsequent theorems are special cases of the one already stated. To see that their work is a consequence of Corollary 4, one need only recognize that $c'/c = q'_+/q$.

Actually, Burton and Grimmer proved the preceding theorem for the equation $x'' + (q(t) + d(t))f(x) = e(t)$, where d was a continuous function which satisfied

$$\int_0^\infty |d(\tau)|/q^{1/2}(\tau)d\tau < \infty .$$

However, the introduction of this perturbation term required a severe restriction on f, in particular, that there existed positive constants c_1, c_2, and γ, $0 \leqq \gamma \leqq 1$, such that

$$f^2(x) \leqq c_1 F^\gamma(x) + c_2 .$$

With this restriction on f, the approach in this paper is applicable to equations with this type of perturbation. Of special interest is the fact that the development is easily adapted to yield corresponding results for the linear equation $x'' + (q(t) + \mu(t))x = e(t)$, where μ is a locally integrable function such that

$$\int_0^\infty |\mu(\tau)|/q^{1/2}(\tau)d\tau < \infty .$$

3. Solutions on bounded intervals. In this section it is assumed that p and q satisfy condition (II) on the interval $[0, \omega)$, where $\omega \leqq \infty$.

THEOREM 2. *The conditions* (I) *and* (II) *assure that all solutions of the equation*

$$(p(t)x')' + q(t)f(x) = 0 \tag{5}$$

are continuable to the interval $[0, \omega)$. *Suppose that there is a function* α *which satisfies the conditions specified in Theorem* 1 *on the interval* $[0, \omega)$, *except possibly for condition* (VI). *If each bounded solution of* (5) *is oscillatory at* $t = \omega$, *then all bounded solutions of* (5) *tend to zero as* $t \to w$.

This theorem shows that in many instances the assumption that

$\alpha(p/q)^{1/2}$ be bounded is unnecessary. The significance of this observation is made clear by the following example. Consider the equation

$$x'' + \frac{1}{\omega - t}x = 0\ , \quad \text{where} \quad 0 < \omega < \infty\ .$$

Letting q be the coefficient of x and defining α as q'/q, it follows that all the conditions of Theorem 2 are satisfied. Consequently, all the solutions of the equation tend to zero as $t \to \omega$. However, observing that $\alpha = q$, we see that $\alpha/q^{1/2}$ is unbounded. Therefore, the assumption that $\alpha(p/q)^{1/2}$ is bounded is unnecessarily restrictive in many cases.

Proof of Theorem 2. The only place where the assumption that $\alpha(p/q)^{1/2}$ is bounded entered into the proof of Theorem 1 was in showing the equivalance of the integrability of $\alpha px'^2/q$ and $\alpha xf(x)$. Therefore, it is sufficient to show that if x is a bounded, oscillatory solution of (5), then $\alpha xf(x)$ is integrable on $[0, \omega)$ if and only if $\alpha px'^2/q$ is integrable on $[0, \omega)$.

Let $\langle t_k \rangle$ be the sequence of zeros of x, and note that $t_k \to \omega$ as $k \to \infty$. Multiplying equation (5) by $\alpha x/q^{-1}$ and integrating by parts yields the identity

$$\begin{aligned}\alpha(\tau)\frac{p(\tau)x'(\tau)}{x(\tau)}x(\tau)\Big|_0^t &- \int_0^t \alpha(\tau)\frac{p(\tau)x'^2(\tau)}{q(\tau)}d\tau \\ &- \int_0^t x(\tau)\frac{p^{1/2}(\tau)x'(\tau)}{q^{1/2}(\tau)}(pq)^{1/2}(\tau)d(\alpha q^{-1})(\tau) \\ &+ \int_0^t \alpha(\tau)x(\tau)f(x(\tau))d\tau = 0\ .\end{aligned}$$

Evaluating this equation at $t = t_k$ causes the first term to vanish. The theorem then follows by letting $k \to \infty$, and observing that the second integral is absolutely convergent as $t \to \omega$.

The proof is complete.

This proof of Theorem 2 mimics one given by Gollwitze [3] for a linear equation.

4. Conclusion. The results given in this paper pose new questions. For example, it would be desirable to be able to cover the most general form of the corollary given by Meir, Willett, and Wong which was discussed following Corollary 2. It would also be of interest to compare Theorem 1 with the theorem given by the author in [9]. And of course, the applicability of the approach to other stability problems is still an open question.

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Received June 25, 1976. The results presented in this paper are included in the author's doctoral dissertation, written at Drexel University under the direction of Prof. Herman E. Gollwitzer.

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