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AN ALGEBRAIC CLOSED GRAPH THEOREM

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In this work we consider the question as to when an everywhere defined closed linear map from a quadratic space H_1 into another such space H_2 is orthocontinuous. The following result is proved:

Let (H_1, Φ_1), (H_2, Φ_2) be quadratic spaces whose $\perp$-closed subspaces are semi-simple. If T is an everywhere defined closed linear map on H_1 into H_2 then T is orthocontinuous.

1. **Introduction.** In [1], [2] Piziak generalized, algebraically, the geometry of Hilbert space. He introduced the notion of quadratic space and with this studied sesquilinear forms in infinite dimensions. He showed that certain general results which are of pure algebra imply standard topological results in the context of Hilbert space (e.g., an analogue of the Riesz representation theorem was proved for these spaces and this implies the Riesz representation theorem for Hilbert spaces).

Now, in Hilbert space an everywhere defined linear operator is continuous iff its graph is closed. It is known that if T is an everywhere defined linear operator on a quadratic space and if T is orthocontinuous then the graph of T is $\perp$-closed. The question is whether or not the converse of this is true. In [2] it is conjectured that this may not be true in general but that it may be true if our quadratic space is such that every $\perp$-closed subspace is splitting. In this work we show that this conjecture is true. In fact we show that if every $\perp$-closed subspace of our quadratic space is semi-simple then T is orthocontinuous. We also consider other cases where T is orthocontinuous but where H_1, H_2 are neither both anisotropic nor are both such that their $\perp$-closed subspaces are semi-simple.

One of the implications of our results is that in the case of inner-product spaces the completeness of the spaces is not necessary for the "algebraic closed graph theorem" to hold. Thus the theorem holds for pre-Hilbert spaces. Surprising as this may seem at first, we point out that the algebraic closed graph theorem does not imply the closed graph theorem. This is because there may be no context in a quadratic space in which to discuss continuity. Even if such a context exists it is possible for an orthocontinuous map not to be continuous as Example 4 shows.

2. Preliminaries.

DEFINITION 2.1. [2] A quadratic space is a triple (K, H, Φ) where K is a division ring with involution $*$, say, H is a left vector space over K and Φ is a nondegenerate orthosymmetric sesquilinear form on H with respect to the involutive anti-automorphism $*$ of K.

DEFINITION 2.2. [2] Let (K, H, Φ) be a quadratic space. For x, y in H we say x is orthogonal to y and write $x \perp y$ iff $\Phi(x, y) = 0$.

We note that since Φ is orthosymmetric, $x \perp y$ implies $y \perp x$ and conversely.

We shall in what follows suppress K, Φ if there is no danger of confusion and refer to a quadratic space (K, H, Φ) as H.

DEFINITION 2.3. [2] Let H be a quadratic space. An element x in H is said to be isotropic iff $x \perp x$ and anisotropic otherwise. If every element in H is anisotropic we say H is anisotropic.

For any subset A of H put

$$A^{\perp} = \{y \in H: y \perp x \text{ for all } x \in A\} .$$

It is easy to see that $A^{\perp}$ is a subspace of H for every subset A of H.

DEFINITION 2.4. [1] A subspace M of H is said to be $\perp$-closed iff $M^{\perp\perp} = M$. If $H = M \oplus M^{\perp}$ we say that M is a splitting subspace, and if $M \cap M^{\perp} = (0)$ we say that M is semi-simple.

We point out that while it is true that every splitting subspace is $\perp$-closed and semi-simple it is not true in general that every semi-simple subspace is splitting. In fact a $\perp$-closed semi-simple subspace of a quadratic space H may not split H, e.g., [1, Proposition 3.2.23]. If however H is finite dimensional then every semi-simple subspace is splitting [1, Corollary 3.5.8].

As pointed out in [2] the nature of the scalars and the possibility of existence of nonzero isotropic vectors are two main differences between Hilbert spaces and general quadratic spaces. (Isotropic vectors play an important role in some physical theories, e.g., in the geometry space-time with the Minkowskii metric [2].)

DEFINITION 2.5. Let (H_1, Φ_1), (H_2, Φ_2) be quadratic spaces. Let $T: H_1 \to H_2$ be a linear map. T is said to be orthocontinuous if

$$T(M^{\perp\perp}) \subseteq T(M)^{\perp\perp}$$

for every subspace M of H_1.

PROPOSITION 2.6. [2] *Let* (H_1, Φ_1), (H_2, Φ_2) *be quadratic spaces and* $T: H_1 \to H_2$ *be a linear map. Then the following are equivalent*:

(i) $M = M^{\perp\perp} \Rightarrow T^{-1}(M) = T^{-1}(M)^{\perp\perp}$ *for all subspaces* M *of* H_2.

(ii) *If* M *is a* $\perp$*-closed subspace of* H_2 *then* $T^{-1}(M)$ *is a* $\perp$*-closed subspace of* H_1.

(iii) $T(M^{\perp\perp}) \subseteq T(M)^{\perp\perp}$ *for all subspaces* M *of* H_1.

(iv) $(T^{-1}(M))^{\perp\perp} \subseteq T^{-1}(M^{\perp\perp})$ *for all subspaces* M *of* H_2.

(v) T *is orthocontinuous.*

In Hilbert space the restriction of a continuous map to a closed subspace is continuous. This is not the case in general in a quadratic space.

PROPOSITION 2.7. *Let* (H_1, Φ_1), (H_2, Φ_2) *be quadratic spaces and* $T: H_1 \to H_2$ *an orthocontinuous linear map. If* M *is a* $\perp$*-closed semi-simple subspace of* H_1 *and* Φ_M *is the restriction of* Φ_1 *to* M *then* (K, M, Φ_M) *is a quadratic space. Further the restriction of* T *to* M *is orthocontinuous.*

Proof. To show that (M, Φ_M) is a quadratic space it suffices to show that Φ_M is nondegenerate. Suppose $x \in M$ and $\Phi_M(x, y) = 0$ for all $y \in M$. Then $x \in M^\perp$. But $M \cap M^\perp = (0)$; hence $x = 0$. Hence Φ_M is nondegenerate.

Now let A be a $\perp$-closed subspace of H which is contained in M. We note that

$$\{x \in M: \Phi_M(x, y) = 0 \text{ for all } y \in A\} = M \cap A^\perp \,.$$

Thus the closure in M of A is $M \cap (M \cap A^\perp)^\perp$. But

$$\begin{aligned} M \cap (M \cap A^\perp)^\perp &= M \cap (M^\perp \vee A^{\perp\perp}) \\ &= M \cap M^\perp \vee M \cap A^{\perp\perp} \\ &= M \cap M^\perp \vee A \\ &= 0 \vee A \,. \end{aligned}$$

i.e. Any $\perp$-closed subspace of H contained in M is $\perp$-closed in M. If T_M is the restriction of T to M and B is a $\perp$-closed subspace of H_2 then $T_M^{-1}(B) = M \cap T^{-1}(B)$. Since M is $\perp$-closed in H_1 and T is orthocontinuous we have that $T^{-1}(B) \cap M$ is a $\perp$-closed subspace of H_1 which is contained in M and hence by the above argument is $\perp$-closed in M. Thus T_M is orth ocontinuous.

3. Algebraic closed graph theorem.

DEFINITION 3.1. Let (H_i, Φ_i) $i = 1, 2$ be quadratic spaces and

$T: H_1 \to H_2$ be a linear map. The graph of T, written $G(T)$, is the set

$$G(T) = \{(x, Tx): x \in D_T \subseteqq H_1\} .$$

PROPOSITION 3.2. *Let (H_i, Φ_i) $i = 1, 2$ be quadratic spaces. Then a subspace of $H_1 \times H_2$ of the form $A \times B$ where A and B are subspaces of H_1 and H_2 respectively is $\perp$-closed in $H_1 \times H_2$ iff A is a $\perp$-closed subspace of H_1 and B a $\perp$-closed subspace of H_2.*

Proof. Let $(x, y) \in (A \times B)^\perp$. Then

$$\begin{aligned} 0 &= \Phi_1 \oplus \Phi_2((x, y), (u, v)) \qquad \text{for all} \quad (u, v) \in A \times B \\ &= \Phi_1(x, u) + \Phi_2(y, v) \qquad \text{for all} \quad (u, v) \in A \times B . \end{aligned}$$

In particular for $u = 0$ we have that $y \in B^\perp \subseteqq H_2$. Similarly $x \in A^\perp \subseteqq H_1$. Hence $(x, y) \in A^\perp \times B^\perp$. It is clear that if $(x, y) \in A^\perp \times B^\perp$ then $(x, y) \in (A \times B)^\perp$. Thus $(A \times B)^\perp = A^\perp \times B^\perp$. If $A \times B$ is a $\perp$-closed subspace of $H_1 \times H_2$ we have that

$$\begin{aligned} (A \times B) &= (A \times B)^{\perp\perp} \\ &= A^{\perp\perp} \times B^{\perp\perp} . \end{aligned}$$

So, $A = A^{\perp\perp}$ and $B = B^{\perp\perp}$. Conversely if $A^{\perp\perp} = A$ and $B^{\perp\perp} = B$ then from $(A \times B)^\perp = A^\perp \times B^\perp$ we have that $A \times B$ is a $\perp$-closed subspace of $H_1 \times H_2$.

COROLLARY 3.3. *Let (H_i, Φ_i) $i = 1, 2$ be quadratic spaces and let $\pi: H_1 \times H_2 \to H_1$* (resp. *$\pi_2: H_1 \times H_2 \to H_2$) be the linear map defined by*

$$\pi((x, y)) = x (\text{resp. } \pi_2((x, y)) = y) \qquad \textit{for all} \quad (x, y) \in H_1 \times H_2 .$$

Then for any subspace A of H_1(resp. B *of* H_2)$\pi^{-1}(A)$(resp. $\pi_2^{-1}(B)$) *is $\perp$-closed in $H_1 \times H_2$ iff A is $\perp$-closed in H_1,* (resp. iff B is $\perp$-closed in H_2).

Proof. $\pi^{-1}(A) = A \times H_2$ for any subset A of H_1. This by Proposition 3.2 we have that $A \times H_2$ is a $\perp$-closed subspace of $H_1 \times H_2$ iff A is a $\perp$-closed subspace of H_1. (The proof for π_2 proceeds similarly.)

PROPOSITION 3.4. *Let M be a splitting subspace of a quadratic space (H, Φ). Then there exists a $*$ sesquilinear form Ψ on $H/M^\perp$ with respect to which $(H/M^\perp, \Psi)$ is a quadratic space. Moreover if p is the canonical map $p: H \to H/M^\perp$ and if a subspace A of $H/M^\perp$ is $\perp$-closed then $p^{-1}(A)$ is a $\perp$-closed subspace of H. If H is of* *the form* $(H_1 \times H_2, \Phi_1 \oplus \Phi_2)$ *where* (H_i, Φ_i) $i = 1, 2$ *are quadratic spaces and* $M = H_1 \times (0)$, *say, then a subspace* A *of* $H/M^\perp$ *is* $\perp$*-closed iff* $p^{-1}(A)$ *is a* $\perp$*-closed subspace of* H.

Proof. Since M is a splitting subspace it is semi-simple and hence the restriction of Φ to M, Φ_M, say is such that (M, Φ_M) is a quadratic space, by Proposition 2.7. Also since M is a splitting subspace there exists a projection P on H such that $M = \mathrm{Im}(P)$ [1]. Thus $M^\perp = \mathrm{Ker}\, P$ and so there exists an isomorphism $\phi: H/M^\perp \to M$. For $[x], [y] \in H/M^\perp$ define

$$\Psi([x], [y]) = \Phi_M(\phi([x]), \phi([y])) .$$

We then have that

$$\begin{aligned} &\Psi([x], [y]) = 0 \quad \text{for all} \quad [x] \in H/M^\perp \\ &\iff \Phi_M(\phi([x]), \phi([y])) = 0 \quad \text{for all} \quad \phi([x]) \in M \\ &\iff \phi([y]) = 0 \quad \text{since } \Phi_M \text{ is nondegenerate and } \phi \text{ is onto} \\ &\iff [y] = 0 \quad \text{since } \phi \text{ is } 1-1 . \end{aligned}$$

Therefore Ψ is nondegenerate. It is easy to see that Ψ is orthosymmetric and $*$ sesquilinear relative to the involutive anti-automorphism $*$ of K. So, $(H/M^\perp, \Psi)$ is a quadratic space. Let $A \subseteq H/M^\perp$.

$$\begin{aligned} [x] \in A^\perp &\iff \Phi_M(\phi([x]), \phi([y])) = 0 \forall [y] \in A \\ &\iff \phi([y]) \in \phi(A)^\perp . \end{aligned}$$

From this we obtain $\phi(A^\perp) = \phi(A)^\perp$ and hence that ϕ maps $\perp$-closed subspaces of $H/M^\perp$ into $\perp$-closed subspaces of M. Therefore ϕ^{-1} is orthocontinuous. Also if $B^{\perp\perp} = B$ in M, we have that there exists an $A \subseteq H/M^\perp$ such that $A^{\perp\perp} = A$ in $H/M^\perp$ and $\phi(A) = B$. Indeed, since ϕ is onto there exists $A \subseteq H/M^\perp$ such that $\varphi(A) = B$.

$$\begin{aligned} B &= B^{\perp\perp} \\ &= \phi(A)^{\perp\perp} \\ &= \phi(A^{\perp\perp}) . \end{aligned}$$

Hence

$$\begin{aligned} A &= \phi^{-1}(B) \\ &= \phi^{-1}(\phi(A^{\perp\perp})) \\ &= A^{\perp\perp} \end{aligned}$$

since ϕ is a bijection. Thus every $\perp$-closed subspace of M is the image of a $\perp$-closed subspace of $H/M^\perp$, i.e., if B is a $\perp$-closed subspace of M then $\phi^{-1}(B)$ is a $\perp$-closed subspace of $H/M^\perp$. Hence φ is orthocontinuous. Now consider the following diagram:

$$\begin{array}{ccc} H & & \\ {\scriptstyle p}\downarrow & \searrow {\scriptstyle P} & \\ H/M^{\perp} & \xrightarrow{\phi} & M\,. \end{array}$$

Suppose A is a $\perp$-closed subspace of $H/M^{\perp}$. Since

$$p^{-1}(A) = P^{-1}(\phi(A))$$

and P being a projection is orthocontinuous [1] we have that $p^{-1}(A)$ is a $\perp$-closed subspace of H. Finally suppose $H = (H_1 \times H_2, \Phi_1 \oplus \Phi_2)$ then $H_1 \times \{0\}$, $\{0\} \times H_2$ are $\perp$-closed subspaces of H which split H. Put $M = H_1 \times \{0\}$. By Corollary 3.3 we have that if B is a subspace of $H_1 \cong H_1 \times \{0\}$ then $P^{-1}(B)$ is $\perp$-closed in $H = H_1 \times H_2$ iff B is $\perp$-closed in H_1. Therefore $p^{-1}(A) = P^{-1}(\phi(A))$ is $\perp$-closed in $H_1 \times H_2$ iff $\phi(A)$ is $\perp$-closed in M iff A is $\perp$-closed in $H/M^{\perp}$.

COROLLARY 3.5. *Let (H_i, Φ_i) $i = 1, 2$ be quadratic spaces. Let $\pi: H_1 \times H_2 \to H_1$ be defined by $\pi((x, y)) = x$ for all (x, y) in $H_1 \times H_2$. Then π maps $\perp$-closed subspaces of $H_1 \times H_2$ onto $\perp$-closed subspaces of H_1.*

Proof. Consider the following diagram:

$$\begin{array}{ccc} H_1 \times H_2 & & \\ {\scriptstyle p}\downarrow & \searrow {\scriptstyle \pi} & \\ H_1 \times H_2/\pi^{-1}(0) & \xrightarrow{\phi} & H_1 \end{array}$$

where p and ϕ are the mappings defined in Proposition 3.4. Let A be a $\perp$-closed subspace of $H_1 \times H_2$. Then $A = p^{-1}(B)$ for some B in $H_1 \times H_2/\pi^{-1}(0)$. By Proposition 3.4 we have that B is $\perp$-closed in $H_1 \times H_2/\pi^{-1}(0)$. Thus

$$\begin{aligned} \pi(A) &= \phi(p(A)) \\ &= \phi(p(p^{-1}(B))) \\ &= \phi(B) \end{aligned}$$

is $\perp$-closed in H_1.

REMARK. We note that if M is a $\perp$-closed subspace of a quadratic space H although it is not true in general that the intersection of a $\perp$-closed subspace of H and M is $\perp$-closed in M it is however true that every $\perp$-closed subspace of M is $\perp$-closed in H. Indeed if $A \subseteq M$ then the closure of A in M is $M \cap (M \cap A^{\perp})^{\perp}$ which is a $\perp$-closed subspace of H. It follows therefore that if T is a linear transformation on H which maps $\perp$-closed subspaces into $\perp$-closed subspaces, the restriction of T to any $\perp$-closed subspace, M, of H maps $\perp$-closed subspaces of M into $\perp$-closed subspaces of the co-domain of T.

THEOREM 3.6. *Let (H_i, Φ_i) $i = 1, 2$ be quadratic spaces such that every $\perp$-closed subspace of H_1, H_2 is semi-simple. Let $T: H_1 \to H_2$ be an every where defined closed linear map. Then T is orthocontinuous.*

Proof. We first note that since every $\perp$-closed subspace of H_1, H_2 is semi-simple the same is true of every $\perp$-closed subspace of $H_1 \times H_2$. Hence, since $G(T)$ is $\perp$-closed we have that $G(T) \cap G(T)^{\perp} = (0)$ and that the restriction of any orthocontinuous linear map on $H_1 \times H_2$ to $G(T)$ is orthocontinuous. Let $\pi_1: H_1 \times H_2 \to H_1$ be defined by $\pi_1(x, y) = x$ for every (x, y) in $H_1 \times H_2$. By Corollary 3.5 π_1 maps $\perp$-closed subspaces onto $\perp$-closed subspaces. The restriction of π_1 to $G(T)$, $\pi_{1/G(T)}$, is 1-1, onto and by the Remark maps $\perp$-closed subspaces of $G(T)$ onto $\perp$-closed subspaces of H_1. Therefore $\pi_{1/G(T)}^{-1}$ is orthocontinuous. Also $\pi_2: H_1 \times H_2 \to H_2$ defined by $\pi_2((x, y)) = y$ for all $(x, y) \in H_1 \times H_2$ is orthocontinuous by Corollary 3.3. Therefore its restriction to $G(T)$, $\pi_{2/G(T)}$, is orthocontinuous.

Now

$$Tx = \pi_{2/G(T)} \circ \pi_{1/G(T)}^{-1}(x)$$

which is orthocontinuous.

An observation of the proof of the theorem shows that if the graph of T is semi-simple then T is orthocontinuous. We now consider other cases where the conditions on H_1, H_2 imply this and hence the orthocontinuity of T.

COROLLARY 3.3. *Let (H_i, Φ_i) $i = 1, 2$ be anisotropic quadratic spaces. If T is an everywhere defined closed linear map on H_1 into H_2 then T is orthocontinuous.*

Proof. Since (H_i, Φ_i) $i = 1, 2$ is anisotropic so also is $(H_1 \times H_2, \Phi_1 \oplus \Phi_2)$ as can be easily checked. Hence every $\perp$-closed subspace of $H_1 \times H_2$ is splitting. Since a splitting subspace is semi-simple we have that every $\perp$-closed subspace of $H_1 \times H_2$ is semi-simple. The result then follows from the theorem.

REMARK. Corollary 3.7 establishes Piziak's conjecture [2].

PROPOSITION 3.8. *Let (H_i, Φ_i) $i = 1, 2$ be quadratic spaces over a division ring K. Suppose there exists a subset R of K with the* *following properties:*

(i) $0 \varepsilon \pm R$

(ii) $R \cap -R = \{0\}$.

If $\Phi_1(x, x)$, $\Phi_2(y, y) \in R$ for all $x \in H_1$, $y \in H_2$ and if H_1 is anisotropic then an everywhere defined closed linear map $T: H_1 \rightarrow H_2$ is orthocontinuous.

Proof. In view of the observation made after the proof of the theorem, it suffices to show that $G(T)$ is semi-simple. Suppose $(x, Tx) \in G(T) \cap G(T)^{\perp}$ then

$$\begin{aligned} \Phi(x, x) + \Phi_2(Tx, Tx) &= \Phi_1 \oplus \Phi_2((x, Tx), (x, Tx)) \\ &= 0 . \\ \therefore \quad \Phi_1(x, x) &= -\Phi_2(Tx, Tx) . \end{aligned}$$

But $\Phi_1(x, x) \in R$ and $\Phi_2(Tx, Tx) \in R$. Thus if $(x, Tx) \in G(T) \cap G(T)^{\perp}$ we have that $\Phi_1(x, x) \in R \cap -R$. But $R \cap -R = \{0\}$. Therefore $\Phi_1(x, x) = 0$. Since H_1 is anisotropic we have that $x = 0$ and so $(x, Tx) = (0, 0)$ so $G(T)$ is semi-simple.

REMARK. (1) If in the above proposition T is 1-1 and H_2 is anisotropic while H_1 is allowed to be arbitrary the same result is obtained.

(2) As pointed out in the introduction, our results imply that in the case of inner product space the completeness of the spaces is not necessary for an everywhere defined closed linear map to be orthocontinuous. We pointed out also that even if there is a context in which to discuss continuity it is possible for an orthocontinuous map not to be continuous. We now give an example.

4. Example. Let $K = R^1$, $H_1 = H_2 = R^4$. Define Φ_1 on H_1 by

$$\Phi_1((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = \sum_1^3 x_i y_i - x_4 y_4$$

and Φ_2 on H_2 by

$$\Phi_2((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = \sum_1^4 x_i y_i .$$

Then (H_i, Φ_i) $i = 1, 2$ are quadratic spaces. In fact H_2 is a Hilbert space as is well known. Let $\mathcal{T}$ be a collection of subsets of H_1 consisting of $\varnothing$, H_1, and all subsets A of H_1 for which there exists a $\perp$-closed subspace $M \subseteq H_1$ with $A \subseteq H_1 \sim M$. Then

(i) $\varnothing, H_1 \in \mathcal{T}$ by definition.

(ii) If $A_\alpha \in \mathcal{T}$ we have that $\bigcup_\alpha A_\alpha \in \mathcal{T}$.

Indeed since for each α there exists a $\perp$-closed subspace M_α such that $A_\alpha \subseteqq H_1 \sim M_\alpha$ we have $\bigcup_\alpha A_\alpha \subseteqq \bigcup_\alpha H_1 \sim M_\alpha = H_1 \sim \bigcap_\alpha M_\alpha$. But $\bigcap_\alpha M_\alpha$ is $\perp$-closed. Therefore $\bigcup_\alpha A_\alpha \in \mathscr{T}$.

(iii) If A_k $k = 1, 2, \cdots, n \in \mathscr{T}$ then $\bigcap_{k=1}^n A_k \in \mathscr{T}$. For $\bigcap_{k=1}^n A_k \subseteqq A_k$ for all $k = 1, 2, \cdots, n$, and $A_k \subseteqq H_1 \sim M_k$ for some $\perp$-closed subspace M_k of H_1.

Therefore $\mathscr{T}$ is a topology for H_1. Let $\mathscr{U}$ be the usual topology of R^4 which as is known arises from Φ_2 in a natural way. Now let $I: (H_1, \mathscr{T}) \to (H_2, \mathscr{U})$ be the map defined by $Ix = x$ for all $x \in H_1$. Then I is orthocontinuous since the inverse image of a $\perp$-closed subspace of (H_2, Φ_2) is a finite dimensional subspace of (H_1, Φ_1) and hence by [1, Corollary 3.5.2] is a $\perp$-closed subspace of (H_1, Φ_1). But I is not continuous for if it were $I^{-1}(\{x \in H_2: ||x|| < 1\})$ would be an open set in $(H_1, \mathscr{T})$ and hence would not contain 0. This is a contradiction since $0 \in I^{-1}(\{x \in H_2: ||x|| < 1\})$.

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