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COMPLETIONS OF REGULAR RINGS II

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This paper continues earlier investigations into the structure of completions of a (von Neumann) regular ring R with respect to pseudo-rank functions, and into the connections between the ring-theoretic structure of such completions and the geometric structure of the compact convex set $P(R)$ of all pseudo-rank functions on R. In particular, earlier results on the completion of R with respect to a single $N \in P(R)$ are extended to completions with respect to any nonempty subset $X \subseteq P(R)$. Completions in this generality are proved to be regular and self-injective by reducing to the case of a single pseudo-rank function, using a theorem that the lattice of σ-convex faces of $P(R)$ forms a complete Boolean algebra. Given a completion $\bar{R}$ with respect to some $X \subseteq P(R)$, it is shown that the Boolean algebra of central idempotents of $\bar{R}$ is naturally isomorphic to the lattice of those σ-convex faces of $P(R)$ which are contained in the σ-convex face generated by X. Consequently, conditions on X are obtained which tell when $\bar{R}$ is a direct product of simple rings, and how many simple ring direct factors $\bar{R}$ must have. Also, it is shown that the X-completion of R contains a natural copy of the completion with respect to any subset of X, so in particular the $P(R)$-completion of R contains copies of all the X-completions of R. The final section investigates the question of when a regular self-injective ring is complete with respect to some family of pseudo-rank functions. It is proved that a regular, right and left self-injective ring R is complete with respect to a family $X \subseteq P(R)$ provided only that the Boolean algebra of central idempotents of R is complete with respect to X.

1. Completions. All rings in this paper are associative with unit, and ring maps are assumed to preserve the unit. This paper is a direct continuation of [7], and the reader should consult [7] for definitions which are not discussed here. A family of pseudo-rank functions on a regular ring R induces a uniform topology on R, and the purpose of this paper is to study the resulting completion of R. We begin by recalling the appropriate topological concepts.

Let S be a nonempty set, and let D be a nonempty family of pseudo-metrics on S. The (uniform) *topology induced by D on S* has as a subbasis the balls $\{x \in S \mid d(x, y) < \varepsilon\}$, for various $y \in S$, $d \in D$, $\varepsilon > 0$. Thus the basic open neighborhoods of a point $y \in S$ are the sets $\{x \in S \mid d_i(x, y) < \varepsilon$ for $i = 1, \cdots, n\}$ for various $\varepsilon > 0$ and $d_1, \cdots, d_n \in D$. A net in S is a *Cauchy net* (with respect to D)

provided it is Cauchy with respect to each $d \in D$. The space S is *complete* (with respect to D) if the topology on S is Hausdorff and every Cauchy net in S converges in S.

The *completion of* S (with respect to D) is constructed from the set of all Cauchy nets in S by factoring out an equivalence relation $\sim$, where $\{x_i\} \sim \{y_j\}$ if and only if $d(x_i, y_j) \to 0$ for all $d \in D$. Each $d \in D$ extends to a pseudo-metric $\bar{d}$ on the completion $\bar{S}$, and the family $\{\bar{d} \mid d \in D\}$ induces a complete Hausdorff uniform topology on $\bar{S}$. There is a natural map $\phi: S \to \bar{S}$, where $\phi(x)$ is the equivalence class of the constant net $\{x, x, \cdots\}$. This map ϕ is continuous, and $\phi(S)$ is dense in $\bar{S}$. For $x, y \in S$, $\phi(x) = \phi(y)$ if and only if $d(x, y) = 0$ for all $d \in D$.

Now consider another space T topologized by a family E of pseudo-metrics. A function $f: S \to T$ is *uniformly continuous* (with respect to D and E) provided that for any $\varepsilon > 0$ and any $e \in E$, there exist $\delta > 0$ and $d_1, \cdots, d_n \in D$ such that for all $x, y \in S$, $\max\{d_i(x, y)\} < \delta$ implies $e(f(x), f(y)) < \varepsilon$. Any such f extends uniquely to a continuous map $\bar{f}$ from the completion $\bar{S}$ to the completion $\bar{T}$, and $\bar{f}$ is uniformly continuous.

DEFINITION. Let R be a regular ring, and let X be a nonempty subset of $\boldsymbol{P}(R)$. Each $N \in X$ induces a pseudo-metric δ_N on R, where $\delta_N(x, y) = N(x - y)$ [19, pp. 231, 232]. The family $\{\delta_N \mid N \in X\}$ then induces a uniform topology on R, which we call the *X-topology*.

In general, the X-topology has a basis of open sets of the form $\{x \in R \mid N_i(x - y) < \varepsilon \text{ for } i = 1, \cdots, k\}$ for various $y \in R$, $\varepsilon > 0$, and $N_1, \cdots, N_k \in X$. However, if X is convex, then the X-topology has a basis of open sets of the form $\{x \in R \mid N(x - y) < \varepsilon\}$. Namely, given an open set $U \subseteq R$ and an element $y \in U$, we first find $\varepsilon > 0$ and $N_1, \cdots, N_k \in X$ such that $y \in V \subseteq U$, where $V = \{x \in R \mid N_i(x - y) < \varepsilon \text{ for } i = 1, \cdots, k\}$. Setting $N = (N_1 + \cdots + N_k)/k \in X$ and $W = \{x \in R \mid N(x - y) < \varepsilon/k\}$, we obtain $y \in W \subseteq V \subseteq U$.

DEFINITION. Let R be a regular ring, and let $X \subseteq \boldsymbol{P}(R)$. The *kernel* of X, denoted $\ker(X)$, is the set $\{x \in R \mid P(x) = 0 \text{ for all } P \in X\}$. If X is empty, then $\ker(X) = R$, while if X is nonempty, then we see from [6, Lemma 5] that $\ker(X)$ is a proper two-sided ideal of R. For nonempty X, note that the X-topology on R is Hausdorff if and only if $\ker(X) = 0$.

LEMMA 1.1. *Let R be a regular ring, and let X, Y be nonempty subsets of $\boldsymbol{P}(R)$. Then the following conditions are equivalent*:

(a) *The identity map* (R, Y-topology) $\to$ (R, X-topology) *is* (uniformly) *continuous.*

(b) *For each* $P \in X$, *the map* $P: (R, Y\text{-topology}) \to [0, 1]$ *is* (uniformly) *continuous.*

(c) *Given* $\varepsilon > 0$ *and* $P \in X$, *there exist* $\delta > 0$ *and* $N_1, \cdots, N_k \in Y$ *such that for all* $x \in R$, $\max\{N_i(x)\} < \delta$ *implies* $P(x) < \varepsilon$.

Proof. (a) $\Leftrightarrow$ (c): It is clear from the definitions that if the identity map $(R, Y) \to (R, X)$ is continuous, then (c) holds; and if (c) holds, then the identity map $(R, Y) \to (R, X)$ is uniformly continuous.

(b) $\Leftrightarrow$ (c): If $P: (R, Y) \to [0, 1]$ is continuous for all $P \in X$, then (c) clearly holds. Conversely, assume (c) and consider any $P \in X$. Given $\varepsilon > 0$, there exist $\delta > 0$ and $N_1, \cdots, N_k \in Y$ as in (c). For any $x, y \in R$, we see that if $\max\{N_i(x - y)\} < \delta$, then $|P(x) - P(y)| \leqq P(x - y) < \varepsilon$, using [19, Corollary, p. 231]. Thus $P: (R, Y) \to [0, 1]$ is uniformly continuous.

DEFINITION. Let R be a regular ring, and let $X, Y \subseteq \boldsymbol{P}(R)$. We say that X is *continuous with respect to* Y, denoted $X \ll Y$, provided condition (c) of Lemma 1.1 is satisfied. In particular, $\varnothing \ll Y$ for any Y, whereas $X \ll \varnothing$ only for $X = \varnothing$. In case $X = \{P\}$, we write $P \ll Y$ in place of $\{P\} \ll Y$, and similarly when $Y = \{Q\}$. Note in general that $X \ll Y$ if and only if $P \ll Y$ for all $P \in X$. Note also that $X \ll Y$ implies $\ker(Y) \leqq \ker(X)$.

THEOREM 1.2. *Let R be a regular ring, and let $X, Y \subseteq \boldsymbol{P}(R)$. Then the following conditions are equivalent*:

(a) $X \ll Y$.

(b) *X is contained in the σ-convex face generated by Y in $\boldsymbol{P}(R)$.*

(c) *X is contained in the σ-convex hull of the face generated by Y in $\boldsymbol{P}(R)$.*

Proof. (b) $\Leftrightarrow$ (c) by [7, Theorem 3.9].

(b) $\Rightarrow$ (a): Given $P \in X$, [7, Theorem 3.9] says that $P \ll Q$ for some Q in the σ-convex hull of Y. There is a σ-convex combination $Q = \sum \alpha_k Q_k$ with all $Q_k \in Y$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $Q(x) < \delta$ implies $P(x) < \varepsilon$. Choose a positive integer n such that $\sum_{k=n+1}^{\infty} \alpha_k < \delta/2$. Then whenever $x \in R$ and $\max\{Q_1(x), \cdots, Q_n(x)\} < \delta/2$, we have

$$Q(x) = \sum_{k=1}^{n} \alpha_k Q_k(x) + \sum_{k=n+1}^{\infty} \alpha_k Q_k(x) \leqq \sum_{k=1}^{n} \alpha_k(\delta/2) + \sum_{k=n+1}^{\infty} \alpha_k < \delta ,$$

whence $P(x) < \varepsilon$. Thus $P \ll Y$. Since this holds for all $P \in X$, we obtain $X \ll Y$.

(a) $\Rightarrow$ (b): Given $P \in X$, we have $P \ll Y$. Thus there exist real numbers $\delta_1, \delta_2, \cdots > 0$, positive integers $n(1) = 1 < n(2) < \cdots$, and $Q_1, Q_2, \cdots \in Y$ with the following property: whenever $x \in R$ and $Q_i(x) < \delta_k$ for $i = n(k), \cdots, n(k+1) - 1$, then $P(x) < 1/k$. Now set $Q = \sum_{i=1}^{\infty} Q_i/2^i$, which lies in the σ-convex hull of Y. We claim that $P \ll Q$.

Given $\varepsilon > 0$, choose a positive integer $k > 1/\varepsilon$, and set $n = n(k+1) - 1$, $\delta = \delta_k/2^n$. Whenever $x \in R$ and $Q(x) < \delta$, we have

$$Q_i(x) \leqq 2^i Q(x) \leqq 2^n Q(x) < 2^n \delta = \delta_k$$

for $i = n(k), \cdots, n$, whence $P(x) < 1/k < \varepsilon$. Thus $P \ll Q$, hence [7, Theorem 3.9] says that P lies in the σ-convex face generated by Q. Therefore P lies in the σ-convex face generated by Y.

COROLLARY 1.3. *Let R be a regular ring, let X and Y be nonempty subsets of $\boldsymbol{P}(R)$, and assume that X and Y generate the same σ-convex face in $\boldsymbol{P}(R)$. Then the X-topology and the Y-topology on R are identical. Moreover, Cauchyness and uniform continuity are the same whether considered relative to X or relative to Y.*

Proof. By Theorem 1.2, $X \ll Y$ and $Y \ll X$, whence Lemma 1.1 shows that the identity map $(R, X\text{-topology}) \to (R, Y\text{-topology})$ is a homeomorphism. Thus the topologies are identical. The equivalence of Cauchyness and uniform continuity relative to X and Y also follows from the relation $X \ll Y \ll X$.

DEFINITION. Let R be a regular ring, and let X be a nonempty subset of $\boldsymbol{P}(R)$. The *X-completion of R* is the completion of R with respect to the uniform topology induced by X. By the standard properties of pseudo-rank functions [19, p. 232], the ring operations on R and the maps $N \in X$ are all uniformly continuous with respect to X. Thus the X-completion $\bar{R}$ is a ring, the natural map $R \to \bar{R}$ is a ring map, and each $N \in X$ extends uniquely to a continuous map $\bar{N}: \bar{R} \to [0, 1]$. The pseudo-metrics $\bar{\delta}_N$ on $\bar{R}$ which are part of the completion construction are of course induced by the $\bar{N}$, i.e., $\bar{\delta}_N(x, y) = \bar{N}(x - y)$ for all $N \in X$ and all $x, y \in \bar{R}$.

Because of the continuity of the ring operations, we obtain a slight simplification in the construction of $\bar{R}$. Namely, the set C of Cauchy nets in R forms a ring, the subset C_0 of null nets (i.e., nets which converge to zero) forms a two-sided ideal in C, and $\bar{R} = C/C_0$. The kernel of the natural map $\phi: R \to \bar{R}$ is thus the ideal $\{x \in R \mid N(x) = 0$ for all $N \in X\}$, i.e., $\ker \phi = \ker(X)$.

These properties of $\bar{R}$ are standard consequences of the general theory of completions of uniform spaces. By analogy with the case of a single pseudo-rank function—[11, Theorem 3.7] and [6, Corollary

15]—we should expect $\bar{R}$ to be a regular self-injective ring, and the maps $\bar{N}$ should be pseudo-rank functions on $\bar{R}$. While these properties do hold, the only one we are able to prove directly is that each $\bar{N}$ is a pseudo-rank function on $\bar{R}$. It is possible to prove self-injectivity in a fairly straightforward manner once it is established that $\bar{R}$ is regular, but regularity seems impossible to prove directly, mainly because the proofs in the case of a single pseudo-rank function depend so heavily on the use of sequences that they do not generalize to nets.

DEFINITION. Let R be a regular ring, let X be a nonempty subset of $\boldsymbol{P}(R)$, and let $\bar{R}$ denote the X-completion of R. Each $N \in X$ extends uniquely to a continuous map $\bar{N}: \bar{R} \rightarrow [0, 1]$. For now, we refer to $\bar{N}$ as the *natural extension of N to $\bar{R}$*. Once we have proved that $\bar{R}$ is regular and that $\bar{N}$ is a pseudo-rank function on $\bar{R}$, we shall refer to $\bar{N}$ as the *natural extension of N to $\boldsymbol{P}(\bar{R})$*. For all $x, y \in R$, we have $N(xy) \leqq N(x), N(y)$ by definition and $N(x + y) \leqq N(x) + N(y)$ by [19, Corollary, p. 231]. By continuity, we obtain $\bar{N}(xy) \leqq \bar{N}(x), \bar{N}(y)$ and $\bar{N}(x + y) \leqq \bar{N}(x) + \bar{N}(y)$ for all $x, y \in \bar{R}$.

LEMMA 1.4. *Let R be a regular ring, let X be a nonempty subset of $\boldsymbol{P}(R)$, and let $\bar{R}$ denote the X-completion of R. Any idempotent $e \in \bar{R}$ can be obtained as the limit of a net of idempotents from R.*

Proof. Let $\phi: R \rightarrow \bar{R}$ be the natural map, and for each $N \in X$ let $\bar{N}$ denote the natural extension of N to $\bar{R}$. Now e has basic open neighborhoods of the form $B = \{x \in \bar{R} \mid \bar{N}_i(x - e) < \varepsilon$ for $i = 1, \cdots, k\}$, for suitable $\varepsilon > 0$ and $N_1, \cdots, N_k \in X$. We must show that for any such B, there exists an idempotent $f \in R$ with $\phi f \in B$.

There exists a net $\{a_j\} \subseteqq R$ such that $\phi a_j \rightarrow e$, and of course $\phi(a_j^2) \rightarrow e^2 = e$ as well. Thus there is some $a = a_j \in R$ such that $\bar{N}_i(\phi a - e) < \varepsilon/3$ and $\bar{N}_i(\phi(a^2) - e) < \varepsilon/3$ for all i. Note that

$$N_i(a^2 - a) = \bar{N}_i(\phi(a^2) - \phi a) \leqq \bar{N}_i(\phi(a^2) - e) + \bar{N}_i(\phi a - e) < 2\varepsilon/3$$

for all i. According to [11, Lemma 2.3], there exists an idempotent $f \in R$ such that $f - a \in aR(a^2 - a)$. Thus $N_i(f - a) \leqq N_i(a^2 - a) < 2\varepsilon/3$ for all i, and consequently

$$\bar{N}_i(\phi f - e) \leqq \bar{N}_i(\phi f - \phi a) + \bar{N}_i(\phi a - e) = N_i(f - a) + \bar{N}_i(\phi a - e) < \varepsilon$$

for all i. Therefore $\phi f \in B$.

LEMMA 1.5. *Let R be a regular ring, let X be a nonempty* *subset of* $\boldsymbol{P}(R)$, *let* $\bar{R}$ *denote the* X*-completion of* R, *and let* $\phi: R \to \bar{R}$ *be the natural map. If* e, f *are orthogonal idempotents in* $\bar{R}$, *then there exists a net* $\{(e_j, f_j)\} \subseteq R \times R$ *such that* $(\phi e_j, \phi f_j) \to (e, f)$ *in* $\bar{R} \times \bar{R}$, *and for all* j, e_j *and* f_j *are orthogonal idempotents.*

Proof. For each $N \in X$, let $\bar{N}$ denote the natural extension of N to $\bar{R}$. In $\bar{R} \times \bar{R}$, (e, f) has basic open neighborhoods of the form

$$B = \{(x, y) \in \bar{R} \times \bar{R} \mid \bar{N}_i(x - e), \bar{N}_i(y - f) < \varepsilon \text{ for } i = 1, \cdots, k\},$$

for suitable $\varepsilon > 0$ and $N_1, \cdots, N_k \in X$. We must show that for any such B, there exist orthogonal idempotents $e', f' \in R$ such that $(\phi e', \phi f') \in B$.

According to Lemma 1.4, there exist nets $\{g_j\}, \{h_j\}$ (which we may arrange to be indexed by the same directed set) of idempotents in R such that $\phi g_j \to e$ and $\phi h_j \to f$. In addition, $\phi(g_j h_j) \to ef = 0$ and $\phi(h_j g_j) \to fe = 0$. Thus there exist idempotents $g = g_j$ and $h = h_j$ in R such that $\bar{N}_i(\phi g - e) < \varepsilon/2$, $\bar{N}_i(\phi h - f) < \varepsilon/2$, $\bar{N}_i(\phi(gh)) < \varepsilon/2$, and $\bar{N}_i(\phi(hg)) < \varepsilon/2$ for all i. Note that $N_i(gh), N_i(hg) < \varepsilon/2$ for all i. According to [11, Lemma 2.4], there exist orthogonal idempotents $e', f' \in R$ such that $e' - g \in ghR$ and $f' - h \in hgR$. Thus $N_i(e' - g) \leqq N_i(gh) < \varepsilon/2$ and likewise $N_i(f' - h) < \varepsilon/2$ for all i. Consequently, $\bar{N}_i(\phi e' - e) \leqq N_i(e' - g) + \bar{N}_i(\phi g - e) < \varepsilon$ and likewise $\bar{N}_i(\phi f' - f) < \varepsilon$ for all i. Therefore $(\phi e', \phi f') \in B$.

PROPOSITION 1.6. *Let* R *be a regular ring, let* X *be a nonempty subset of* $\boldsymbol{P}(R)$, *and let* $\bar{R}$ *denote the* X*-completion of* R. *For each* $N \in X$, *let* $\bar{N}$ *denote the natural extension of* N *to* $\bar{R}$. *Then* $\bar{N}$ *is a pseudo-rank function on* $\bar{R}$.

Proof. Let $\phi: R \to \bar{R}$ denote the natural map, and note that $\bar{N}(1) = \bar{N}(\phi(1)) = N(1) = 1$. We have observed above that $\bar{N}(xy) \leqq \bar{N}(x), \bar{N}(y)$ for all $x, y \in \bar{R}$. Now consider any orthogonal idempotents $e, f \in \bar{R}$. By Lemma 1.5, there exists a net $\{(e_j, f_j)\} \subseteq R \times R$ such that $\phi e_j \to e$, $\phi f_j \to f$, and e_j, f_j are orthogonal idempotents for each j. Observing that $\bar{N}\phi(e_j + f_j) = \bar{N}\phi(e_j) + \bar{N}\phi(f_j)$ for all j, we conclude that $\bar{N}(e + f) = \bar{N}(e) + \bar{N}(f)$. Thus $\bar{N}$ is a pseudo-rank function on $\bar{R}$.

In order to prove that the X-completion $\bar{R}$ of a regular ring R is a regular self-injective ring, we must use the following circuitous procedure. The first step, which we develop in the next section, is to prove that the lattice of σ-convex faces of $\boldsymbol{P}(R)$ is a complete Boolean algebra. Using this, we reduce the problem to the case when the $N \in X$ are facially independent. In this case, we prove that $\bar{R}$ is isomorphic to the direct product of the N-completions of

R, from which the required properties of $\bar{R}$ are immediate.

PROPOSITION 1.7. *Let R be a regular ring, let X and Y be nonempty subsets of $\boldsymbol{P}(R)$, and assume that X and Y generate the same σ-convex face in $\boldsymbol{P}(R)$. Then* $\ker(X) = \ker(Y)$ *and the X-completion of R coincides with the Y-completion.*

Proof. By Theorem 1.2, $X \ll Y$ and $Y \ll X$, hence we see that $\ker(X) = \ker(Y)$. In addition, Corollary 1.3 shows that the X-topology and the Y-topology on R are the same, and that nets in R are Cauchy (null) with respect to X exactly when they are Cauchy (null) with respect to Y. Thus the two completions of R, constructed as the ring of Cauchy nets modulo the ideal of null nets, are identical.

THEOREM 1.8. *Let R be a regular ring, let X be a nonempty subset of $\boldsymbol{P}(R)$, and let $\bar{R}$ denote the X-completion of R. For each $N \in X$, let $\bar{R}_N$ denote the N-completion of R. If X is a facially independent subset of $\boldsymbol{P}(R)$, then $\bar{R} \cong \prod_{N \in X} \bar{R}_N$.*

Proof. For each $N \in X$, let $\phi_N\colon R \to \bar{R}_N$ be the natural map, and let $\bar{N}$ be the natural extension of N to $\boldsymbol{P}(\bar{R}_N)$. Set $S = \prod_{N \in X} \bar{R}_N$, and for each $N \in X$ let p_N denote the projection $S \to \bar{R}_N$. The maps ϕ_N induce a map $\phi\colon R \to S$, and we note that $\ker \phi = \bigcap \ker \phi_N = \bigcap \ker(N) = \ker(X)$.

For each $N \in X$, we have a pseudo-rank function $N^* = \bar{N} p_N$ on S, and we note that $N^*\phi = \bar{N} p_N \phi = \bar{N}\phi_N = N$, i.e., N^* is an extension of N to $\boldsymbol{P}(S)$. Setting $X^* = \{N^* \mid N \in X\}$, we see also that S is complete with respect to X^*. Thus to show that $S \cong \bar{R}$, it suffices to show that $\phi(R)$ is dense in S in the X^*-topology.

Now let $s \in S$, $\varepsilon > 0$, and $N_1, \cdots, N_k \in X$. Set $N = (N_1 + \cdots + N_k)/k$ in $\boldsymbol{P}(R)$. Inasmuch as the N_i are facially independent, [7, Theorem 4.3] says that the natural map from the N-completion of R into $T = \bar{R}_{N_1} \times \cdots \times \bar{R}_{N_k}$ is an isomorphism. We have a natural map $\psi\colon R \to T$ (induced by $\phi_{N_1}, \cdots, \phi_{N_k}$), and we have a rank function N' on T defined by the rule $N'(x_1, \cdots, x_k) = [\bar{N}_1(x_1) + \cdots + \bar{N}_k(x_k)]/k$. By virtue of the isomorphism of the N-completion of R onto T, we see that $\psi(R)$ is dense in T in the N'-metric. Applying this information to the element $t = (p_{N_1}(s), \cdots, p_{N_k}(s))$ in T, there must exist an element $r \in R$ such that $N'(\psi(r) - t) < \varepsilon/k$. Inasmuch as

$$\begin{aligned} N'(\psi(r) - t) &= N'(\phi_{N_1}(r) - p_{N_1}(s), \cdots, \phi_{N_k}(r) - p_{N_k}(s)) \\ &= [\bar{N}_1(\phi_{N_1}(r) - p_{N_1}(s)) + \cdots + \bar{N}_k(\phi_{N_k}(r) - p_{N_k}(s))]/k \\ &= [\bar{N}_1 p_{N_1}(\phi(r) - s) + \cdots + \bar{N}_k p_{N_k}(\phi(r) - s)]/k \\ &= [N_1^*(\phi(r) - s) + \cdots + N_k^*(\phi(r) - s)]/k\ , \end{aligned}$$

we conclude that $N_i^*(\phi(r) - s) < \varepsilon$ for all $i = 1, \cdots, k$.

Therefore $\phi(R)$ is dense in S in the X^*-topology, as desired.

2. σ-Convex faces in $\boldsymbol{P}(R)$. We show in this section that for any regular ring R, the lattice of σ-convex faces of $\boldsymbol{P}(R)$ forms a complete Boolean algebra.

LEMMA 2.1. *Let R be a regular ring, and let $\{F_i\}$ be a collection of faces of $\boldsymbol{P}(R)$.*

(a) *The convex hull of $\bigcup F_i$ is a face of $\boldsymbol{P}(R)$.*

(b) *The σ-convex hull of $\bigcup F_i$ is a σ-convex face of $\boldsymbol{P}(R)$.*

(c) *If the F_i are all σ-convex and only finitely many of them are nonempty, then the convex hull of $\bigcup F_i$ is a σ-convex face of $\boldsymbol{P}(R)$.*

Proof. (a) Since $\boldsymbol{P}(R)$ is a Choquet simplex by [7, Corollary 3.6], this follows from [2, Proposition 3].

(b) In view of (a), the σ-convex hull of $\bigcup F_i$ is also the σ-convex hull of a face of $\boldsymbol{P}(R)$. By Theorem 1.2, this is a σ-convex face of $\boldsymbol{P}(R)$.

(c) We may assume that there are only finitely many F_i, say $F_1, \cdots, F_n$. Let F denote the convex hull of $\bigcup F_i$, which is a face of $\boldsymbol{P}(R)$ by (a).

Now consider any σ-convex combination $N = \sum \alpha_k P_k$ where all $P_k \in F$. For each k, there is a convex combination $P_k = \beta_{k1}P_{k1} + \cdots + \beta_{kn}P_{kn}$ with each $P_{ki} \in F_i$. Set $\gamma_i = \sum_k \alpha_k\beta_{ki} \geqq 0$ for each $i = 1, \cdots, n$, and note that $\gamma_1 + \cdots + \gamma_n = 1$. After renumbering, we may assume that $\gamma_1, \cdots, \gamma_r > 0$ and $\gamma_{r+1}, \cdots, \gamma_n = 0$, for some $1 < r \leqq n$. For each $i = 1, \cdots, r$, set $N_i = \sum_k (\alpha_k\beta_{ki}/\gamma_i)P_{ki}$, which lies in F_i because F_i is σ-convex. Then $N = \gamma_1 N_1 + \cdots + \gamma_r N_r$ is a convex combination of the N_i, whence $N \in F$.

Therefore F is σ-convex.

DEFINITION. As in [7, 8], we use $B(R)$ to denote the Boolean algebra of central idempotents in a ring R. In case R is regular and right (or left) self-injective, $B(R)$ is complete [8, Proposition 4.1]: for $\{e_i\} \subseteq B(R)$, $\bigwedge e_i$ is the central idempotent which generates the ideal $\bigcap e_iR$.

LEMMA 2.2. *Let R be a regular ring, let $N \in \boldsymbol{P}(R)$, and let $E \subseteq B(R)$. If $e_0R \cap \ker(N) = 0$ for some $e_0 \in E$, then there exists a countable sequence $\{e_1, e_2, \cdots\} \subseteq E$ such that $\bigcap_{e \in E} eR = \bigcap_{n=1}^{\infty} e_nR$.*

Proof. Replacing E by $\{e_0e \mid e \in E\}$, we may assume that $eR \cap$

$\ker(N) = 0$ for all $e \in E$. Thus we may transfer the problem to $R/\ker(N)$, i.e., we may assume, without loss of generality, that $\ker(N) = 0$. Now N is a rank function on R, from which it follows that R does not contain any uncountable direct sums of nonzero right ideals.

Set $F = \{1 - e \mid e \in E\}$ and $X = \{xR \mid x \in \bigcup_{f \in F} fR\}$. Given any nonzero $y \in FR = \sum_{f \in F} fR$, we must have $yf \neq 0$ for some $f \in F$, whence yfR is a nonzero member of X which is contained in yR. Thus every nonzero submodule of $(FR)_R$ contains a nonzero member of X, hence $(FR)_R$ must have an essential submodule which is a direct sum of members of X. Inasmuch as R contains no uncountable direct sums of nonzero right ideals, this direct sum must be countable, hence we obtain an independent sequence $\{x_1R, x_2R, \cdots\} \subseteq X$ such that $\bigoplus x_nR$ is an essential submodule of $(FR)_R$.

Since R is a right nonsingular ring, the left annihilator of $\bigoplus x_nR$ must coincide with the left annihilator of FR. For each n, $x_nR \leqq (1 - e_n)R$ for some $e_n \in E$. Consequently, we see that $\bigcap_{n=1}^{\infty} e_nR$ is contained in the left annihilator of FR, i.e., $\bigcap_{n=1}^{\infty} e_nR \subseteq \bigcap_{e \in E} eR$. The opposite inclusion is automatic.

PROPOSITION 2.3. *Let R be a regular ring, let $N \in \boldsymbol{P}(R)$, and let $\bar{R}$ denote the N-completion of R. Let $X, Y \subseteq \boldsymbol{P}(R)$ such that $X, Y \ll N$, and for each $P \in X \cup Y$ let $\bar{P}$ be the continuous extension of P to $\boldsymbol{P}(\bar{R})$. Set $\bar{X} = \{\bar{P} \mid P \in X\}$ and $\bar{Y} = \{\bar{P} \mid P \in Y\}$. Then the following conditions are equivalent*:

(a) $X \ll Y$.

(b) $\bar{X} \ll \bar{Y}$.

(c) $\ker(\bar{Y}) \leqq \ker(\bar{X})$.

Proof. Let ϕ denote the natural map $R \to \bar{R}$, and let $\bar{N}$ denote the natural extension of N to $\boldsymbol{P}(\bar{R})$.

(a) $\Rightarrow$ (b): Given $P \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $Q_1, \cdots, Q_k \in Y$ such that for all $y \in R$, $\max\{Q_i(y)\} < \delta$ implies $P(y) < \varepsilon/2$. Since $\bar{P}, \bar{Q}_1, \cdots, \bar{Q}_k \ll \bar{N}$, there also exists $\delta' > 0$ such that for all $z \in \bar{R}$, $\bar{N}(z) < \delta'$ implies both $\bar{P}(z) < \varepsilon/2$ and $\max\{\bar{Q}_i(z)\} < \delta/2$.

Now consider any $x \in \bar{R}$ for which $\max\{\bar{Q}_i(x)\} < \delta/2$. There is some $y \in R$ for which $\bar{N}(\phi y - x) < \delta'$, whence $\bar{P}(\phi y - x) < \varepsilon/2$ and $\max\{\bar{Q}_i(\phi y - x)\} < \delta/2$. Then $Q_i(y) = \bar{Q}_i(\phi y) \leqq \bar{Q}_i(\phi y - x) + \bar{Q}_i(x) < \delta$ for all $i = 1, \cdots, k$, whence $P(y) < \varepsilon/2$ and so $\bar{P}(x) \leqq \bar{P}(x - \phi y) + \bar{P}(\phi y) = \bar{P}(\phi y - x) + P(y) < \varepsilon$. Thus for all $x \in \bar{R}$, $\max\{\bar{Q}_i(x)\} < \delta/2$ implies $\bar{P}(x) < \varepsilon$.

(b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c) are clear.

(c) $\Rightarrow$ (b): According to [7, Lemma 3.7], each $\ker(\bar{Q})$ (for $Q \in Y$) is generated by a central idempotent. Since we have a rank function $\bar{N}$ on $\bar{R}$, we see from Lemma 2.2 that there exists a countable sequence $\{Q_1, Q_2, \cdots\} \subseteq Y$ such that $\ker(\bar{Y}) = \bigcap_{n=1}^{\infty} \ker(\bar{Q}_n)$. Set $\bar{Q} = \sum_{n=1}^{\infty} \bar{Q}_n/2^n$, which lies in the σ-convex hull of $\bar{Y}$, and note from Theorem 1.2 that $\bar{Q} \ll \bar{Y}$. Inasmuch as each $\bar{Q}_n \ll \bar{N}$, we also see from Theorem 1.2 that $\bar{Q} \ll \bar{N}$.

Now $\ker(\bar{Q}) = \bigcap_{n=1}^{\infty} \ker(\bar{Q}_n) = \ker(\bar{Y}) \leqq \ker(\bar{X})$, hence $\ker(\bar{Q}) \leqq \ker(\bar{P})$ for all $P \in X$. According to [7, Proposition 3.8], $\bar{X}$ is contained in the σ-convex hull of the face generated by $\bar{Q}$. Therefore $\bar{X} \ll \bar{Q}$ by Theorem 1.2, whence $\bar{X} \ll \bar{Y}$.

LEMMA 2.4. *Let R be a regular ring, and let $F \subseteq G$ be σ-convex faces of $\boldsymbol{P}(R)$. If $F \neq G$, then there exists $Q \in G$ such that the σ-convex face generated by Q is disjoint from F.*

Proof. Choose some $N \in G - F$, and let H be the intersection of F with the σ-convex face generated by N. We are done if H is empty, hence we may assume that H is nonempty. Let $\bar{R}$ denote the N-completion of R, and let $\bar{N}$ denote the natural extension of N to $\boldsymbol{P}(\bar{R})$. By Theorem 1.2, $H \ll N$, hence each $P \in H$ extends continuously to some $\bar{P} \in \boldsymbol{P}(\bar{R})$. Set $\bar{H} = \{\bar{P} \mid P \in H\}$.

Inasmuch as N does not lie in the σ-convex face H, we see from Theorem 1.2 that N is not continuous with respect to H. According to Proposition 2.3, it follows that $\ker(\bar{H}) \nleqq \ker(\bar{N})$, whence $\ker(\bar{H}) \neq 0$. Using [7, Lemma 3.7], we thus obtain a nonzero central idempotent $e \in B(\bar{R})$ such that $e\bar{R} = \ker(\bar{H})$.

Since $e \neq 0$, $\bar{N}(e) \neq 0$, hence we can define a pseudo-rank function $\bar{Q} \in \boldsymbol{P}(\bar{R})$ by the rule $\bar{Q}(x) = \bar{N}(ex)/\bar{N}(e)$. Pulling $\bar{Q}$ back to $Q \in \boldsymbol{P}(R)$, we have $Q \leqq [1/\bar{N}(e)]N$, whence Q lies in the face generated by N [7, Corollary 3.3]. Thus $Q \in G$.

Now consider any P in the σ-convex face generated by Q, and note that P also lies in the σ-convex face generated by N. By Theorem 1.2, $P \ll Q, N$, hence P extends continuously to some $\bar{P} \in \boldsymbol{P}(\bar{R})$. According to Proposition 2.3, $(1-e)\bar{R} = \ker(\bar{Q}) \leqq \ker(\bar{P})$, hence $\bar{P}(e) = 1$. Since $e\bar{R} = \ker(\bar{H})$, we conclude that $P \notin H$ and so $P \notin F$.

Therefore the σ-convex face generated by Q is disjoint from F.

LEMMA 2.5. *Let R be a regular ring, let F, G be faces in $\boldsymbol{P}(R)$, and let F_1, G_1 be the σ-convex hulls of F, G. If F and G are disjoint, then F_1 and G_1 are disjoint.*

Proof. Suppose there exists $N \in F_1 \cap G_1$. Then there is a σ-convex combination $N = \sum \alpha_k P_k$ with all $P_k \in F$. By renumbering, we may assume that $\alpha_1 > 0$. If $\alpha_1 = 1$, then $P_1 = N \in G_1$. If $\alpha_1 < 1$, then

$$\alpha_1 P_1 + (1 - \alpha_1) \sum_{k=2}^{\infty} \alpha_k P_k/(1 - \alpha_1) = \sum_{k=1}^{\infty} \alpha_k P_k = N \in G_1 .$$

Since G_1 is a face of $\boldsymbol{P}(R)$ by Theorem 1.2, $P_1 \in G_1$ in this case also, hence we obtain a σ-convex combination $P_1 = \sum \beta_k Q_k$ with all $Q_k \in G$. Again, we may assume that $\beta_1 > 0$. Since $\sum \beta_k Q_k = P_1$ lies in the face F, we conclude as above that $Q_1 \in F$. But then $Q_1 \in F \cap G$, which is impossible.

THEOREM 2.6. *Let R be a regular ring, and let $\mathscr{F}$ denote the lattice of σ-convex faces of $\boldsymbol{P}(R)$. Then $\mathscr{F}$ is a complete Boolean algebra. For $\{F_i\} \subseteq \mathscr{F}$, $\wedge F_i = \cap F_i$ and $\vee F_i$ is the σ-convex hull of $\cup F_i$. For $F, G \in \mathscr{F}$, $F \vee G$ is the convex hull of $F \cup G$.*

Proof. It is clear that $\mathscr{F}$ is a complete lattice in which arbitrary infima are given by intersections. For $\{F_i\} \subseteq \mathscr{F}$, Lemma 2.1 shows that the σ-convex hull of $\cup F_i$ belongs to $\mathscr{F}$, whence the σ-convex hull of $\cup F_i$ equals $\vee F_i$. For $F, G \in \mathscr{F}$, Lemma 2.1 shows that the convex hull of $F \cup G$ belongs to $\mathscr{F}$, whence the convex hull of $F \cup G$ equals $F \vee G$.

Given $F, G, H \in \mathscr{F}$, we automatically have $(F \wedge G) \vee (F \wedge H) \subseteq F \wedge (G \vee H)$. Now consider any $N \in F \wedge (G \vee H)$. Since $N \in G \vee H$, we obtain a convex combination $N = \alpha P + (1 - \alpha)Q$ with $P \in G$, $Q \in H$. If $\alpha = 0$, then $N = Q \in F \wedge H$, while if $\alpha = 1$, then $N = P \in F \wedge G$. If $0 < \alpha < 1$, then since N lies in the face F, we obtain $P, Q \in F$, and consequently $P \in F \wedge G$, $Q \in F \wedge H$. Thus $N \in (F \wedge G) \vee (F \wedge H)$ in any case, whence $F \wedge (G \vee H) = (F \wedge G) \vee (F \wedge H)$. Therefore $\mathscr{F}$ is a distributive lattice.

Now let $F \in \mathscr{F}$, and let X denote the set of those $G \in \mathscr{F}$ which are disjoint from F. Given any nonempty chain $\{G_i\} \subseteq X$, we see that $\cup G_i$ is a face of $\boldsymbol{P}(R)$ which is disjoint from F. According to Lemma 2.5, the σ-convex hull of $\cup G_i$ is also disjoint from F. Thus $\vee G_i \in X$, which provides the chain $\{G_i\}$ with an upper bound in X. Now Zorn's Lemma gives us a maximal element $G \in X$.

If $F \vee G \neq \boldsymbol{P}(R)$, then by Lemma 2.4 there is a nonempty $H \in \mathscr{F}$ which is disjoint from $F \vee G$. In particular, H is disjoint from both F and G. Inasmuch as $\mathscr{F}$ is distributive, we obtain $F \wedge (G \vee H) = (F \wedge G) \vee (F \wedge H) = \varnothing$ and so $G \vee H \in X$, which contradicts the maximality of G. Thus $F \vee G = \boldsymbol{P}(R)$, whence G is a complement for F in $\mathscr{F}$.

Therefore $\mathscr{F}$ is a complete, complemented, distributive lattice, i.e., a complete Boolean algebra.

COROLLARY 2.7. *Let R be a regular ring, and let $X \subseteq \boldsymbol{P}(R)$.*

Then there exists a facially independent set $Y \subseteq \boldsymbol{P}(R)$ such that Y and X generate the same σ-convex face in $\boldsymbol{P}(R)$. In particular, any σ-convex face in $\boldsymbol{P}(R)$ can be generated by a facially independent subset of $\boldsymbol{P}(R)$.

Proof. If X is empty, then X itself is facially independent, hence we may assume that X is nonempty. Let $\mathscr{F}$ denote the lattice of σ-convex faces of $\boldsymbol{P}(R)$, which is a complete Boolean algebra by Theorem 2.6. For each $P \in \boldsymbol{P}(R)$, let $F(P)$ denote the σ-convex face generated by P in $\boldsymbol{P}(R)$, and set $\mathscr{F}_0 = \{F(P) \mid P \in \boldsymbol{P}(R)\}$.

Note that every nonempty face in $\mathscr{F}$ contains a (nonempty) face from $\mathscr{F}_0$. Since $\mathscr{F}$ is a complete Boolean algebra, we may thus express F as the supremum of some family $\{F(P_i)\}$ of pairwise disjoint members of $\mathscr{F}_0$. Since the $F(P_i)$ are pairwise disjoint, the set $Y = \{P_i\}$ is facially independent. Since each P_i generates $F(P_i)$, we see that Y generates $\vee F(P_i) = F$. Thus Y and X generate the same σ-convex face in $\boldsymbol{P}(R)$.

The results of Theorem 2.6 and Corollary 2.7 do not hold in general for non-σ-convex faces. That is, the lattice of faces of $\boldsymbol{P}(R)$ may not be a complete Boolean algebra (although it must be a complete distributive lattice), and there may exist faces in $\boldsymbol{P}(R)$ which cannot be generated by facially independent sets. In fact, in the following example we construct a regular ring R such that $\boldsymbol{P}(R)$ cannot be generated (as a face) by facially independent pseudo-rank functions.

Let K be a field, let $K_1, K_2, \cdots$ be copies of K, and let R be the K-subalgebra of $\prod K_n$ generated by 1 and $J = \bigoplus K_n$. Note that R is regular and that $R/J \cong K$.

Since $R/J \cong K$, there exists a unique $P_0 \in \boldsymbol{P}(R)$ such that $\ker(P_0) = J$. For $n = 1, 2, \cdots$, let e_n denote the identity of K_n. Since $R/(1 - e_n)R \cong K$, there exists a unique $P_n \in \boldsymbol{P}(R)$ such that $\ker(P_n) = (1 - e_n)R$. Given any $P \in \boldsymbol{P}(R)$ and $n = 1, 2, \cdots$, we claim that $P(e_n x) = P(e_n)P_n(x)$ for all $x \in R$, which is clear if $P(e_n) = 0$. If $P(e_n) \neq 0$, then the rule $Q(x) = P(e_n x)/P(e_n)$ defines $Q \in \boldsymbol{P}(R)$ such that $\ker(Q) = (1 - e_n)R$. In this case, we obtain $Q = P_n$ by uniqueness of P_n, from which the claim follows.

We now claim that every $P \in \boldsymbol{P}(R)$ is a σ-convex combination of $P_0, P_1, P_2, \cdots$. More specifically, we claim that

$$P = [1 - \sum_{n=1}^{\infty} P(e_n)]P_0 + \sum_{n=1}^{\infty} P(e_n)P_n \, .$$

If $\sum P(e_n) = 0$, then $P(J) = 0$, whence $P = P_0$ by uniqueness and the claim holds. Now assume that $\sum P(e_n) = \gamma > 0$, and set $Q =$

$\sum_{n=1}^{\infty} [P(e_n)/\gamma]P_n$ in $\boldsymbol{P}(R)$. Given $x \in R$ and $n = 1, 2, \cdots$, we have $(e_1x + \cdots + e_nx)R \leqq xR$ and so $P(e_1x) + \cdots + P(e_nx) \leqq P(x)$, whence

$$\gamma Q(x) = \sum_{n=1}^{\infty} P(e_n)P_n(x) = \sum_{n=1}^{\infty} P(e_nx) \leqq P(x) ,$$

using the claim above. As a result, $\gamma Q \leqq P$, hence [7, Proposition 3.2] says that $P - \gamma Q = \beta Q'$ for some $\beta \geqq 0$ and some $Q' \in \boldsymbol{P}(R)$. Note that $\beta = 1 - \gamma = 1 - \sum_{n=1}^{\infty} P(e_n)$. If $\beta = 0$, then $\gamma = 1$ and $P = Q = \sum_{n=1}^{\infty} P(e_n)P_n$. If $\beta > 0$, then $Q'(J) = \beta^{-1}(P - \gamma Q)(J) = 0$, whence $Q' = P_0$ (by uniqueness) and so

$$P = \gamma Q + \beta P_0 = \beta P_0 + \sum_{n=1}^{\infty} P(e_n)P_n ,$$

as required.

Since every $P \in \boldsymbol{P}(R)$ is a σ-convex combination of the P_n, every nonempty face of $\boldsymbol{P}(R)$ must contain at least one P_n. As a result, any collection of nonempty pairwise disjoint faces of $\boldsymbol{P}(R)$ must be countable, whence every facially independent subset of $\boldsymbol{P}(R)$ must be countable.

Now consider any facially independent subset $X \subseteq \boldsymbol{P}(R)$. We claim that the face F generated by X is not equal to $\boldsymbol{P}(R)$. Write $X = \{Q_1, Q_2, \cdots\}$, repeating some Q_i if necessary in order to get an infinite sequence. For each $i = 1, 2, \cdots$, there is a σ-convex combination $Q_i = \sum_{n=0}^{\infty} \alpha_{in}P_n$. Inasmuch as $\lim_{n\to\infty} \alpha_{in} = 0$ for each i, we can find positive integers $n(1) < n(2) < \cdots$ such that for all $k = 1, 2, \cdots, \alpha_{1,n(k)}, \alpha_{2,n(k)}, \cdots, \alpha_{k,n(k)} < 1/2^{2k}$. Define $\beta_1, \beta_2, \cdots$ by setting $\beta_{n(k)} = 1/2^k$ for all k and all other $\beta_n = 0$, and set $Q = \sum_{n=1}^{\infty} \beta_nP_n$ in $\boldsymbol{P}(R)$. We shall prove that $Q \notin F$.

If $Q \in F$, then by [1, (1.9)] there are convex combinations $\alpha Q + (1 - \alpha)Q' = \alpha_1Q_1 + \cdots + \alpha_tQ_t$ for some $0 < \alpha < 1$, some $Q' \in \boldsymbol{P}(R)$, and some t. Now choose a positive integer $k \geqq t$ such that $2^k > \alpha^{-1}$. Then

$$\beta_{n(k)} = 1/2^k = 2^k/2^{2k} > \alpha^{-1}\alpha_{i,n(k)}$$

for $i = 1, \cdots, k$, whence

$$\begin{aligned} Q(e_{n(k)}) &= \beta_{n(k)} = (\alpha_1 + \cdots + \alpha_t)\beta_{n(k)} \\ &> \alpha^{-1}(\alpha_1\alpha_{1,n(k)} + \cdots + \alpha_t\alpha_{t,n(k)}) \\ &= \alpha^{-1}[\alpha_1Q_1(e_{n(k)}) + \cdots + \alpha_tQ_t(e_{n(k)})] \\ &= \alpha^{-1}[\alpha Q(e_{n(k)}) + (1 - \alpha)Q'(e_{n(k)})] \geqq Q(e_{n(k)}) , \end{aligned}$$

which is impossible. Thus $Q \notin F$, hence $F \neq \boldsymbol{P}(R)$.

Thus the faces generated by facially independent subsets of $\boldsymbol{P}(R)$ are all proper, so that $\boldsymbol{P}(R)$ cannot be generated (as a face) by facially independent pseudo-rank functions.

Also, if X is a maximal facially independent subset of $\boldsymbol{P}(R)$, then X generates a face F which is proper, yet F intersects every nonempty face of $P(R)$. Thus F has no complement in the lattice of faces of $\boldsymbol{P}(R)$, hence the lattice of faces of $\boldsymbol{P}(R)$ is not a Boolean algebra.

3. Structure of completions.

THEOREM 3.1. *Let R be a regular ring, let X be a nonempty subset of $\boldsymbol{P}(R)$, and let $\bar{R}$ denote the X-completion of R. For each $N \in X$, let $\bar{N}$ denote the natural extension of N to $\bar{R}$.*

(a) *$\bar{R}$ is a regular, right and left self-injective ring.*

(b) *For each $N \in X$, $\bar{N}$ is a pseudo-rank function on $\bar{R}$.*

(c) *If $\bar{X} = \{\bar{N} \mid N \in X\}$, then $\ker(\bar{X}) = 0$ and $\bar{R}$ is complete with respect to $\bar{X}$.*

Proof. (a) According to Corollary 2.7, there exists a facially independent set $Y \subseteq \boldsymbol{P}(R)$ such that Y and X generate the same σ-convex face in $\boldsymbol{P}(R)$. Then Proposition 1.7 shows that $\bar{R}$ coincides with the Y-completion of R. For each $N \in Y$, let $\bar{R}_N$ denote the N-completion of R, which by [11, Theorem 3.7] and [6, Corollary 15] is a regular, right and left self-injective ring. According to Theorem 1.8, $\bar{R} \cong \prod_{N \in Y} \bar{R}_N$, whence $\bar{R}$ is regular and right and left self-injective.

(b) Proposition 1.6.

(c) is clear from the completion process.

Our major tool for investigating the structure of an X-completion $\bar{R}$ is Theorem 3.7, which provides a complete description of the Boolean algebra $B(\bar{R})$ of central idempotents of $\bar{R}$. In order to prove this theorem, we first require generalizations of several of the results of [7].

DEFINITION. Let $\{e_i \mid i \in I\}$ be a nonempty family of pairwise orthogonal idempotents in a ring R. There is a standard net of idempotents in R formed from $\{e_i\}$ as follows. For index set, we take the family $\mathscr{F}$ of all nonempty finite subsets of I, ordered by inclusion. For each $F \in \mathscr{F}$, we write $e_F = \sum_{i \in F} e_i$, thus obtaining a net $\{e_F\}$ of idempotents indexed by the directed set $\mathscr{F}$. We abbreviate this net as $\sum e_i$, and refer to it as the *net of partial sums of the e_i*.

LEMMA 3.2. *Let R be a regular ring, let X be a nonempty subset of $\boldsymbol{P}(R)$ such that $\ker(X) = 0$, and assume that R is complete* *with respect to X. Let J be a right ideal of R which is closed in the X-topology, and let $\{e_i\}$ be a nonempty family of orthogonal idempotents in J.*

(a) *$\sum e_i$ converges to an idempotent $e \in J$.*

(b) *If $\bigoplus e_iR$ is essential in J, then $eR = J$. If, in addition, J is a two-sided ideal, then e is central in R.*

(c) *J is generated by an idempotent. If J is a two-sided ideal, then J is generated by a central idempotent.*

Proof. (a) Let I be the index set for the e_i, let $\mathscr{F}$ be the family of all nonempty finite subsets of I, and set $e_F = \sum_{i \in F} e_i$ for all $F \in \mathscr{F}$. We claim that the net $\sum e_i = \{e_F\}$ is Cauchy with respect to any $N \in X$.

Whenever $F \subseteqq G$ in $\mathscr{F}$, we have $e_F = e_F e_G$ and so $N(e_F) \leqq N(e_G) \leqq 1$. Thus the net $\{N(e_F)\}$ of real numbers is increasing and bounded above, hence it must converge. As a result, given any $\varepsilon > 0$ there must exist $F \in \mathscr{F}$ such that $|N(e_G) - N(e_H)| < \varepsilon/2$ whenever $G, H \supseteqq F$ in $\mathscr{F}$. In particular, when $G \supseteqq F$ we see that e_F and $e_G - e_F$ are orthogonal idempotents, whence $N(e_G - e_F) = N(e_G) - N(e_F) < \varepsilon/2$. Consequently, $N(e_G - e_H) < \varepsilon$ whenever $G, H \supseteqq F$ in $\mathscr{F}$. Thus the net $\sum e_i$ is indeed Cauchy with respect to N.

By completeness, $\sum e_i$ converges to some $e \in R$, and of course e is an idempotent. Since each e_F lies in the closed set J, we also have $e \in J$.

(b) Given any $i \in I$, we have $e_F e_i = e_i$ for all $F \supseteqq \{i\}$ in $\mathscr{F}$, whence $ee_i = e_i$. Thus $\bigoplus e_iR \leqq eR \leqq J$. Since $\bigoplus e_iR$ is essential in J, it follows that eR is essential in J, from which we infer that $eR = J$.

If J is two-sided, then eR is a two-sided ideal in a semiprime ring, whence e must be central.

(c) Choose a maximal independent family $\{x_jR\}$ of principal right ideals contained in J, so that $\bigoplus x_jR$ is essential in J. Also, choose a right ideal K such that $J \bigoplus K$ is essential in R_R, whence $(\bigoplus x_jR) \bigoplus K$ is essential as well. Inasmuch as R is regular and right self-injective by Theorem 3.1, we see that for each k,

$$R_R = E((\bigoplus x_jR) \bigoplus K) = x_kR \bigoplus E((\bigoplus_{j \neq k} x_jR) \bigoplus K) .$$

As a result, there exists an idempotent $f_k \in R$ such that $f_kR = x_kR$ and $f_kx_j = 0$ for all $j \neq k$. Thus we obtain orthogonal idempotents f_j such that $\bigoplus f_jR = \bigoplus x_jR$ is essential in J.

According to (a) and (b), $\sum f_j$ converges to an idempotent f such that $fR = J$, and if J is two-sided, then f is central.

LEMMA 3.3. *Let R be a regular ring, let X be a nonempty* *subset of* $\boldsymbol{P}(R)$, *and let* $\bar{R}$ *denote the* X*-completion of* R. *If* $P \in \boldsymbol{P}(R)$ and $P \ll X$, *then* P *extends* (uniquely) *to a continuous* $\bar{P} \in \boldsymbol{P}(\bar{R})$. *In addition,* $\ker(\bar{P})$ *is generated by a central idempotent in* $\bar{R}$.

Proof. By continuity, P extends uniquely to a continuous map $\bar{P}: \bar{R} \to [0, 1]$. Exactly as in Proposition 1.6, we infer that $\bar{P} \in \boldsymbol{P}(\bar{R})$. Now $\ker(\bar{P})$ is a two-sided ideal of $\bar{R}$ which is topologically closed, hence Lemma 3.2 says that $\ker(\bar{P})$ is generated by a central idempotent.

LEMMA 3.4. *Let* R *be a regular ring, let* X *be a nonempty subset of* $\boldsymbol{P}(R)$ *such that* $\ker(X) = 0$, *and assume that* R *is complete with respect to* X. *Let* $P \in \boldsymbol{P}(R)$ *such that* $P \ll X$.

(a) *If* $x, x_1, x_2, \cdots \in R$ *such that* $x_1R \leqq x_2R \leqq \cdots$ *and* $\bigcup x_nR$ *is essential in* xR, *then* $P(x) = \sup P(x_n)$.

(b) *If* $y, y_1, y_2, \cdots \in R$ *such that* $y_1R \geqq y_2R \geqq \cdots$ *and* $\bigcap y_nR = yR$, *then* $P(y) = \inf P(y_n)$.

Proof. (a) Proceeding as in [6, Lemma 12], we construct orthogonal idempotents $e_1, e_2, \cdots \in R$ such that $e_1R \oplus \cdots \oplus e_nR = x_nR$ for all n. Each $e_n \in xR$, and xR is closed in the X-topology (because it is an annihilator). Thus by Lemma 3.2, $\sum e_n$ converges to an idempotent $e \in R$ such that $eR = xR$. Since P is continuous, we thus obtain

$$P(x) = P(e) = \sum P(e_n) = \sup\{P(e_1) + \cdots + P(e_n)\} = \sup P(x_n) .$$

(b) Choose idempotents $e_1, e_2, \cdots \in R$ such that $(1 - e_n)R = y_nR$ for all n, and note that $Re_1 \leqq Re_2 \leqq \cdots$. Since R is left self-injective by Theorem 3.1, some left ideal of R is an injective hull for $\bigcup Re_n$. Thus there is an idempotent $e \in R$ such that $\bigcup Re_n$ is essential in Re. Observing that Re and $\bigcup Re_n$ have the same right annihilator, we see that $(1 - e)R = yR$. According to (a), $1 - P(y) = P(e) = \sup P(e_n) = \sup\{1 - P(y_n)\}$, whence $P(y) = \inf P(y_n)$.

PROPOSITION 3.5. *Let* R *be a regular ring, let* X *be a nonempty subset of* $\boldsymbol{P}(R)$ *such that* $\ker(X) = 0$, *and assume that* R *is complete with respect to* X. *Let* $P, Q \in \boldsymbol{P}(R)$ *such that* $P, Q \ll X$. *If* $\ker(Q) \leqq \ker(P)$, *then* $P \ll Q$.

Proof. If not, then there exist $\varepsilon > 0$ and $x_1, x_2, \cdots \in R$ such that for all n, $Q(x_n) < 1/2^n$ but $P(x_n) \geqq \varepsilon$. Set $y_{kn}R = x_kR + \cdots + x_nR$ for all $n \geqq k$. Since R is right self-injective by Theorem 3.1, there exist elements $z_1, z_2, \cdots \in R$ such that $\bigcup_{n=k}^{\infty} y_{kn}R$ is essential in z_kR for all k, and there exists $z \in R$ such that $\bigcap_{k=1}^{\infty} z_kR = zR$. Using

Lemma 3.4, we obtain

$$Q(z) \leqq Q(z_k) = \sup\{Q(y_{kk}), Q(y_{k,k+1}), \cdots\}$$
$$\leqq \sup\{Q(x_k) + \cdots + Q(x_n)\} = \sum_{n=k}^{\infty} Q(x_n) < \sum_{n=k}^{\infty} 1/2^n = 1/2^{k-1}$$

for all $k = 1, 2, \cdots$. Thus $Q(z) = 0$, whence $P(z) = 0$. However, $z_k R \geqq y_{kk} R = x_k R$ for all k and so $P(z_k) \geqq P(x_k) \geqq \varepsilon$ for all k, hence Lemma 3.4 says that $P(z) = \inf P(z_k) \geqq \varepsilon$, which is a contradiction. Therefore $P \ll Q$.

COROLLARY 3.6. *Let R be a regular ring, let X be a nonempty subset of $\boldsymbol{P}(R)$, and let $\bar{R}$ denote the X-completion of R. Let $Y, W \subseteq \boldsymbol{P}(R)$ such that $Y, W \ll X$, and for each $P \in Y \cup W$ let $\bar{P}$ be the continuous extension of P to $\boldsymbol{P}(\bar{R})$. Set $\bar{Y} = \{\bar{P} \mid P \in Y\}$ and $\bar{W} = \{\bar{P} \mid P \in W\}$. Then the following conditions are equivalent:*

(a) $Y \ll W$.

(b) $\bar{Y} \ll \bar{W}$.

(c) $\ker(\bar{W}) \leqq \ker(\bar{Y})$.

Proof. Let $\phi: R \to \bar{R}$ be the natural map. For each $N \in X$, let $\bar{N}$ denote the natural extension of N to $\boldsymbol{P}(\bar{R})$, and set $\bar{X} = \{\bar{N} \mid N \in X\}$

(a) $\Rightarrow$ (b): Given $P \in Y$ and $\varepsilon > 0$, there exist $\delta > 0$ and $Q_1, \cdots, Q_k \in W$ such that for all $y \in R$, $\max\{Q_i(y)\} < \delta$ implies $P(y) < \varepsilon/2$. Since $\bar{P}, \bar{Q}_1, \cdots, \bar{Q}_k \ll \bar{X}$, there also exist $\delta' > 0$ and $N_1, \cdots, N_s \in X$ such that for all $z \in \bar{R}$, $\max\{\bar{N}_j(z)\} < \delta'$ implies both $\bar{P}(z) < \varepsilon/2$ and $\max\{\bar{Q}_i(z)\} < \delta/2$.

Now consider any $x \in \bar{R}$ for which $\max\{\bar{Q}_i(x)\} < \delta/2$. There is some $y \in R$ for which $\max\{\bar{N}_j(\phi y - x)\} < \delta'$, whence $\bar{P}(\phi y - x) < \varepsilon/2$ and $\max\{\bar{Q}_i(\phi y - x)\} < \delta/2$. Then $\bar{Q}_i(y) = \bar{Q}_i(\phi y) \leqq \bar{Q}_i(\phi y - x) + \bar{Q}_i(x) < \delta$ for all $i = 1, \cdots, k$, whence $P(y) < \varepsilon/2$ and so $\bar{P}(x) \leqq \bar{P}(x - \phi y) + \bar{P}(\phi y) = \bar{P}(\phi y - x) + P(y) < \varepsilon$. Thus for all $x \in \bar{R}$, $\max\{\bar{Q}_i(x)\} < \delta/2$ implies $\bar{P}(x) < \varepsilon$.

(b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c) are clear.

(c) $\Rightarrow$ (b): Given any $P \in Y$, Lemma 3.3 gives us a central idempotent $e \in \bar{R}$ such that $(1 - e)\bar{R} = \ker(\bar{P})$. Since $\ker(\bar{W}) \leqq \ker(\bar{P})$, we thus obtain $\bigcap_{Q \in W} e[\ker(\bar{Q})] = 0$. Lemma 3.3 also shows that each of the ideals $e[\ker(\bar{Q})]$ is generated by a central idempotent, hence Lemma 2.2 says that there exists a countable sequence $\{Q_1, Q_2, \cdots\} \subseteq W$ such that $\bigcap_{n=1}^{\infty} e[\ker(\bar{Q}_n)] = 0$, i.e., $\bigcap_{n=1}^{\infty} \ker(\bar{Q}_n) \leqq \ker(\bar{P})$. Set $\bar{Q} = \sum_{n=1}^{\infty} \bar{Q}_n/2^n$, which lies in the σ-convex hull of $\bar{W}$, and note from Theorem 1.2 that $\bar{Q} \ll \bar{W}$. Inasmuch as each $\bar{Q}_n \ll \bar{X}$ we also see from Theorem 1.2 that $\bar{Q} \ll \bar{X}$. Observing that $\ker(\bar{Q}) \leqq \ker(\bar{P})$, we see from Proposition 3.5 that $\bar{P} \ll \bar{Q}$, whence $\bar{P} \ll \bar{W}$. Therefore $\bar{Y} \ll \bar{W}$.

Let K be a convex subset of a real vector space, and let F be a face of K. It is clear from the definitions that a subset of F is a face of F if and only if it is a face of K. Thus the lattice of faces of F is just the lattice of those faces of K which are contained in F.

THEOREM 3.7. *Let R be a regular ring, let X be a nonempty subset of $\boldsymbol{P}(R)$, and let $\bar{R}$ denote the X-completion of R. Let F be the σ-convex face generated by X in $\boldsymbol{P}(R)$, and let $\mathscr{F}$ be the lattice of σ-convex faces of F. Then $B(\bar{R}) \cong \mathscr{F}$.*

Proof. For each $N \in F$, we have $N \ll X$ by Theorem 1.2, and we let $\bar{N}$ denote the continuous extension of N to $\boldsymbol{P}(\bar{R})$.

Given $e \in B(\bar{R})$, set $\theta(e) = \{N \in F \mid \bar{N}(e) = 1\}$, and note that $\theta(e)$ is a σ-convex subset of F. Suppose that we have $0 < \alpha < 1$ and $N_1, N_2 \in F$ with $\alpha N_1 + (1-\alpha)N_2 \in \theta(e)$. Then $\alpha\bar{N}_1(e) + (1-\alpha)\bar{N}_2(e) = 1$, whence $\bar{N}_1(e) = \bar{N}_2(e) = 1$ and so $N_1, N_2 \in \theta(e)$. Thus $\theta(e)$ is a face of F, i.e., $\theta(e) \in \mathscr{F}$. Now suppose that $e \leqq f$ in $B(\bar{R})$, i.e., $e = ef$. For any $N \in \theta(e)$, we have $1 = \bar{N}(e) \leqq \bar{N}(f)$ and so $\bar{N}(f) = 1$, whence $N \in \theta(f)$. Thus $\theta(e) \subseteq \theta(f)$. Therefore we have a monotone map $\theta: B(\bar{R}) \to \mathscr{F}$.

Given any $G \in \mathscr{F}$, set $\bar{G} = \{\bar{N} \mid N \in G\}$. According to Lemma 3.2, there is some $\mu(G) \in B(\bar{R})$ such that $\ker(\bar{G}) = (1 - \mu(G))\bar{R}$. If $G \subseteq H$ in $\mathscr{F}$, then $(1 - \mu(H))\bar{R} = \ker(\bar{H}) \leqq \ker(\bar{G}) = (1 - \mu(G))\bar{R}$ and so $1 - \mu(H) \leqq 1 - \mu(G)$, whence $\mu(G) \leqq \mu(H)$. Therefore we have a monotone map $\mu: \mathscr{F} \to B(\bar{R})$.

Consider any $e \in B(\bar{R})$. Since $\bar{N}(e) = 1$ for all $N \in \theta(e)$, we obtain $\bar{N}(1-e) = 0$ for all $N \in \theta(e)$, whence $1 - e \in \ker(\overline{\theta(e)}) = (1 - \mu\theta(e))\bar{R}$. Thus $1 - e \leqq 1 - \mu\theta(e)$, hence $\mu\theta(e) \leqq e$. Set $f = e - \mu\theta(e)$, which is a central idempotent in $\bar{R}$, and assume that $f \neq 0$. Then $\bar{Q}(f) > 0$ for some $Q \in X$, and we may define $P^* \in \boldsymbol{P}(\bar{R})$ by the rule $P^*(x) = \bar{Q}(fx)/\bar{Q}(f)$. Pulling P^* back to $P \in \boldsymbol{P}(R)$, we see that $P \leqq [1/\bar{Q}(f)]Q$, whence [7, Corollary 3.3] shows that $P \in F$. Clearly $P^* \ll \bar{Q}$ and so $P^* \ll \{\bar{N} \mid N \in X\}$, hence $P^* = \bar{P}$. Thus $\bar{P}(x) = \bar{Q}(fx)/\bar{Q}(f)$ for all $x \in \bar{R}$. Since $ef = f$, we obtain $\bar{P}(e) = 1$, whence $P \in \theta(e)$ and so $1 - \mu\theta(e) \in \ker(\bar{P})$. Now $f = f(1 - \mu\theta(e))$ belongs to $\ker(\bar{P})$, which is impossible, because $\bar{P}(f) = 1$. Therefore $f = 0$, i.e., $\mu\theta(e) = e$.

Finally, consider any $G \in \mathscr{F}$. Since $1 - \mu(G) \in \ker(\bar{N})$ for all $N \in G$, we have $\bar{N}(\mu(G)) = 1$ for all $N \in G$, whence $G \subseteq \theta\mu(G)$. Given any $P \in \theta\mu(G)$, we have $\bar{P}(\mu(G)) = 1$, hence $\ker(\bar{G}) = (1 - \mu(\bar{G}))\bar{R} \leqq \ker(\bar{P})$. According to Corollary 3.6, $P \ll G$, and consequently $P \in G$, by Theorem 1.2. Therefore $\theta\mu(G) = G$.

Therefore θ and μ are inverse order isomorphisms, hence lattice isomorphisms.

LEMMA 3.8. *Let R be a regular ring, let X be a nonempty subset of $\boldsymbol{P}(R)$, let $\bar{R}$ denote the X-completion of R, and let $e \in B(\bar{R})$. Then $e\bar{R}$ is a simple ring if and only if e is an atom of $B(\bar{R})$.*

Proof. Obviously simplicity of $e\bar{R}$ implies atomicity of e. Conversely, assume that e is an atom, so that $e\bar{R}$ is indecomposable as a ring. Since $\bar{R}$ is a regular, right and left self-injective ring by Theorem 3.1, [18, Theorems 4.7, 5.1] show that $e\bar{R}$ is directly finite, whence [16, Proposition 2.7] shows that $e\bar{R}$ is simple.

The following corollaries of Theorem 3.7 extend [6, Theorems 19, 22, 23] to the case of X-completions.

COROLLARY 3.9. *Let R be a regular ring, and let X be a nonempty subset of $\boldsymbol{P}(R)$. Then the following conditions are equivalent:*

(a) *The X-completion of R is a simple ring.*

(b) *X consists of a single extreme point of $\boldsymbol{P}(R)$.*

(c) *The σ-convex face generated by X is minimal among the nonempty σ-convex faces of $\boldsymbol{P}(R)$.*

Proof. Let $\bar{R}$ denote the X-completion of R, let F denote the σ-convex face generated by X in $\boldsymbol{P}(R)$, and let $\mathscr{F}$ denote the lattice of σ-convex faces of F.

(b) $\Rightarrow$ (c): We have $X = \{N\}$ for some extreme point $N \in \boldsymbol{P}(R)$, hence $F = \{N\}$ as well, from which minimality is clear.

(c) $\Rightarrow$ (a): According to (c), $\mathscr{F} = \{\varnothing, F\}$, hence Theorem 3.7 shows that $B(\bar{R}) = \{0, 1\}$. By Lemma 3.8, $\bar{R}$ is simple.

(a) $\Rightarrow$ (b): Obviously $B(\bar{R}) = \{0, 1\}$, hence $\mathscr{F} = \{\varnothing, F\}$, by Theorem 3.7. Choosing $N \in F$, we see that F is the σ-convex face generated by N. According to Proposition 1.7, $\bar{R}$ equals the N-completion of R, whence [6, Corollary 20] shows that N is an extreme point of $\boldsymbol{P}(R)$. Then $\{N\} \in \mathscr{F}$, whence $F = \{N\}$, and consequently $X = \{N\}$.

COROLLARY 3.10. *Let R be a regular ring, let X be a nonempty subset of $\boldsymbol{P}(R)$, and let F be the σ-convex face generated by X in $\boldsymbol{P}(R)$. Then the set of simple ring direct factors of the X-completion of R has the same cardinality as the set of extreme points of F. This is also the same cardinality as that of the set of extreme points of the face generated by X.*

Proof. Let $\bar{R}$ denote the X-completion of R. According to Lemma 3.8, the set of simple ring direct factors of $\bar{R}$ has the same cardinality as the set of atoms of $B(\bar{R})$. Using Theorem 3.7, we can put the atoms of $B(\bar{R})$ in one-to-one correspondence with the minimal (nonempty) σ-convex faces of F. Finally, we see from Corollary 3.9 that the set of minimal σ-convex faces of F has the same cardinality as the set of extreme points of F.

If G is the face generated by X, then clearly any extreme point of G is also an extreme point of F. Inasmuch as F is the σ-convex hull of G (by Theorem 1.2), we conclude that any extreme point of F must lie in G. Therefore F and G have the same extreme points.

COROLLARY 3.11. *Let R be a regular ring, let X be a nonempty subset of $\boldsymbol{P}(R)$, and let k be a positive integer. Then the following conditions are equivalent*:

(a) *The X-completion of R is a direct product of k simple rings.*

(b) *The σ-convex face generated by X can be generated by k distinct extreme points of $\boldsymbol{P}(R)$.*

(c) *The face generated by X is the convex hull of k distinct extreme points of $\boldsymbol{P}(R)$.*

(d) *The face generated by X has dimension $k-1$.*

Proof. Let $\bar{R}$ denote the X-completion of R, let F denote the σ-convex face generated by X in $\boldsymbol{P}(R)$, and let G denote the face generated by X in $\boldsymbol{P}(R)$.

(a) $\Rightarrow$ (b): Clearly $B(\bar{R})$ is an atomic Boolean algebra with k atoms, hence by Theorem 3.7 the same is true of the lattice of σ-convex faces of F. Thus F contains k distinct minimal (nonempty) σ-convex faces $F_1, \cdots, F_k$, and F is generated by $F_1 \cup \cdots \cup F_k$. According to Corollary 3.9, each $F_i = \{N_i\}$ for some extreme point $N_i \in \boldsymbol{P}(R)$. Then $N_1, \cdots, N_k$ are distinct extreme points of $\boldsymbol{P}(R)$, and F is the σ-convex face generated by $\{N_1, \cdots, N_k\}$.

(b) $\Rightarrow$ (a): There exist distinct extreme points $N_1, \cdots, N_k \in \boldsymbol{P}(R)$ such that F is the σ-convex face generated by $\{N_1, \cdots, N_k\}$. Then the lattice of σ-convex faces of F is atomic with k atoms (namely $\{N_1\}, \cdots, \{N_k\}$), hence by Theorem 3.7 the same is true of $B(\bar{R})$. Thus $\bar{R}$ is a direct product of k nonzero indecomposable rings, and by Lemma 3.8 each of these indecomposable rings is simple.

(b) $\Rightarrow$ (c): There exist distinct extreme points $N_1, \cdots, N_k \in \boldsymbol{P}(R)$ such that F is the σ-convex face generated by $\{N_1, \cdots, N_k\}$. Since each $\{N_i\}$ is a σ-convex face of $\boldsymbol{P}(R)$, we see from Lemma 2.1 that F equals the convex hull of $\{N_1, \cdots, N_k\}$. Thus $F = G$, so that G is the convex hull of $\{N_1, \cdots, N_k\}$.

(c) $\Rightarrow$ (b): There exist distinct extreme points $N_1, \cdots, N_k \in \boldsymbol{P}(R)$ such that G is the convex hull of $\{N_1, \cdots, N_k\}$. Since each $\{N_i\}$ is a σ-convex face of $\boldsymbol{P}(R)$, we see from Lemma 2.1 that G is σ-

convex. Thus $F = G$, and F is the σ-convex face generated by $\{N_1, \cdots, N_k\}$.

(c) $\Rightarrow$ (d): There exist distinct extreme points $N_1, \cdots, N_k \in \boldsymbol{P}(R)$ such that G is the convex hull of $\{N_1, \cdots, N_k\}$. Thus the affine span of G equals the affine span of $\{N_1, \cdots, N_k\}$, whence $\dim(G) \leqq k-1$. If $\dim(G) < k-1$, then the N_i must be affinely dependent. After renumbering, we obtain $N_1 = \alpha_2 N_2 + \cdots + \alpha_k N_k$ for some real numbers $\alpha_2, \cdots, \alpha_k$ whose sum is 1. Renumbering once again, we obtain an index t with $2 \leqq t < k$ such that $\alpha_2, \cdots, \alpha_t \leqq 0$ and $\alpha_{t+1}, \cdots, \alpha_k > 0$. Now $N_1 - \alpha_2 N_2 - \cdots - \alpha_t N_t = \alpha_{t+1} N_{t+1} + \cdots + \alpha_k N_k$, and we note that $1 - \alpha_2 - \cdots - \alpha_t = \alpha_{t+1} + \cdots + \alpha_k = \beta > 0$. Thus

$$(\alpha_{t+1}/\beta)N_{t+1} + \cdots + (\alpha_k/\beta)N_k = \beta^{-1}N_1 - (\alpha_2/\beta)N_2 - \cdots - (\alpha_t/\beta)N_t ,$$

so that some positive convex combination of $N_{t+1}, \cdots, N_k$ equals a convex combination of $N_1, \cdots, N_t$.

Let H be the convex hull of $\{N_1, \cdots, N_t\}$, which is a face of $\boldsymbol{P}(R)$ by Lemma 2.1. Since a positive convex combination of $N_{t+1}, \cdots, N_k$ lies in this face, we obtain $N_{t+1}, \cdots, N_k \in H$, whence $G = H$. Using the implication (c) $\Rightarrow$ (a), we find that $\bar{R}$ is a direct product of t simple rings as well as a direct product of k simple rings. Since $t < k$, this is impossible. Therefore $\dim(G) = k-1$.

(d) $\Rightarrow$ (c): Let A denote the affine span of G in $\boldsymbol{R}^R$. Since $\dim(A) = k - 1 < \infty$, A is closed in $\boldsymbol{R}^R$, hence $A \cap \boldsymbol{P}(R)$ is closed in $\boldsymbol{P}(R)$. Given any $P \in A \cap \boldsymbol{P}(R)$, we have $P = \alpha_1 N_1 + \cdots + \alpha_s N_s$ for some $N_1, \cdots, N_s \in G$ and some real numbers $\alpha_1, \cdots, \alpha_s$ whose sum is 1. After renumbering, we obtain an index $t < s$ such that $\alpha_1, \cdots, \alpha_t \leqq 0$ and $\alpha_{t+1}, \cdots, \alpha_s > 0$. Proceeding as above, we obtain a convex combination $\beta_0 P + \beta_1 N_1 + \cdots + \beta_t N_t$ with $\beta_0 > 0$ which equals a convex combination of $N_{t+1}, \cdots, N_s$. Thus $\beta_0 P + \beta_1 N_1 + \cdots + \beta_t N_t$ lies in the face G, whence $P \in G$. Therefore $A \cap \boldsymbol{P}(R) = G$, so that G is closed in $\boldsymbol{P}(R)$.

Now G is a compact convex subset of $\boldsymbol{R}^R$, hence the Krein-Milman Theorem [14, p. 131] says that G is the closure of the convex hull of its extreme points. Suppose G contains $k+1$ distinct extreme points $P_1, \cdots P_{k+1}$. If H is the convex hull of these extreme points, then H is a face of $\boldsymbol{P}(R)$ by Lemma 2.1, and the implication (c) $\Rightarrow$ (d) says that $\dim(H) = k$. Since $H \subseteqq G$, this is impossible. Thus G must have only $h \leqq k$ distinct extreme points $P_1, \cdots, P_h$. Since the convex hull of the finite set $\{P_1, \cdots, P_h\}$ is closed, G must be the convex hull of $\{P_1, \cdots, P_h\}$. Using the implication (c) $\Rightarrow$ (d) again, we find that $\dim(G) = h - 1$, whence $h = k$.

COROLLARY 3.12. *Let R be a regular ring, let X be a nonempty* *subset of* $\boldsymbol{P}(R)$, *and let* F *be the* σ*-convex face generated by* X *in* $\boldsymbol{P}(R)$. *Then the* X*-completion of* R *is a direct product of simple rings if and only if* F *can be generated by some collection of extreme points of* $\boldsymbol{P}(R)$.

Proof. Let $\bar{R}$ denote the X-completion of R, and let $\mathscr{F}$ denote the lattice of σ-convex faces of F.

If $\bar{R}$ is a direct product of simple rings, then $B(\bar{R})$ must be atomic, whence Theorem 3.7 shows that $\mathscr{F}$ is atomic. Thus there exist minimal (nonempty) σ-convex faces $F_i \subseteq F$ such that $F = \vee F_i$ in $\mathscr{F}$. According to Corollary 3.9, each F_i consists of a single extreme point N_i, hence F is the σ-convex face generated by the collection $\{N_i\}$ of extreme points.

Conversely, assume that F is generated by a collection of extreme points of $\boldsymbol{P}(R)$. Then F is the supremum of a collection of atoms in $\mathscr{F}$, whence $\mathscr{F}$ is atomic. By Theorem 3.7, $B(\bar{R})$ is atomic, hence there exist orthogonal atoms $e_j \in B(\bar{R})$ such that $\vee e_j = 1$. Each $e_j\bar{R}$ is a simple ring by Lemma 3.8. Since $\wedge (1 - e_j) = 0$ generates the ideal $\cap (1 - e_j)\bar{R}$, we see that the ideal $\bigoplus e_j\bar{R}$ has zero annihilator in $\bar{R}$. Consequently, we obtain an injective ring map $\phi: \bar{R} \rightarrow \prod e_j\bar{R}$. As in [5, Theorem 18], we conclude that ϕ is an isomorphism, whence $\bar{R}$ is a direct product of simple rings.

Let R be the simple regular ring of [6, Example C]. According to [6, Lemma 31], $\boldsymbol{P}(R)$ has uncountably many distinct extreme points. If F is the σ-convex face generated by the extreme points of $\boldsymbol{P}(R)$, then Corollaries 3.12 and 3.10 show that the F-completion of R is a direct product of uncountably many simple rings.

4. Decomposition of completions.

PROPOSITION 4.1. *Let* R *be a regular ring, let* X_1, X_2 *be nonempty subsets of* $\boldsymbol{P}(R)$ *such that* $X_1 \ll X_2$, *and let* $\bar{R}_i$ *denote the* X_i*-completion of* R.

(a) *The natural map* $R/\ker(X_2) \rightarrow R/\ker(X_1)$ *extends uniquely to a continuous map* $\phi: \bar{R}_2 \rightarrow \bar{R}_1$. *Moreover,* ϕ *is a ring map.*

(b) *For each* $N \in X_1$, *let* $\bar{N}$ *denote the natural extension of* N *to* $\boldsymbol{P}(\bar{R}_1)$ *and let* N^* *denote the continuous extension of* N *to* $\boldsymbol{P}(\bar{R}_2)$. *Then* $N^* = \bar{N}\phi$.

(c) *If* $X_1^* = \{N^* \mid N \in X_1\}$, *then* $\ker\phi = \ker(X_1^*)$.

Proof. (a) The existence and uniqueness of ϕ are standard properties of completions. Since the ring operations in each $\bar{R}_i$ are continuous, ϕ is a ring map.

(b) is exactly analogous to [7, Lemma 2.4].

(c) If $\bar{X}_1 = \{\bar{N} \mid N \in X_1\}$, then $\ker(\bar{X}_1) = 0$ because $\bar{R}_1$ is the X-completion of R. Thus it follows from (b) that $\ker \phi = \bigcap_{N \in X_1} \ker(\bar{N}\phi) = \ker(X_1^*)$.

DEFINITION. In the situation of Proposition 4.1, we refer to ϕ as the *natural map* from $\bar{R}_2$ to $\bar{R}_1$.

THEOREM 4.2. *Let R be a regular ring, let X_1, X_2 be nonempty subsets of $\boldsymbol{P}(R)$, and let $\bar{R}_i$ denote the X_i-completion of R. If $X_1 \ll X_2$, then the natural map $\phi: \bar{R}_2 \to \bar{R}_1$ is surjective.*

Proof. For all $N \in X_i$, let $\bar{N}$ denote the natural extension of N to $\boldsymbol{P}(\bar{R}_i)$. For all $N \in X_1$, let N^* denote the continuous extension of N to $\boldsymbol{P}(\bar{R}_2)$, and note from Proposition 4.1 that $N^* = \bar{N}\phi$. Set $\bar{X}_i = \{\bar{N} \mid N \in X_i\}$ and $X_1^* = \{N^* \mid N \in X_1\}$, and note from Proposition 4.1 that $\ker \phi = \ker(X_1^*)$.

According to Lemma 3.2, there is a central idempotent $e \in \bar{R}_2$ such that $(1 - e)\bar{R}_2 = \ker(X_1^*)$, and we note that $e \neq 0$. Set $X_2' = \{N \in X_2 \mid \bar{N}(e) \neq 0\}$, which is nonempty because $\ker(\bar{X}_2) = 0$. For each $N \in X_2'$, we may define $\bar{N}' \in \boldsymbol{P}(\bar{R}_2)$ by the rule $\bar{N}'(x) = \bar{N}(ex)/\bar{N}(e)$. Since $\bar{N}' \leqq [1/\bar{N}(e)]\bar{N}$, we have $\bar{N}' \ll \bar{N}$, hence $\bar{N}' \ll \bar{X}_2$. Setting $\bar{X}_2' = \{\bar{N}' \mid N \in X_2'\}$, we thus have $\bar{X}_2' \ll \bar{X}_2$.

Obviously $1 - e \in \ker(\bar{X}_2')$. Given any $x \in \bar{R}_2$ for which $ex \neq 0$, we have $\bar{N}(ex) \neq 0$ for some $N \in X_2$. For this N, $\bar{N}(e) \neq 0$ as well, whence $N \in X_2'$ and $\bar{N}'(x) \neq 0$. Thus $\ker(\bar{X}_2') = (1 - e)\bar{R}_2 = \ker(X_1^*)$, hence Corollary 3.6 shows that $\bar{X}_2' \ll X_1^*$.

Now let ψ_i denote the natural map $R \to \bar{R}_i$, and note that $\phi\psi_2 = \psi_1$. Given any $x \in \bar{R}_1$, there exists a net $\{x_j\} \subseteqq R$ such that $\phi\psi_2(x_j) = \psi_1(x_j) \to x$ in the $\bar{X}_1$-topology. Since $(1 - e)\bar{R}_2 = \ker(X_1^*) = \ker \phi$, we see that $\phi(e\psi_2(x_j)) \to x$ as well. Now

$$N^*(e\psi_2(x_j) - e\psi_2(x_k)) = \bar{N}(\phi(e\psi_2(x_j)) - \phi(e\psi_2(x_k)))$$

for all j, k and all $N \in X_1$, hence the net $\{e\psi_2(x_j)\} \subseteqq \bar{R}_2$ must be Cauchy with respect to X_1^*. Inasmuch as $\bar{X}_2' \ll X_1^*$, it follows that $\{e\psi_2(x_j)\}$ is also Cauchy with respect to $\bar{X}_2'$. Since

$$\bar{N}(e\psi_2(x_j) - e\psi_2(x_k)) = \bar{N}(e)\bar{N}'(e\psi_2(x_j) - e\psi_2(x_k))$$

for all j, k and all $N \in X_2'$, $\{e\psi_2(x_j)\}$ is Cauchy with respect to $\bar{N}$ for all $N \in X_2'$. In addition, we have $\bar{N}(e\psi_2(x_j) - e\psi_2(x_k)) \leqq \bar{N}(e) = 0$ for all j, k and all $N \in X_2 - X_2'$, hence $\{e\psi_2(x_j)\}$ is Cauchy with respect to $\bar{N}$ in this case as well. Therefore the net $\{e\psi_2(x_j)\} \subseteqq \bar{R}_2$ is Cauchy with respect to $\bar{X}_2$.

By completeness, there exists $y \in \bar{R}_2$ such that $e\psi_2(x_j) \to y$ in the $\bar{X}_2$-topology. Since ϕ is continuous, $\phi(e\psi_2(x_j)) \to \phi(y)$ in the $\bar{X}_1$-topology, and consequently $\phi(y) = x$.

Therefore ϕ is surjective.

DEFINITION. Let R be a regular ring, let $\{X_i\}$ be a nonempty family of nonempty subsets of $\boldsymbol{P}(R)$, and for each i let $\bar{R}_i$ denote the X_i-completion of R. If $\bar{R}$ denotes the $(\bigcup X_i)$-completion of R, then we have natural maps $\phi_i\colon \bar{R} \to \bar{R}_i$ for each i. Together, these maps induce a map $\phi\colon \bar{R} \to \prod \bar{R}_i$, which we of course call the *natural map*.

COROLLARY 4.3. *Let R be a regular ring, let $\{X_i\}$ be a nonempty family of nonempty subsets of $\boldsymbol{P}(R)$, and for each i let $\bar{R}_i$ denote the X_i-completion of R. If $\bar{R}$ denotes the $(\bigcup X_i)$-completion of R, then the natural map $\phi\colon \bar{R} \to \prod \bar{R}_i$ yields an isomorphism of $\bar{R}$ onto a subdirect product of the $\bar{R}_i$.*

Proof. For each $N \in \bigcup X_i$, let $\bar{N}$ denote the natural extension of N to $\boldsymbol{P}(\bar{R})$. Set $\bar{X}_i = \{\bar{N} \mid N \in X_i\}$ for each i, and note from Proposition 4.1 that $\ker(\bar{X}_i)$ equals the kernel of the natural map ϕ_i: $\bar{R} \to \bar{R}_i$. As a result, $\ker \phi = \bigcap \ker \phi_i = \bigcap \ker(\bar{X}_i) = \ker(\bigcup \bar{X}_i) = 0$, hence ϕ is injective. Inasmuch as each ϕ_i is surjective by Theorem 4.2, $\phi(\bar{R})$ is a subdirect product of the $\bar{R}_i$.

THEOREM 4.4. *Let R be a regular ring, let F be a nonempty σ-convex face of $\boldsymbol{P}(R)$, and let $\bar{R}$ denote the F-completion of R. Let $\mathscr{F}$ denote the lattice of σ-convex faces of F, and for each nonempty $G \in \mathscr{F}$ let $\bar{R}_G$ denote the G-completion of R. Then there is a lattice isomorphism $\mu\colon \mathscr{F} \to B(\bar{R})$ such that $\mu(G)\bar{R} \cong \bar{R}_G$ for all nonempty $G \in \mathscr{F}$.*

Proof. Set $\bar{G} = \{\bar{N} \mid N \in G)$ for all $G \in \mathscr{F}$. Using Theorem 3.7, we obtain a lattice isomorphism $\mu\colon \mathscr{F} \to B(\bar{R})$ such that $(1 - \mu(G))\bar{R} = \ker(\bar{G})$ for all $G \in \mathscr{F}$. Given a nonempty $G \in \mathscr{F}$, the natural map $\phi_G\colon \bar{R} \to \bar{R}_G$ is surjective by Theorem 4.2. Since $\ker(\phi_G) = \ker(\bar{G}) = (1 - \mu(G))\bar{R}$ by Proposition 4.1, we conclude that ϕ_G restricts to an isomorphism of $\mu(G)\bar{R}$ onto $\bar{R}_G$.

Taking account of Proposition 1.7, Theorem 4.4 shows that whenever $X \subseteqq Y$ are nonempty subsets of $\boldsymbol{P}(R)$, then the Y-completion of R contains a copy of the X-completion of R. In particular, the $\boldsymbol{P}(R)$-completion of R is the "largest" completion, since it contains copies of all the X-completions of R.

PROPOSITION 4.5. *Let R be a regular ring, let $\{X_k\}$ be a nonempty family of nonempty subsets of $\boldsymbol{P}(R)$, amd for each k let $\bar{R}_k$ denote the X_k-completion of R. Let $\bar{R}$ denote the $(\cup X_k)$-completion of R, for each $N \in \cup X_k$ let $\bar{N}$ denote the natural extension of N to $\boldsymbol{P}(\bar{R})$, and for each k set $\bar{X}_k = \{\bar{N} \mid N \in X_k\}$. Then the natural map $\phi: \bar{R} \to \prod \bar{R}_k$ is an isomorphism if and only if* $\ker(\bar{X}_i) + \ker(\bar{X}_j) = \bar{R}$ *for all $i \neq j$.*

Proof. Note that the natural map $\phi_i: \bar{R} \to \bar{R}_i$ is the composition of ϕ with the projection $\prod \bar{R}_k \to \bar{R}_i$. If ϕ is an isomorphism, then clearly $\ker(\phi_i) + \ker(\phi_j) = \bar{R}$ for all $i \neq j$, whence Proposition 4.1 shows that $\ker(\bar{X}_i) + \ker(\bar{X}_j) = \bar{R}$ for all $i \neq j$.

Conversely, assume that $\ker(\bar{X}_i) + \ker(\bar{X}_j) = \bar{R}$ for all $i \neq j$. Using Lemma 3.2, we obtain central idempotents $e_k \in \bar{R}$ such that $(1 - e_k)\bar{R} = \ker(\bar{X}_k)$. Inasmuch as $(1 - e_i)\bar{R} + (1 - e_j)\bar{R} = \bar{R}$ for all $i \neq j$, we see that the e_k are pairwise orthogonal. Since $\bar{R}$ is the $(\cup X_k)$-completion of R, we have $\cap \ker(\bar{X}_k) = 0$, so that $\cap (1 - e_k)\bar{R} = 0$. Thus the annihilator of the ideal $\bigoplus e_k\bar{R}$ is zero. Proceeding as in [5, Theorem 18], we see that the natural map $\psi: \bar{R} \to \prod e_k\bar{R}$ is an isomorphism.

For each k, $\ker(\phi_k) = \ker(\bar{X}_k) = (1 - e_k)\bar{R}$ by Proposition 4.1, hence ϕ_k induces a monomorphism $\theta_k: e_k\bar{R} \to \bar{R}/\ker(\bar{X}_k) \to \bar{R}_k$. According to Theorem 4.2, ϕ_k is surjective, whence θ_k is an isomorphism. As a result, these θ_k induce an isomorphism $\theta: \prod e_k\bar{R} \to \prod \bar{R}_k$. Observing that $\phi = \theta\psi$, we conclude that ϕ is an isomorphism.

THEOREM 4.6. *Let R be a regular ring, let $\{X_k\}$ be a nonempty family of nonempty subsets of $\boldsymbol{P}(R)$, and let $\bar{R}$ denote the $(\cup X_k)$-completion of R. For each k, let $\bar{R}_k$ denote the X_k-completion of R, and let F_k be the face generated by X_k in $\boldsymbol{P}(R)$. Then the natural map $\phi: \bar{R} \to \prod \bar{R}_k$ is an isomorphism if and only if the faces F_k are pairwise disjoint.*

Proof. For each $N \in \cup X_k$, let $\bar{N}$ denote the natural extension of N to $\boldsymbol{P}(\bar{R})$. For each k, set $\bar{X}_k = \{\bar{N} \mid N \in X_k\}$.

First assume that there exists $P \in F_i \cap F_j$ for some $i \neq j$. By [7, Corollary 3.3], there exist Q_i in the couvex hull of X_i and Q_j in the convex hull of F_j such that $P \leqq \alpha Q_i, \alpha Q_j$ for some $\alpha > 0$. Now $P \ll Q_i \ll X_i \ll \cup X_k$, hence P has a continuous extension $\bar{P} \in \boldsymbol{P}(\bar{R})$. By continuity, $\bar{P} \leqq \alpha\bar{Q}_i, \alpha\bar{Q}_j$, whence

$$\ker(\bar{X}_i) + \ker(\bar{X}_j) \leqq \ker(\bar{Q}_i) + \ker(\bar{Q}_j) \leqq \ker(\bar{P}) < \bar{R} .$$

Then Proposition 4.5 says that ϕ is not an isomorphism.

Conversely, if ϕ is not an isomorphism, then by Proposition 4.5

we must have $\ker(\bar{X}_i) + \ker(\bar{X}_j) \neq \bar{R}$ for some $i \neq j$. By Lemma 3.2, $\ker(\bar{X}_i)$ and $\ker(\bar{X}_j)$ are each generated by a central idempotent, hence there is a central idempotent $e \neq 0$ in $\bar{R}$ such that $(1-e)\bar{R} = \ker(\bar{X}_i) + \ker(\bar{X}_j)$. Then $\bar{N}(e) \neq 0$ for some $N \in \cup X_k$, hence we may define $\bar{Q} \in \boldsymbol{P}(\bar{R})$ by the rule $\bar{Q}(x) = \bar{N}(ex)/\bar{N}(e)$. Pulling $\bar{Q}$ back to $Q \in \boldsymbol{P}(R)$, we see that $Q \leqq [1/\bar{N}(e)]N$, whence $Q \ll N \ll \cup X_k$. Inasmuch as $\ker(\bar{X}_i) + \ker(\bar{X}_j) = (1-e)\bar{R} \leqq \ker(\bar{Q})$, Corollary 3.6 says that $Q \ll X_i, X_j$. According to Theorem 1.2, Q lies in the σ-convex hulls of F_i and F_j. Therefore F_i and F_j are not disjoint, by Lemma 2.5.

COROLLARY 4.7. *Let R be a regular ring, and let $\{F_k\}$ be a nonempty family of nonempty faces of $\boldsymbol{P}(R)$. Let $\bar{R}$ denote the $(\cup F_k)$-completion of R, and for each k let $\bar{R}_k$ denote the F_k-completion of R. If the F_k are pairwise disjoint, then $\bar{R} \cong \prod \bar{R}_k$.*

Theorem 4.6 and Corollary 4.7 are generalizations of [7, Theorem 4.3 and Corollary 4.4], for if $N \in \boldsymbol{P}(R)$ is a positive σ-convex combination of some $P_k \in \boldsymbol{P}(R)$, then the σ-convex face generated by N coincides with the σ-convex face generated by the P_k.

5. Extending pseudo-rank functions to completions.

[7, Theorem 7.4] gives a description of the closure of the face generated by a subset $X \subseteq \boldsymbol{P}(R)$. This theorem is a bit awkward, because it is not constructed in terms of the X-completion of R. A more natural description of closures of faces is given by the following theorem.

THEOREM 5.1. *Let R be a regular ring, let X be a nonempty subset of $\boldsymbol{P}(R)$, and let $\bar{R}$ denote the X-completion of R. Let ϕ: $R \to \bar{R}$ be the natural map, and let $P \in \boldsymbol{P}(R)$. Then P lies in the closure of the face generated by X in $\boldsymbol{P}(R)$ if and only if $P = P'\phi$ for some $P' \in \boldsymbol{P}(\bar{R})$.*

Proof. Since $\bar{R}$ is a regular, right and left self-injective ring by Theorem 3.1, [17, Theorems 4.7, 5.1] show that $\bar{R}$ is directly finite.

Assume first that $P = P'\phi$ for some $P' \in \boldsymbol{P}(\bar{R})$. If $\bar{X} = \{\bar{N} \mid N \in X\}$ (where $\bar{N}$ denotes the natural extension of N to $\boldsymbol{P}(\bar{R})$), then $\ker(\bar{X}) = 0 \leqq \ker(P')$, hence [7, Theorem 7.1] says that P' lies in the closure of the face generated by $\bar{X}$ in $\boldsymbol{P}(\bar{R})$. As a result, we infer that $P = P'\phi$ lies in the closure of the face generated by $\bar{X}\phi = X$.

Conversely, let F denote the face generated by X in $\boldsymbol{P}(R)$, and assume that P lies in the closure of F. By Theorem 1.2, $N \ll X$ for each $N \in F$, hence each such N has a continuous extension $\bar{N} \in \boldsymbol{P}(\bar{R})$ such that $\bar{N}\phi = N$. If $\phi^*: \boldsymbol{P}(\bar{R}) \to \boldsymbol{P}(R)$ is the map induced by

ϕ, we thus have $F \subseteq \phi^*(\boldsymbol{P}(\bar{R}))$. Now $\phi^*(\boldsymbol{P}(\bar{R}))$ is a continuous image of a compact space and so is compact, hence closed in $\boldsymbol{P}(R)$. Therefore $\phi^*(\boldsymbol{P}(\bar{R}))$ contains the closure of F, whence $P \in \phi^*(\boldsymbol{P}(\bar{R}))$, i.e., $P = P'\phi$ for some $P' \in \boldsymbol{P}(\bar{R})$.

6. Completeness versus self-injectivity. Theorem 3.1 shows that any regular ring R which is complete with respect to a nonempty set X of pseudo-rank functions is right and left self-injective. Since self-injectivity may be viewed as an algebraic completeness property, it is natural to ask about the converse implication: If R is a regular, right and left self-injective ring, must R be complete with respect to some family of pseudo-rank functions? For indecomposable rings, the next theorem shows that the answer is yes. In general, we show that the answer depends on whether or not $B(R)$ is complete, and can be negative.

THEOREM 6.1. *Let R be a regular, right and left self-injective ring which is indecomposable* (as a ring). *Then there exists a unique rank function N on R, and R is complete in the N-metric.*

Proof. By [18, Theorems 4.7, 5.1], R is directly finite, whence [16, Proposition 2.7] shows that R is a simple ring. In addition, [5, Lemma 5′, p. 832] shows that for any $x, y \in R$, either $xR \lesssim yR$ or $yR \lesssim xR$, i.e., R satisfies the "comparability axiom" of [9, p. 812]. As a result, [9, Corollary 3.15] shows that there exists a unique rank function N on R.

According to [17, Corollary to Theorem 1], the lattice $L(R)$ of principal right ideals of R is continuous, i.e., $L(R)$ is a continuous geometry. Since R is indecomposable, $L(R)$ is irreducible [19, Theorem 2.9, p. 76]. As a result, [19, Theorem 17.4, p. 230] says that R is complete in the N-metric.

In general, a regular ring may be complete with respect to some families of pseudo-rank functions but not others. As the following example shows, there exists a regular, right and left self-injective ring R with rank functions N, N' such that R is complete in the N-metric but not in the N'-metric.

Choose fields $F_1, F_2, \cdots$ and set $R = \prod F_n$, which is a regular self-injective ring. If e_n denotes the unit of F_n, then $R/(1-e_n)R \cong F_n$, hence there exists a unique pseudo-rank function $P_n \in \boldsymbol{P}(R)$ with $\ker(P_n) = (1 - e_n)R$. Setting $N = \sum_{n=1}^{\infty} P_n/2^n$, we obtain a rank function N on R, and it is clear that R is complete in the N-metric. Now choose a maximal ideal M of R which contains $\bigoplus F_n$. There is a unique pseudo-rank function $P \in \boldsymbol{P}(R)$ with $\ker(P) = M$, and we set $N' = (N + P)/2$ which is a rank function on R. If R is complete in the N'-metric, then we see from Lemma 3.2 then $\sum e_n \rightarrow 1$ in the N'-metric. However, $\sum_{n=1}^{\infty} N'(e_n) = 1/2$, hence this is impossible. Therefore R is not complete in the N'-metric.

We now proceed to show that a regular ring R is complete with respect to a family X of pseudo-rank functions provided only that $B(R)$ is complete with respect to X. As with Theorem 3.1, we must first prove the case of a single pseudo-rank function. In this case, the proof of [19, Theorem 17.4, p. 230] may be applied, once we have shown that the pseudo-rank function involved satisfies a certain countable additivity property, as follows.

DEFINITION. Let R be a regular ring, let $N \in \boldsymbol{P}(R)$, and let J be a right ideal of R. We shall say that N is *countably additive on* J provided that whenever $x_1R, x_2R, \cdots$ is a countable sequence of independent principal right ideals contained in J and $\bigoplus x_nR$ is essential in xR for some $x \in J$, then $N(x) = \sum N(x_n)$. If this holds for $J = R$, then we simply say that N is *countably additive*.

LEMMA 6.2. *Let R be a regular ring, let $N \in \boldsymbol{P}(R)$, let J be a right ideal of R, and assume that N is countably additive on J. If $x, x_1, x_2, \cdots \in J$ and $\sum x_nR$ is essential in xR, then $N(x) \leqq \sum N(x_n)$.*

Proof. We may choose independent principal right ideals $y_1R, y_2R, \cdots \leqq J$ such that $y_1R \oplus \cdots \oplus y_kR = x_1R + \cdots + x_kR$ for all k. Since N is countably additive on J, we obtain $N(x) = \sum N(y_n)$. In addition, we have $y_1R \oplus \cdots \oplus y_kR \lesssim x_1R \oplus \cdots \oplus x_kR$ for each k and so $N(y_1) + \cdots + N(y_k) \leqq N(x_1) + \cdots + N(x_k)$, by [7, Lemma 6.6]. Thus $N(x) = \sum N(y_n) \leqq \sum N(x_n)$.

LEMMA 6.3. *Let R be a regular, right and left self-injective ring with a rank function N. Let e be an idempotent in R such that N is countably additive on $(1 - e)R$. Then $(1 - e)Re$ is complete in the N-metric.*

Proof. Let $L(R)$ denote the lattice of principal right ideals of R, which is continuous by [17, Corollary to Theorem 1].

Let $\{x_n\}$ be a Cauchy sequence in $(1 - e)Re$. By passing to a subsequence, we may assume that $N(x_i - x_j) < 1/2^{k+1}$ whenever $i, j \geqq k$. Now define $a_nR, b_kR, cR \in L(R)$ as follows: $a_nR = (e + x_n)R$, $b_kR = E(\sum_{n=k}^{\infty} a_nR)$, $cR = \bigcap_{k=1}^{\infty} b_kR$. Note that $a_kR \leqq b_kR$ for all k and that $b_1R \geqq b_2R \geqq \cdots$. Since $x_k \in (1 - e)R$ and $e + x_k \in a_kR$, we see that $a_kR + (1 - e)R = R$, whence $b_kR + (1 - e)R = R$ for all k. Inasmuch as $L(R)$ is lower continuous, we thus obtain

$$cR + (1-e)R = \left(\bigcap_{k=1}^{\infty} b_kR\right) + (1-e)R = \bigcap_{k=1}^{\infty} [b_kR + (1-e)R] = R .$$

As a result, there exists an idempotent $f \in R$ such that $fR \leqq cR$ and $(1-f)R = (1-e)R$.

Since $(1-f)R = (1-e)R$, we have $Rf = Re$, hence $f = fe$ and $e = ef$. As a result, we see that the element $x = f - e$ lies in $(1-e)Re$. Note also that $e + x = f \in cR$. We shall show that $x_n \to x$.

Whenever $n \geqq k$,

$$\begin{aligned} a_nR = (e + x_n)R &= [e + x_k + \sum_{j=k+1}^{n} (x_j - x_{j-1})]R \\ &\leqq (e + x_k)R + \sum_{j=k+1}^{\infty} (x_j - x_{j-1})R \leqq a_kR + \sum_{j=k+1}^{\infty} (x_j - x_{j-1})R . \end{aligned}$$

Defining $d_kR = E(\sum_{j=k+1}^{\infty} (x_j - x_{j-1})R) \leqq (1-e)R$, we thus have $a_nR \leqq a_kR + d_kR$ for all $n \geqq k$. As a result, $\sum_{n=k}^{\infty} a_nR \leqq a_kR + d_kR$, whence $b_kR \leqq a_kR + d_kR$. We also have $b_kR = a_kR \oplus u_kR$ for some u_k, whence $a_kR \oplus u_kR \leqq a_kR + d_kR \lesssim a_kR \oplus d_kR$. According to [18, Theorems 4.7, 5.1], R is directly finite, hence [8, Corollary 3.9] implies that $u_kR \lesssim d_kR$. Since $d_k \in (1-e)R$ and all $x_j - x_{j-1} \in (1-e)R$, we may use Lemma 6.2 to obtain

$$N(u_k) \leqq N(d_k) \leqq \sum_{j=k+1}^{\infty} N(x_j - x_{j-1}) < \sum_{j=k+1}^{\infty} 1/2^j = 1/2^k$$

for all k.

Now $f \in cR \leqq b_kR = a_kR + u_kR = (e + x_k)R + u_kR$, hence $f = (e + x_k)r + u_ks$ for some $r, s \in R$. Since $x_k \in (1-e)Re$, $e + x_k$ is idempotent, so that $(e + x_k)f = (e + x_k)r + (e + x_k)u_ks$. We also have $e + x_k \in Re = Rf$, hence $e + x_k = (e + x_k)f = (e + x_k)r + (e + x_k)u_ks = f - u_ks + (e + x_k)u_ks = f + (e + x_k - 1)u_ks$. Consequently,

$$x_k - x = (e + x_k) - (e + x) = e + x_k - f = (e + x_k - 1)u_ks ,$$

and so $N(x_k - x) \leqq N(u_k) < 1/2^k$.

Therefore $x_n \to x$.

THEOREM 6.4. *Let R be a regular, right and left self-injective ring with a rank function N. Then N is countably additive if and only if R is complete in the N-metric.*

Proof. First assume that R is complete, and let $x_1R, x_2R, \cdots$ be independent principal right ideals such that $\oplus x_nR$ is essential in some principal right ideal xR. For each k, choose $y_k \in R$ such that $y_kR = x_1R \oplus \cdots \oplus x_kR$. Then $y_1R \leqq y_2R \leqq \cdots$ and $\cup y_kR$ is essential in xR, whence Lemma 3.4 says that $N(x) = \sup N(y_k)$. Since $N(y_k) = N(x_1) + \cdots + N(x_k)$ for all k, we obtain $N(x) = \sum N(x_n)$. Thus N

is countably additive.

Conversely, assume that N is countably additive, and let T denote the ring of all 2×2 matrices over R. By [12, Theorem 1], N induces a rank function P on T such that $P\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = N(x)$ for all $x \in R$. Given $x \in R$, $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}T$ and $\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}T$ are isomorphic principal right ideals of T such that $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}T \oplus \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}T = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}T$, from which we see that $P\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = N(x)/2$. Also, $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}T \cong \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}T$, hence $P\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = N(x)/2$ as well.

The rule $xR \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}T$ defines an isomorphism from the lattice of principal right ideals of R onto the lattice of those principal right ideals of T which are contained in $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}T$. Inasmuch as $P\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = N(x)/2$ for all $x \in R$, we infer from the countable additivity of N that P must be countably additive on $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}T$. As a result, Lemma 6.3 shows that $\begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$ is complete in the P-metric, from which we conclude that R is complete in the N-metric.

The result of Theorem 6.4 is used in the proof of [10, Corollaire 2.8], although the reference quoted there only covers the case in which the ring is indecomposable.

DEFINITION. Let R be a regular ring, and let X be a nonempty subset of $\boldsymbol{P}(R)$ such that $\ker(X) = 0$. We shall say that $B(R)$ is *orthogonally complete with respect to* X provided that for any orthogonal family $\{e_i\} \subseteq B(R)$, $\sum e_i$ converges to some $e \in B(R)$. Note that when $\sum e_i \to e$, we have $e = \vee e_i$. Thus if $B(R)$ is orthogonally complete with respect to X, then $B(R)$ is also complete as a lattice.

For the case of a rank function N, we proceed to show that if R is self-injective and $B(R)$ is orthogonally complete with respect to N, then R is complete in the N-metric. In order to accomplish this, we must consider the Type I and Type II cases separately. (See [8, 15] for the definitions.)

PROPOSITION 6.5. *Let R be a regular, right and left self-injective ring of Type I with a rank function N. If $B(R)$ is orthogonally complete with respect to N, then R is complete in the N-metric.*

Proof. Case I. R is abelian.

Let $x_1R, x_2R, \cdots$ be an independent family of principal right ideals of R, and let $\bigoplus x_nR$ be essential in some principal right ideal xR. Choose idempotents $e, e_1, e_2, \cdots \in R$ such that $eR = xR$ and $e_nR = x_nR$ for all n. Since R is abelian, we have $e, e_1, e_2, \cdots \in B(R)$, the e_n are pairwise orthogonal, and $e = \vee e_n$. Inasmuch as $B(R)$ is orthogonally complete with respect to N, $\sum e_n \to \vee e_n = e$ in the N-metric, whence $\sum N(x_n) = \sum N(e_n) = N(e) = N(x)$. Therefore N is countably additive, hence Theorem 6.4 says that R is complete in the N-metric.

Case II. R is Type I_n for some n.

There exist $n \times n$ matrix units $e_{ij} \in R$ such that the ring $T = e_{11}Re_{11}$ is abelian. We may define a rank function P on T by the rule $P(x) = N(x)/N(e_{11})$. Inasmuch as the rule $e \mapsto e_{11}e$ defines an isomorphism of $B(R)$ onto $B(T)$, we infer that $B(T)$ must be orthogonally complete with respect to P. As a result, Case I shows that T is complete in the P-metric, hence also in the N-metric. For any i, j, there is an additive isomorphism of T onto $e_{ii}Re_{jj}$ given by the rule $x \mapsto e_{i1}xe_{1j}$, and we observe that $N(x) = N(e_{i1}xe_{1j})$ for all $x \in T$. Thus each $e_{ii}Re_{jj}$ must be complete in the N-metric, whence R is complete in the N-metric.

Case III. General case.

According to [17, Theorems 4.7, 5.1], R is directly finite, hence Type I_f. Consequently, R is isomorphic to a direct product of rings of Type I_n [8, Corollary 6.5], [16, Corollaire 3.5]. Thus there exist orthogonal central idempotents $e_1, e_2, \cdots \in B(R)$ such that $\vee e_n = 1$, each e_nR is Type I_n, and $R = \prod e_nR$.

Whenever $e_n \neq 0$, we may define a rank function P_n on e_nR by the rule $P_n(x) = N(x)/N(e_n)$. Since $B(e_nR) = B(R) \cap e_nR$, $B(e_nR)$ is orthogonally complete with respect to P_n, whence Case II shows that e_nR is complete in the P_n-metric and thus in the N-metric.

Given any Cauchy sequence $\{x_n\} \subseteq R$, it follows that for each n, the sequence $\{e_nx_1, e_nx_2, \cdots\}$ converges to some $y_n \in e_nR$. Inasmuch as $R = \prod e_nR$, we thus have $y \in R$ such that $e_ny = y_n$ for all n, i.e., $e_nx_k \to e_ny$ for each n. Also, because $B(R)$ is orthogonally complete, we have $\sum e_n \to \vee e_n = 1$, whence $\sum_n e_nx_k \to x_k$ for all k and $\sum_n e_ny \to y$. Thus $x_k \to y$.

LEMMA 6.6. *Let R be a regular, right self-injective ring, and let X be a nonempty subset of $\boldsymbol{P}(R)$ such that* $\ker(X) = 0$. *Let $x, y \in R$ and $g \in B(R)$.*

(a) *If $N(ex) \leqq N(ey)$ for all $e \leqq g$ in $B(R)$ and all $N \in X$, then $gxR \lesssim gyR$.*

(b) *If $N(ex) = N(ey)$ for all $e \leqq g$ in $B(R)$ and all $N \in X$, then $gxR \cong gyR$.*

Proof. (a) By [16, Théorème 1.1] or [8, Theorem 3.3], there exists $e \in B(R)$ such that $egyR \lesssim egxR$ and $(1-e)gxR \lesssim (1-e)gyR$. Then $egxR = aR \oplus bR$ with $aR \cong egyR$, and $N(b) = N(egx) - N(egy) \leqq 0$

for all $N \in X$. Since $\ker(X) = 0$, we obtain $b = 0$, hence $egxR \cong egyR$. Thus $gxR \lesssim gyR$.

(b) is proved in the same manner.

LEMMA 6.7. *Let R be a regular ring, let X be a nonempty subset of $\boldsymbol{P}(R)$ such that $\ker(X) = 0$, and assume that $B(R)$ is orthogonally complete with respect to X. Let $\phi: B(R) \to \boldsymbol{R}$ be a continuous map such that $\phi(e+f) = \phi(e) + \phi(f)$ for all orthogonal $e, f \in B(R)$. Then there exists $g \in B(R)$ such that $\phi(e) \geqq 0$ for all $e \leqq 1 - g$ in $B(R)$ and $\phi(e) < 0$ for all nonzero $e \leqq g$ in $B(R)$.*

Proof. Set $A = \{f \in B(R) \mid \phi(e) \geqq 0$ for all $e \leqq f$ in $B(R)\}$, and choose a maximal orthogonal family $\{h_i\} \subseteqq A$. By orthogonal completeness, $\sum h_i$ converges to some $h \in B(R)$. Given $e \leqq h$ in $B(R)$, we note that $\{eh_i\}$ is an orthogonal family in $B(R)$ such that $\sum eh_i \to e$. For any finite set F of indices, we have $\phi(\sum_{i \in F} eh_i) = \sum_{i \in F} \phi(eh_i) \geqq 0$ since each $h_i \in A$. Thus $\phi(e) \geqq 0$, by continuity.

Setting $g = 1 - h \in B(R)$, we now have $\phi(e) \geqq 0$ for all $e \leqq 1 - g$ in $B(R)$.

Now consider any nonzero $e \leqq g$ in $B(R)$. Since e is orthogonal to each h_i, it follows from the maximality of the family $\{h_i\}$ that e does not lie above any nonzero member of A. As a result, each nonzero $f \leqq e$ in $B(R)$ must lie above some member of the set $B = \{f \in B(R) \mid \phi(f) < 0\}$. Consequently, there exists an orthogonal family $\{f_j\} \subseteqq B$ such that $\vee f_j = e$, and by orthogonal completeness we obtain $\sum f_j \to e$. Choose a particular index k. Given any finite set F of indices such that $k \in F$, we have $\phi(\sum_{j \in F} f_j) = \sum_{j \in F} \phi(f_j) \leqq \phi(f_k)$ since each $f_j \in B$. By continuity, $\phi(e) \leqq \phi(f_k) < 0$.

PROPOSITION 6.8. *Let R be a regular, right self-injective ring of Type II with a rank function N. If $B(R)$ is orthogonally complete with respect to N, then N is countably additive.*

Proof. Let $x_1R, x_2R, \cdots$ be independent principal right ideals of R, and let $\bigoplus x_nR$ be essential in some principal right ideal xR. For $k = 1, 2, \cdots$, we have $x_1R \oplus \cdots \oplus x_kR \leqq xR$, whence $N(x_1) + \cdots + N(x_k) \leqq N(x)$. Thus $\sum N(x_n) \leqq N(x)$. Suppose that $\sum N(x_n) < N(x)$, and choose a positive integer t such that $\sum N(x_n) < N(x) - (1/t)$.

The rule $\phi(e) = \sum N(ex_n) - N(ex) + N(e)/t$ defines a continuous map $\phi: B(R) \to \boldsymbol{R}$ such that $\phi(e+f) = \phi(e) + \phi(f)$ for all orthogonal idempotents $e, f \in B(R)$. Applying Lemma 6.7, we obtain $g \in B(R)$ such that $\sum N(ex_n) \geqq N(ex) - N(e)/t$ for all $e \leqq 1 - g$ in $B(R)$ and $\sum N(ex_n) < N(ex) - N(e)/t$ for all nonzero $e \leqq g$ in $B(R)$. Inasmuch as $\sum N(x_n) < N(x) - (1/t)$, we see that $g \neq 0$.

Since R is Type II, it contains no nonzero abelian idempotents, hence [8, Proposition 5.8] says that there is some $y \in R$ for which $t(yR) \cong R$. Note that $t(gyR) \cong gR \neq 0$, whence $gy \neq 0$. Note also that $N(ey) = N(e)/t$ for all $e \in B(R)$. For all nonzero $e \leqq g$ in $B(R)$,

$$N(ey) = N(e)/t \leqq N(e)/t + \sum N(ex_n) < N(ex) ,$$

hence $N(ey) \leqq N(ex)$ for all $e \leqq g$ in $B(R)$. According to Lemma 6.6, $gyR \lesssim gxR$, hence $gyR \cong zR$ for some nonzero $z \in gxR$. Write $gxR = zR \oplus wR$ for some w, and note that

$$\sum N(ex_n) < N(ex) - N(e)/t = N(ex) - N(ez) = N(ew)$$

for all nonzero $e \leqq g$ in $B(R)$.

In particular, $N(ex_1) \leqq N(ew)$ for all $e \leqq g$ in $B(R)$, hence Lemma 6.6 shows that $gx_1R \cong w_1R$ for some $w_1 \in wR$. Next, $wR = w_1R \oplus u_1R$ for some u_1, and

$$N(ex_2) \leqq \sum N(ex_n) - N(ex_1) \leqq N(ew) - N(ew_1) = N(eu_1)$$

for all $e \leqq g$ in $B(R)$, hence Lemma 6.6 shows that $gx_2R \cong w_2R$ for some $w_2 \in u_1R$. Continuing in this manner, we obtain an independent sequence $w_1R, w_2R, \cdots \leqq wR$ such that $gx_nR \cong w_nR$ for all n. Thus $\bigoplus gx_nR \lesssim wR$. Inasmuch as $\bigoplus gx_nR$ is essential in gxR, it follows that $gxR \lesssim wR$. But then $N(z) + N(w) = N(gx) \leqq N(w)$ and so $N(z) = 0$, which contradicts the fact that $z \neq 0$.

Therefore $\sum N(x_n) = N(x)$, so that N is countably additive.

THEOREM 6.9. *Let R be a regular, right and left self-injective ring with a rank function N. Then R is complete in the N-metric if and only if $B(R)$ is orthogonally complete with respect to N.*

Proof. Obviously completeness of R implies orthogonal completeness of $B(R)$. Conversely, assume that $B(R)$ is orthogonally complete.

According to [18, Theorems 4.7, 5.1], R is directly finite, hence [8, Corollary 7.6] shows that there is some $g \in B(R)$ such that gR is Type I_f and $(1-g)R$ is Type II_f. If $g \neq 0$, then we may define a rank function P on gR by the rule $P(x) = N(x)/N(g)$. Observing that $B(gR)$ is orthogonally complete with respect to P, we see from Proposition 6.5 that gR is complete in the P-metric, hence also in the N-metric. If $1 - g \neq 0$, then we may define a rank function Q on $(1-g)R$ by the rule $Q(x) = N(x)/N(1-g)$. According to Proposition 6.8, Q is countably additive, whence Theorem 6.4 shows that $(1-g)R$ is complete in the Q-metric, and thus also in the N-metric.

Therefore gR and $(1-g)R$ are both complete in the N-metric, whence R is complete in the N-metric.

THEOREM 6.10. *Let R be a regular, right and left self-injective ring, and let X be a nonempty subset of $\boldsymbol{P}(R)$ such that* $\ker(X) = 0$. *Then the following conditions are equivalent*:

(a) *R is complete with respect to X.*

(b) *$B(R)$ is orthogonally complete with respect to X.*

(c) *Every ideal of $B(R)$ which is closed in the X-topology is principal.*

Proof. (a) $\Rightarrow$ (c): If I is an ideal of $B(R)$ which is closed in the X-topology, then we check that IR is a two-sided ideal of R which is closed in the X-topology. According to Lemma 3.2, $IR = eR$ for some $e \in B(R)$, whence $I = eB(R)$.

(c) $\Rightarrow$ (b): Let $\{e_i \mid i \in I\}$ be a family of pairwise orthogonal idempotents in $B(R)$. Let $\mathscr{F}$ be the family of nonempty finite subsets of I, and set $e_F = \sum_{i \in F} e_i$ for all $F \in \mathscr{F}$. Set $J = \{e \in B(R) | e \leqq e_F$ for some $F \in \mathscr{F}\}$, and note that J is an ideal of $B(R)$. If K is the X-closure of J, then K is an ideal of $B(R)$, and (c) says that K is generated by some $f \in B(R)$. In particular, note that $e_F \leqq f$ for all $F \in \mathscr{F}$.

Given $N \in X$ and $\varepsilon > 0$, there is some $e \in J$ such that $N(e - f) < \varepsilon$, and $e \leqq e_F$ for some $F \in \mathscr{F}$. Whenever $G \supseteqq F$ in $\mathscr{F}$, we have $e \leqq e_F \leqq e_G \leqq f$, hence $f - e_G = (f - e_G)(f - e)$ and so $N(f - e_G) \leqq N(f - e) < \varepsilon$. Thus $\sum e_i \rightarrow f$, so that $B(R)$ is orthogonally complete.

(b) $\Rightarrow$ (a): According to Corollary 2.7, there exists a facially independent set $Y = \{N_k\} \subseteq \boldsymbol{P}(R)$ such that Y and X generate the same σ-convex face in $\boldsymbol{P}(R)$. In view of Corollary 1.3 and Proposition 1.7, we see that $B(R)$ is orthogonally complete with respect to Y, and that it suffices to prove that R is complete with respect to Y. Therefore we may assume, without loss of generality, that $X - Y$. For each k, let F_k be the face generated by N_k in $\boldsymbol{P}(R)$.

For each k, $\ker(N_k)$ is a two-sided ideal of R which is closed in the X-topology. Using (b), we see (as in Lemma 3.2) that $\ker(N_k) = (1 - e_k)R$ for some $e_k \in B(R)$. Now N_k restricts to a rank function on e_kR, and since $B(R)$ is orthogonally complete with respect to X we see that $B(e_kR)$ is orthogonally complete with respect to N_k. As a result, Theorem 6.9 shows that e_kR is complete in the N_k-metric. If ϕ_k denotes the natural map from R into its N_k-completion $\bar{R}_k$, we thus have shown that ϕ_k is surjective. Recall that $\ker(\phi_k) = \ker(N_k) = (1 - e_k)R$.

Suppose that $e_je_k \neq 0$ for some $j \neq k$. Then we may define pseudo-rank functions $N'_j, N'_k \in \boldsymbol{P}(R)$ by the rules $N'_j(x) = N_j(e_je_kx)/N_j(e_je_k)$ and $N'_k(x) = N_k(e_je_kx)/N_k(e_je_k)$. By [7, Corollary 3.3], $N'_j \in F_j$ and

$N'_k \in F_k$. Set $N = (N'_j + N'_k)/2$, and note that N, N'_j, and N'_k all restrict to rank functions on e_je_kR. Given orthogonal idempotents $\{f_n\} \subseteq B(e_je_kR)$, (b) says that $\sum f_n$ must converge (in the X-topology) to some $f \in B(R)$, and we note that $f \in B(e_je_kR)$. In particular, $\sum f_n \to f$ in the N_j-metric and the N_k-metric, from which we infer the $\sum f_n \to f$ in the N-metric. Therefore $B(e_je_kR)$ is orthogonally complete with respect to N, hence Theorem 6.9 says that e_je_kR is complete in the N-metric. Note that $N'_j, N'_k \ll N$. Inasmuch as N'_j and N'_k both restrict to rank functions on e_je_kR, it now follows from [7, Lemma 4.1] that these restrictions are facially dependent in $\boldsymbol{P}(e_je_kR)$. Consequently, there exist $P \in \boldsymbol{P}(e_je_kR)$ and $\alpha > 0$ such that $P \leqq \alpha N'_j, \alpha N'_k$ on e_je_kR. Defining P to be zero on $(1 - e_je_k)R$, we obtain $P \in \boldsymbol{P}(R)$ such that $P \leqq \alpha N'_j, \alpha N'_k$. Using [7, Corollary 3.3] again, we find that $P \in F_j \cap F_k$, which is impossible.

Therefore $e_je_k = 0$ for all $j \neq k$. We thus have pairwise orthogonal central idempotents e_k such that the annihilator of the ideal $\bigoplus e_kR$ is $\cap (1 - e_k)R = \cap \ker(N_k) = \ker(X) = 0$. As in [5, Theorem 18], it follows that the natural map $R \to \prod e_kR$ is an isomorphism. Inasmuch as each $\phi_k: R \to \bar{R}_k$ is surjective with kernel $(1 - e_k)R$, we now see that the map $\phi: R \to \prod \bar{R}_k$ induced by the ϕ_k must be an isomorphism.

Finally, let $\bar{R}$ denote the X-completion of R, let $\psi: R \to \bar{R}$ and $\theta: \bar{R} \to \prod \bar{R}_k$ be the natural maps, and note that $\theta\psi = \phi$. Since the faces F_k are pairwise disjoint, we conclude from Theorem 4.6 that θ is an isomorphism. Therefore the inclusion map $\psi = \theta^{-1}\phi: R \to \bar{R}$ is an isomorphism, whence R is complete with respect to X.

Returning to our original question, we now see that in order for a regular self-injective ring R to be complete with respect to some nonempty $X \subseteq \boldsymbol{P}(R)$, we need only find such an X such that $B(R)$ is orthogonally complete with respect to X. However, this is not always possible, as the following example shows.

By [4, Theorem 2.2], there exists a nonzero Boolean algebra B with the countable chain condition such that no direct summand of B has a strictly positive finitely additive measure. Considering B as a (commutative) regular ring in the usual way, this says that B contains no uncountable direct sums of nonzero ideals, and that there does not exist a rank function on any direct summand of B.

Now let R be the maximal quotient ring of B, which is a regular self-injective ring. In fact, R is the Boolean completion of B [3, Theorem 5], so that $B(R) = R$. Since B_B is essential in R_B, we see that R does not contain any uncountable direct sums of nonzero ideals (i.e., as a Boolean algebra, R satisfies the countable chain condition). Suppose there is an idempotent $e \in R$ such that there is a rank function N on eR. Then $e \neq 0$, hence there exists a nonzero idempotent $f \in eR \cap B$. But then N induces a rank function on fB, which cannot happen. Thus there does not exist a rank function on any direct summand of R.

If R is complete with respect to some family of pseudo-rank functions, then using Theorem 4.6 we see that R must be isomorphic to a direct product $\prod R_k$, where each R_k is complete with respect to a rank function N_k. But then there exist rank functions on some direct summands of R, which is false. Therefore R is not complete with respect to any family of pseudo-rank functions.

Returning to the general case, we are left with the following problem: Given a regular, right and left self-injective ring R, when is $B(R)$ orthogonally complete with respect to some family of pseudo-rank functions? Since all pseudo-rank functions on $B(R)$ extend to pseudo-rank functions on R by [7, Corollary 6.10], we need only look for a suitable family of pseudo-rank functions on $B(R)$. This reduces the problem to Boolean algebras. For the case of a single pseudo-rank function, we thus have the following problem: Given a Boolean algebra B, when does there exist a rank function N on B such that B is complete in the N-metric? Obviously B must be complete and satisfy the countable chain condition, but the example above shows that these conditions are not sufficient. Rather complicated necessary and sufficient conditions on B may be found in [13, Theorems 4, 9] and [15, Theorem 4].

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